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Results on Interval-Valued Optimization Problems for Vector Variational-like Inequalities

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Abstract The objective of this article is to present a new class of vector interval-valued variational-like inequality problems. Based on the concepts of LU optimal and weakly LU optimal solutions, we derive some relations between the interval-valued programming and variational-like inequality problems. The study of the interplay between interval-valued optimization problems (IVOP) and vector variational-like inequalities (VVLI) combines theoretical advancements under the concept of differentiability and μ- invexity. Furthermore, to demonstrate the established linkages, we provide examples and an example demonstrates how well the vector variational inequality problems may be applied to deal with (MOIVP) problems.

Keywords Interval-valued programming, invexity, LU-optimal, variational-like inequality problem

MSC(2010) 90C26, 90C30, 90C33.

1. Introduction

In 1980, Giannessi [11] initially introduced vector variational inequality (VVI) in Euclidean space via finite dimensions. The relationship between variational-like inequality and certain mathematical programming problems was identified by Parida et al. [25], who also offered a theory for the existence of a solution to variational-like inequalities. Later on Deng [9] enunciated necessary and sufficient conditions for the existence of weak minima in constrained convex vector optimization problems. New iterative techniques that, under certain conditions, can be used to solve mixed variational-like inequalities were studied by Noor et al. [22] using convergence analysis. In Variational Inequalities, along with their spectrum of applications, Kinderlehrer et al. [17] and Mordukhovich [20] introduced an applicative approach along with exciting emerging fields like medicine, finance, optimization, system stability, environmental, science and phase transformations (also see [34]).

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Oveisiha et al. [23] considered two generalized minty vector variational like inequalities and investigated the relations between their solutions and vector optimization problems for non-differentiable α-invex functions. In general mathematical programming coefficients are always considered as deterministic value, but in reality it observed that the parameters may not be known certainly. Numerous techniques, such as fuzzy numbers and stochastic processes, can be used to manage uncertainty in the real world. Still, given the complexity and incompleteness of the data, it can be challenging to identify a suitable probability distribution or membership function. Because of this, there has been a lot of interest in the uncertain optimization problem recently. An approach to overcome the uncertain optimization problem where only the range of the coefficients is known, is interval-valued programming and it does not require the assumption of probabilistic (stochastic programming) and possibility distributions (fuzzy programming).

As an extension of convexity, Zhang et al. [36] explored LU-convexity to figure out the optimality criteria for real-valued maps with the consideration for optimal solutions with interval values. The role of invexity in variational-like inequalities is the same as that of classical convexity in variational inequalities; this indicates that variational-like inequalities are well-defined in terms of invexity. It has been noted that all the results for variational-like inequalities are derived in the setting of classical convexity, (see Gupta et al. [12], and Jennane et al. [15]). Subsequently, numerous analogous inequalities were obtained for various categories of preinvex functions; (refer to Noor et al. [21]).

Very recently, the equivalence between generalized Stampacchia vector variational inequality and quasi LU-efficient solutions to interval-valued vector optimization challenges was established through Upadhyay et al. [30], who additionally defined a generalized LU-approximately convex function. In the meantime, Laha et al. [18] linked multi-objective optimization problems and vector variational-like inequalities in this chain. Since optimization algorithms usually offer only approximate solutions and terminate in a finite number of steps in a wide range of real-world problems, Evtushenko [10] provided an association between Farkas' theorem and linear and quadratic programming. It has recently been established that vector variational inequalities and vector optimization problems are related to convexificators by Bhardwaj et al., [4] khan et al. [16], Pany et al. [24], and Upadhyay et al. [31] .

Over the past few years, various extensions and generalizations of the variational inequalities have been introduced. A number of outstanding reviews, including those by Chang et al. [6] , Mohapatra et al. [19], have been published to provide an overview of the state of knowledge on the variational-like inequalities. In the recent past, Antczak et al. [2,3], Abdulaleem N. [1], Huy et al. [14], and Treanţă [29] have shown that vector variational inequalities can describe the optimality conditions for vector optimization problems under some certain conditions. Recently, many researchers have opened up a new dimension of best proximity point (BPP) results for an applicative approach to optimization. For this, Younis et al. [33] invoked the BPP for multivalued mappings with the application of the equation of motion. For further applications in this set up, one can see the noteworthy work done in Dar et al. [8].

In this paper, by utilizing the concepts of LU optimal and weakly LU optimal solutions some relations are investigated between the interval-valued optimization and variational-like inequality problems. Moreover, examples are given to validate the derived relations. The rest of the paper is organized as follows. Sections 2 and 3 review some terminology and basic definitions needed for the following sections, while Section 4 focuses on establishing some connections between variational-like inequality problems and interval-valued optimization, supported by relevant examples. In Section 5, a conclusion and additional developments are provided.

2. Notation and preliminaries

Let $\Im$ represents the collection of all bounded and closed intervals in $\Re$. Suppose $\mathcal{A} \in \Im$. After that, we write $\mathcal{A} = [\rho^L, \rho^U]$, where ρ^L and ρ^U denotes, respectively $\mathcal{A}'s$ lower and upper bounds. We make use of the following closed interval attribute throughout this paper.

$$\kappa\mathcal{A} = \{\kappa\rho : \rho \in \mathcal{A}\} = \begin{cases} \left[\kappa\rho^L, \kappa\rho^U\right] & \text{if } \kappa \geq 0, \\ \left[\kappa\rho^U, \kappa\rho^L\right] & \text{if } \kappa < 0, \end{cases}$$

where κ is a real number.

A non-empty open subset of $\Re^n$ is denoted by $\mathfrak{X}$. The n-dimensional Euclidean space is written by $\Re^n$. It is referred to as an interval-valued function, $\mathcal{F} : \Re^n \to \Im$. The interval-valued function $\mathcal{F}$ may be expressed as follows: $\mathcal{F}(\nu) = [\mathcal{F}^L(\nu), \mathcal{F}^U(\nu)]$ for each $\nu \in \Re^n$, where $\mathcal{F}^L(\nu)$ and $\mathcal{F}^U(\nu)$ are real valued functions defined on $\Re^n$ and meet the following condition: $\mathcal{F}^L(\nu) \leq \mathcal{F}^U(\nu)$.

Given that $\mathcal{A} = [\rho^L, \rho^U]$ and $\mathcal{B} = [\wp^L, \wp^U]$ as two closed intervals, we can use $\mathcal{A} \leq_{LU} \mathcal{B}$ if and only if $\rho^L \leq \wp^L$ and $\rho^U \leq \wp^U$. Also we can put $\mathcal{A} <_{LU} \mathcal{B}$ if and only if $\mathcal{A} \leq_{LU} \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. Equivalently, $\mathcal{A} <_{LU} \mathcal{B}$ if and only if

$$\begin{array}{lll} & \rho^L < \wp^L, & \rho^U < \wp^U, \\ \text{or,} & \rho^L \leq \wp^L, & \rho^U < \wp^U, \\ \text{or,} & \rho^L < \wp^L, & \rho^U \leq \wp^U. \end{array}$$

Remark 2.1. Let $\mathcal{A} = [\rho^L, \rho^U]$ be a closed interval. Then

$$\kappa\mathcal{A} \geq_{LU} \mathbf{0},$$

which means that $\kappa\rho^L \geq 0$ and $\kappa\rho^U \geq 0$, where κ is a positive real number and $\mathbf{0} = [0, 0]$.

Definition 2.1. [13] A non-empty set $\mathfrak{X}$ is said to be an invex set at $\upsilon \in \mathfrak{X}$, if there exists $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re^n$ such that

$$\upsilon + \aleph\, \mu(\nu, \upsilon) \in \mathfrak{X}, \text{ for any } \nu \in \mathfrak{X},\ \aleph \in [0, 1].$$

Remark 2.2. We say that the set $\mathfrak{X}$ is invex, if $\mathfrak{X}$ is invex at any $\upsilon \in \mathfrak{X}$.

Definition 2.2. [13] $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re^n$ at $\upsilon \in \mathfrak{X}$ is said to be invex for the differentiable function $\mathfrak{f} : \mathfrak{X} \to \Re$ if for all $\nu \in \mathfrak{X}$,

$$\mathfrak{f}(\nu) - \mathfrak{f}(\upsilon) \geq \mu(\nu, \upsilon)^{\mathcal{T}} \nabla \mathfrak{f}(\upsilon),$$

where "$\mathcal{T}$"is a vector or matrix transposition symbol.

Definition 2.3. [13] $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re^n$ at $\upsilon \in \mathfrak{X}$ is said to be strictly invex for the differentiable function $\mathfrak{f} : \mathfrak{X} \to \Re$ if for all $\nu \in \mathfrak{X}$,

$$\mathfrak{f}(\nu) - \mathfrak{f}(\upsilon) > \mu(\nu, \upsilon)^{\mathcal{T}} \nabla \mathfrak{f}(\upsilon),\ \nu \neq \upsilon.$$

Remark 2.3. If the functions $\mathcal{F}^L$ and $\mathcal{F}^U$ are both differentiable at υ, then the interval-valued function $\mathcal{F} : \mathfrak{X} \to \Im$ is considered to be differentiable at $\upsilon \in \mathfrak{X}$.

We are now going to focus on an interval-valued function's invexity.

Definition 2.4. [32] The function $\mathcal{F} : \mathfrak{X} \to \Im$, which is differentiable and interval-valued, is considered invex at $\upsilon \in \mathfrak{X}$ if and only if both $\mathcal{F}^L$ and $\mathcal{F}^U$ are invex with respect to the same μ at υ. The real valued function $\mathfrak{f}$ is next considered.The class of interval valued functions defined on μ-invex set K will be denoted as F_K in the remaining portions of this work.

Now, we can define the Continuous gH-Differentiable function.

Definition 2.5. [28] A function $\mathcal{F} : \mathfrak{X} \to \Im$ is continuously gH- differentiable if its gH-derivative exists (Hukuhara difference) and is continuous on $\mathfrak{X}$.

3. Extended concepts

Consequently, in the LU-sense, Zhang et al. [35] extended the notions of preinvexity, invexity, pseudo-invexity, and quasi-invexity to interval valued functions in the following ways.

Definition 3.1. [35] Consider that $\mathfrak{f} \in F_K$. Next, we state that

1. If $\mathfrak{f}(\nu+\aleph\mu(\nu^*,\nu)) \precsim_{LU} \aleph\mathfrak{f}(\nu^*)+(1-\aleph)\mathfrak{f}(\nu)$, for each $\nu \in \mathcal{K}$ and every $\aleph \in [0,1]$, then function $\mathfrak{f}$ is LU-preinvex at ν^* with respect to μ. Additionally, we show that $\mathfrak{f}$ is a LU-μ-preinvex function at ν^*.
2. If the real valued functions $\mathfrak{f}^L$ and $\mathfrak{f}^U$ are μ-invex at ν^*, then the function $\mathfrak{f}$ is invex (μ-invex) at ν^*. We further state that $\mathfrak{f}$ is a LU-μ-invex function at ν^* in this case.
3. If the real valued functions $\mathfrak{f}^L$, $\mathfrak{f}^U$ and $\aleph^L\mathfrak{f}^L + \aleph^U\mathfrak{f}^U$ are μ-pseudo-invex at ν^*, then function $\mathfrak{f}$ is pseudo-invex at ν^* where $0 < \aleph^L, \aleph^U \in \Re$. We continue by stating that $\aleph$ is a LU-μ-pseudo-invex function at ν^* in this case.
4. If the real valued functions $\mathfrak{f}^L$, $\mathfrak{f}^U$ and $\aleph^L\mathfrak{f}^L+\aleph^U\mathfrak{f}^U$ are μ-quasi-invex at ν^*,then function $\mathfrak{f}$ is quasi-invex at ν^* where $0 < \aleph^L, \aleph^U \in \Re$. Additionally, we show that in this case, $\mathfrak{f}$ is a LU-μ-quasi-invex function at ν^*.

According to Additional [27], interval valued functions in the LS-sense are covered by the previously mentioned ideas.

Definition 3.2. [27] [28] Let $\mathfrak{f} \in F_K$. Then we state that function

1. $\mathfrak{f}$ is LS-μ-preinvex at ν^* if $\mathfrak{f}(\nu+\aleph\mu(\nu^*,\nu)) \precsim_{LS} \aleph\mathfrak{f}(\nu^*)+(1-\aleph)\mathfrak{f}(\nu)$, for every $\aleph \in [0,1]$ and each $\nu \in K$;
2. $\mathfrak{f}$ is LS-μ-invex at ν^* if the real valued functions $\mathfrak{f}^L$ and $\mathfrak{f}^S$ are μ-invex at ν^*;
3. $\mathfrak{f}$ is LS-μ-pseudo-invex at ν^* if the real valued functions $\mathfrak{f}^L$, $\mathfrak{f}^S$ and $\aleph^L\mathfrak{f}^L+\aleph^S\mathfrak{f}^S$ are μ-pseudo-invex at ν^*, where $0 < \aleph^L, \aleph^S \in \Re$;
4. $\mathfrak{f}$ is LS-μ-quasi-invex at ν^* if the real valued functions $\mathfrak{f}^L$, $\mathfrak{f}^S$ and $\aleph^L\mathfrak{f}^L+\aleph^U\mathfrak{f}^S$ are μ-quasi-invex at ν^*, where $0 < \aleph^L, \aleph^S \in \Re$.

Proposition 3.1. *Consider the differentiable interval-valued function $\mathfrak{f} \in \digamma_K$, which is defined on the convex set $\mathfrak{X} \subseteq \Re^n$ and $\nu^* \in \mathfrak{X}$. Then the following arguments are true.*

1. *$\mathfrak{f}$ is LU-μ-preinvex at ν^* if $\mathfrak{f}^L$ and $\mathfrak{f}^U$ are μ-preinvex at ν^* [35].*
2. *$\mathfrak{f}$ is LS-μ-preinvex at ν^* if $\mathfrak{f}^L$ and $\mathfrak{f}^S$ are μ-preinvex at ν^*.*
3. *If $\mathfrak{f}$ is LS-μ-preinvex at ν^*. Then $\mathfrak{f}$ is $LU-\mu$-preinvex at ν^*.*

Giannessi [11]) presented the Stampacchia vector variational inequalities that we examine.
(VVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \nabla\mathfrak{f}_1^L(\nu^*) + \nabla\mathfrak{f}_1^U(\nu^*), \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \nabla\mathfrak{f}_r^L(\nu^*) + \nabla\mathfrak{f}_r^U(\nu^*), \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} \leq 0.$$

(WVVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \nabla\mathfrak{f}_1^L(x^*) + \nabla\mathfrak{f}_1^U(\nu^*), \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \nabla\mathfrak{f}_r^L(\nu^*) + \nabla\mathfrak{f}_r^U(\nu^*), \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} < 0.$$

(SVVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \nabla\mathfrak{f}_1^L(\nu^*), \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \nabla\mathfrak{f}_r^L(\nu^*), \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} > 0,$$
$$\left(\left\langle \nabla\mathfrak{f}_1^U(\nu^*), \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \nabla\mathfrak{f}_r^U(\nu^*), \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} > 0.$$

Now, for $\zeta_i^L \in \partial\mathfrak{f}_i^L(\nu^*)$ and $\zeta_i^U \in \partial\mathfrak{f}_i^U(\nu^*)$ we define the following problems.
(NVVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \zeta_1^L + \zeta_1^U, \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \zeta_r^L + \zeta_r^U, \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} \leq 0.$$

(NWVVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \zeta_1^L + \zeta_1^U, \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \zeta_r^L + \zeta_r^U, \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} < 0.$$

(NSVVIP) Identify a point $\nu^* \in \mathfrak{X}$ where no $\nu \in \mathfrak{X}$ exists so that

$$\left(\left\langle \zeta_1^L, \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \zeta_r^L, \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} > 0,$$
$$\left(\left\langle \zeta_1^U, \mu(\nu,\nu^*)\right\rangle, ..., \left\langle \zeta_r^U, \mu(\nu,\nu^*)\right\rangle\right)^{\mathcal{T}} > 0.$$

Let S_{SVVIP}, S_{VVIP} and S_{WVVIP} denote the solution set of the problem (S_{VVIP}), (V_{VIP}) and (W_{VVIP}), respectively. It can be shown that $S_{SVVIP} \subseteq S_{VVIP} \subseteq S_{WVVIP}$ by the definitions, but the converse may be not true. Similar to the case of smooth, let S_{NSVVIP}, S_{NVVIP} and S_{NWVVIP} denote the solution set of the problem S_{NSVVIP}, S_{NVVIP} and S_{NWVVIP}, respectively. It also can be shown that $S_{NSVVIP} \subseteq S_{NVVIP} \subseteq S_{NWVVIP}$ by the definitions, but the converse may be not true.

Definition 3.3. The differentiable interval-valued function $\mathcal{F} : \mathfrak{X} \to \Im$ is said to be strictly invex with respect to $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re^n$ at $\upsilon \in \mathfrak{X}$ if the functions $\mathcal{F}^L$ and $\mathcal{F}^U$ both are strictly invex or at least one of $\mathcal{F}^L$ or $\mathcal{F}^U$ is strictly invex with respect to the same μ at υ.

In this paper, we consider the following interval-valued optimization problem:

$$\text{(IVOP)} \qquad \min \ \mathcal{F}(\nu) = (\mathcal{F}_1(\nu), ...\mathcal{F}_r(\nu))$$
$$\text{subject to} \ \ \nu \in \mathfrak{X},$$

where $\mathcal{F} : \mathfrak{X} \to \Im^r, \mathcal{F}_i = [\mathcal{F}_i^L(\nu), \mathcal{F}_i^U(\nu)], i \in \theta_r$ is a differentiable interval-valued vector function. A vector $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, ..., \mathcal{A}_r)$ is said to be an interval valued vector if $\mathcal{A}_k \in \mathcal{K}_c, k = 1, ..., r$. Also for any two interval valued vectors $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, ..., \mathcal{A}_r)$ and $\boldsymbol{\mathcal{B}} = (\mathcal{B}_1, ..., \mathcal{B}_r)$ we write $\boldsymbol{\mathcal{A}} \preceq_{LU} \boldsymbol{\mathcal{B}}$ if and only if $\mathcal{A}_k \preceq_{LU} \mathcal{B}_k$ for each $k = 1, ..., r$. We also write $\boldsymbol{\mathcal{A}} \prec_{LU} \boldsymbol{\mathcal{B}}$ if and only if $\mathcal{A}_k \preceq_{LU} \mathcal{B}_k$ for each $k = 1, ..., r$ and $\mathcal{A}_h \prec_{LU} \mathcal{B}_h$ for at least one index h (Wu [32]).

Definition 3.4. (Wu [32]) Let $\boldsymbol{\nu}^*$ be a feasible solution of (MIP1). We say that $\boldsymbol{\nu}^*$ is

1. LU-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{\boldsymbol{\nu}} \in \mathfrak{X}$, such that, $\mathcal{F}(\bar{\boldsymbol{\nu}}) \prec_{LU} \mathcal{F}(\boldsymbol{\nu}^*)$;
2. strongly LU-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{\boldsymbol{\nu}} \in \mathfrak{X}$. such that., $\mathcal{F}(\bar{\boldsymbol{\nu}}) \preceq_{LU} \mathcal{F}(\boldsymbol{\nu}^*)$;
3. weakly LU-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{\boldsymbol{\nu}} \in \mathfrak{X}$. such that, $f_k(\bar{\boldsymbol{\nu}}) \prec_{LU} f_k(\boldsymbol{\nu}^*)$ for $k = 1, \cdots, r$.

Remark 3.1. (Wu [32]) Let us denote by $\mathfrak{X}_{WP}^{LU}, \mathfrak{X}_P^{LU}, \mathfrak{X}_{SP}^{LU}$ the set of weakly LU-Pareto optimal solutions, LU-Pareto optimal solutions, and strongly LU-Pareto optimal solutions respectively. Then $\mathfrak{X}_{SP}^{LU} \subseteq \mathfrak{X}_P^{LU} \subseteq \mathfrak{X}_{WP}^{LU}$.

For another solution concept, we denote by $w(\mathcal{A}) = \rho^S = \rho^U - \rho^L$ the width (spread) of $\mathcal{A} = [\rho^L, \rho^U]$. In this paper we shall consider only the minimization problem without loss of generality. In this sense, for $\mathcal{A}, \mathcal{B} \in \mathcal{K}_c$ we write $\mathcal{A} \preceq_{LS} \mathcal{B}$ if and only if $\rho^L \leq \wp^L$ and $\rho^S \leq \wp^S$. We also write $A \prec_{LS} \mathcal{B}$ if and only if $\mathcal{A} \preceq_{LS} \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, i.e., $\mathcal{A} \prec_{LS} \mathcal{B}$ if and only if

$$\begin{cases} \rho^L < \wp^L \\ \rho^S \leq \wp^S, \end{cases} or \begin{cases} \rho^L \leq \wp^L \\ \rho^S < \wp^S, \end{cases} or \begin{cases} \rho^L < \wp^L \\ \rho^S < \wp^S. \end{cases} \tag{3.1}$$

For details one is referred to Chalco et al. [5].

Let $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ be interval valued vectors. Then we write $\boldsymbol{\mathcal{A}} \preceq_{LS} \boldsymbol{\mathcal{B}}$ if and only if $\mathcal{A}_k \preceq_{LS} \mathcal{B}_k, k = 1, ..., r$. We also write $\boldsymbol{\mathcal{A}} \prec_{LS} \boldsymbol{\mathcal{B}}$ if and only if $\mathcal{A}_k \preceq_{LS} \mathcal{B}_k$ for $k = 1, ..., r$ and $\mathcal{A}_h \prec_{LS} \mathcal{B}_h$ for at least one index h.

Definition 3.5. [27] Let $\boldsymbol{\nu}^*$ be a feasible solution of $(MIP1)$. We say that $\boldsymbol{\nu}^*$ is

1. LS-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{\boldsymbol{\nu}} \in X$, such that, $\mathcal{F}(\bar{\boldsymbol{\nu}}) \prec_{LS} \mathcal{F}(\boldsymbol{\nu}^*)$;
2. strongly LS-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{\boldsymbol{\nu}} \in \mathfrak{X}$. such that, $\mathcal{F}(\bar{\boldsymbol{\nu}}) \preceq_{LS} \mathcal{F}(\boldsymbol{\nu}^*)$;
3. weakly LS-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{\boldsymbol{\nu}} \in \mathfrak{X}$. such that, $f_k(\bar{\boldsymbol{\nu}}) \prec_{LS} f_k(\boldsymbol{\nu}^*), k = 1, \cdots, r$.

Remark 3.2. [27] Let us denote by $\mathfrak{X}_{WP}^{LS}, \mathfrak{X}_P^{LS}, \mathfrak{X}_{SP}^{LS}$ the set of weakly LS-Pareto optimal solutions, LS-Pareto optimal solutions, and strongly LS-Pareto optimal solutions respectively. Then $\mathfrak{X}_{SP}^{LS} \subseteq \mathfrak{X}_P^{LS} \subseteq \mathfrak{X}_{WP}^{LU}$.

Proposition 3.2. *[27] Let $\mathcal{A}, \mathcal{B}$ be two closed intervals in $\mathcal{K}_c$.*

1. *If $\mathcal{A} \preceq_{LS} \mathcal{B}$ then $\mathcal{A} \preceq_{LU} \mathcal{B}$. (Chalco et al. [5]).*
2. *If $\mathcal{A} \prec_{LS} \mathcal{B}$ then $\mathcal{A} \prec_{LU} \mathcal{B}$.(Singh et al. [27]).*

Proposition 3.3. *[27] Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, ..., \mathcal{A}_r)$ and $\boldsymbol{\mathcal{B}} = (\mathcal{B}_1, ..., \mathcal{B}_r)$ be interval valued vectors.*

1. *If $\boldsymbol{\mathcal{A}} \preceq_{LS} \boldsymbol{\mathcal{B}}$ then $\boldsymbol{\mathcal{A}} \preceq_{LU} \boldsymbol{\mathcal{B}}$.*
2. *If $\boldsymbol{\mathcal{A}} \prec_{LS} \boldsymbol{\mathcal{B}}$ then $\boldsymbol{A} \prec_{LU} \boldsymbol{\mathcal{B}}$.*

Theorem 3.1. *[27] Let $\mathfrak{X}$ be a feasible set of $(MIP1)$. Then*

1. $\mathfrak{X}^{LU}_{SP} \subseteq \mathfrak{X}^{LS}_{SP}$,
2. $\mathfrak{X}^{LU}_{P} \subseteq \mathfrak{X}^{LS}_{P}$,
3. $\mathfrak{X}^{LU}_{WP} \subseteq \mathfrak{X}^{LS}_{WP}$.

Definition 3.1. [36] 1. Assume that ν^* is a feasible solution to the IVOP problem. For any $\bar{\nu} \in \mathfrak{X} \cap \mathcal{U}$, ν^* is considered a local weak efficient solution to the (IVOP) problem if there is a neighbourhood $\mathcal{U}$ of ν^* such that the following cannot hold for all $k = 1, 2, ..., q$

$$\mathfrak{f}_k(\bar{\nu}) \prec_{LU} \mathfrak{f}_k(\nu^*).$$

2. Assume that ν^* is a feasible solution to the IVOP problem. If there exists $\alpha \in int(R^r_+)$ and a neighbourhood $\mathcal{U}$ of ν^* such that for any $\bar{\nu} \in \mathfrak{X} \cap \mathcal{U}$, ν^* is said to be a local quasi efficient solution to the IVOP problem, the following is not true.

$$\mathfrak{f}(\bar{\nu}) + \alpha||\bar{\nu} - \nu^*|| \prec_{LU} \mathfrak{f}(\nu^*).$$

3. Assume that ν^* is a feasible solution to the IVOP problem. If there exists $\alpha_k \in int(R_+)$ and a neighbourhood $\mathcal{U}$ of ν^* such that for each $\bar{\nu} \in \mathfrak{X} \cap \mathcal{U}$, then ν^* is a local weak quasi efficient solution to the problem of (IVOP), and the following is not true for all $k = 1, 2, ..., r$

$$\mathfrak{f}_k(\bar{\nu}) + \alpha_k||\bar{\nu} - \nu^*|| \prec_{LU} \mathfrak{f}_k(\nu^*).$$

4. Assume that ν^* is a feasible solution to the IVOP problem. If there exists $\alpha_k \in int(R_+)$ such that for each $\bar{\nu} \in \mathfrak{X}$, then ν^* is considered a weak quasi efficient solution to the problem of (IVOP) , and the following is not true for all $k = 1, 2, ..., r$

$$\mathfrak{f}_k(\bar{\nu}) + \alpha_k||\bar{\nu} - \nu^*|| \prec_{LU} \mathfrak{f}_k(\nu^*).$$

5. Assume that ν^* is a feasible solution to the IVOP problem. If there is $\alpha \in int(R^r_+)$ and a neighborhood $\mathcal{U}$ of ν^* such that for each $\bar{\nu} \in \mathfrak{X} \cap \mathcal{U}$, then ν^* is a local strong quasi efficient solution to the problem of (IVOP), and the following is not true

$$\mathfrak{f}(\bar{\nu}) + \alpha||\bar{\nu} - \nu^*|| \preceq_{LU} \mathfrak{f}(\nu^*).$$

Remark 3.3. [36] 1. The definitions make it clear that any local efficient solution to the IVOP problem is also a local quasi efficient solution , although the converse may not always hold true. Similarly, every local weak efficient solution to the IVOP problem is a local weak quasi efficient solution, although the converse may not always hold true.

2. Let $\mathfrak{X}_{lsqe}, \mathfrak{X}_{lqe}$ and $\mathfrak{X}_{lwqe}$ represent the collection of all local strong quasi efficient solutions, local quasi efficient solutions and local weak quasi efficient solutions to the IVOP problem , respectively. It is clear that $\mathfrak{X}_{lsqe} \subseteq \mathfrak{X}_{lqe}$ by the definitions, but the converse may be not true.

3. From the definitions, it can be clear that $\mathfrak{X}_{lqe} \subseteq \mathfrak{X}_{lwqe}$. However, the relationship of $\mathfrak{X}_{lqe} \subset \mathfrak{X}_{lwqe}$ between $\mathfrak{X}_{lqe}$ and $\mathfrak{X}_{lwqe}$ does not hold true . Assume that ν^* represents a local weak quasi-efficient solution, but not a local quasi-efficient one, to the IVOP problem. $\alpha \in int(\Re^r_+)$ and a neighborhood $\mathcal{U}$ of ν^* exist, by Definition 3.1, such that for each $\bar{\nu} \in \mathfrak{X} \cap \mathcal{U}$, the following holds

$$\mathfrak{f}(\bar{\nu}) + \alpha||\bar{\nu} - \nu^*|| \prec_{LU} \mathfrak{f}(\nu^*),$$

which means that for all $k = 1, 2, ..., r$, the following

$$\mathfrak{f}_k(\bar{\nu}) + \alpha_k||\bar{\nu} - \nu^*|| \preceq_{LU} \mathfrak{f}_k(\nu^*)$$

is satisfied and

$$\mathfrak{f}_h(\bar{\nu}) + \alpha_h||\bar{\nu} - \nu^*|| \prec_{LU} \mathfrak{f}_h(\nu^*)$$

is satisfied with respect to each index h., which is in opposition to $\nu^* \in \mathfrak{X}_{lwqe}$.

Now, we introduce the following interval-valued variational-like inequality problems:

(IVVLIP) An interval-valued variational-like inequality problem is to find a point $\upsilon \in \mathfrak{X}$, such that

$$\langle \mu(\nu, \upsilon),\ \nabla \mathcal{F}(\upsilon) \rangle \geq_{LU} \mathbf{0}, \quad \forall\ \nu \in \mathfrak{X},$$

where $\nabla \mathcal{F} = [\nabla \mathcal{F}^L,\ \nabla \mathcal{F}^U]$ is an interval-valued function.

(IVWVLIP) An interval-valued weak variational-like inequality problem is to find a point $\upsilon \in \mathfrak{X}$, such that

$$\langle \mu(\nu, \upsilon),\ \nabla \mathcal{F}(\upsilon) \rangle >_{LU} \mathbf{0}, \quad \forall\ \nu \in \mathfrak{X} \text{ and } \nu \neq \upsilon.$$

4. Relationships between interval-valued optimization and variational like inequality problems

In this section, based on the idea of LU and LS μ-optimal and weakly LU and LS optimal solutions, some relations are derived between the interval-valued optimization and (weak) variational-like inequality problems.

Theorem 4.1. *Let $\mathcal{F} : \mathfrak{X} \to \Im^r$ be a continuously gH-differentiable function on $\mathfrak{X}$. Suppose that $\mathcal{F}_i, i \in \theta_r$ are LU-μ-invex at $\vartheta \in \mathfrak{X}$.*

- *(1): If ϑ solves interval-valued variational-like inequality problem (VVIP), then ϑ is a LU-efficient solution to (IVOP).*
- *(2): If ϑ solves interval-valued variational-like inequality problem (SVVIP), then ϑ is a strong LU-efficient solution to (IVOP).*

Proof. ((1):). Suppose contrary to the result that y is not a LU efficient solution to (IVOP). Then there exists a point $\nu \in \mathfrak{X}$, such that

$$\mathcal{F}(\nu) \prec_{LU} \mathcal{F}(\vartheta).$$

That is, $\mathcal{F}_i^L(\nu) \leq \mathcal{F}_i^L(\vartheta)$ and $\mathcal{F}_i^U(\nu) \leq \mathcal{F}_i^U(\vartheta), i \in \theta_r$,
and

$$\mathcal{F}_h^L(\nu) < \mathcal{F}_h^L(\vartheta) \text{ and } \mathcal{F}_h^U(\nu) < \mathcal{F}_i^U(\vartheta),$$

or

$$\mathcal{F}_h^L(\nu) < \mathcal{F}_h^L(\vartheta) \text{ and } \mathcal{F}_h^U(\nu) \leq \mathcal{F}_i^U(\vartheta),$$

or

$$\mathcal{F}_h^L(\nu) \leq \mathcal{F}_h^L(\vartheta) \text{ and } \mathcal{F}_h^U(\nu) < \mathcal{F}_i^U(\vartheta),$$

for at least one index h.

The above inequalities together with the LU-μ-invexity of $\mathcal{F}_i, i \in \theta_r$ at ϑ, yield

$$\left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_i^L(\vartheta) \right\rangle \leq 0 \text{ and } \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_i^U(\vartheta) \right\rangle \leq 0 \tag{4.1}$$

and

$$\begin{cases} \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^L(\nu) \right\rangle < 0 & \text{and} \quad \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^U(\nu) \right\rangle \leq 0, \\ \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^L(\nu) \right\rangle < 0 & \text{and} \quad \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^U(\nu) \right\rangle \leq 0, \\ \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^L(\nu) \right\rangle \leq 0 & \text{and} \quad \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_h^U(\nu) \right\rangle < 0, \end{cases} \tag{4.2}$$

for at least one index h.

Combining the above two inequalities, we have

$$\left(\left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_1^L + \mathcal{F}_1^U)(\vartheta) \right\rangle, ..., \left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_r^L + \mathcal{F}_r^U)(y) \right\rangle \right) \leq 0,$$

which shows that ϑ cannot be a solution to interval-valued weak variational-like inequality problem (VVIP). This contradiction leads to the result.

((2):). Suppose contrary to the result that ϑ is not a strong LU efficient solution to (IVOP). Then there exists a point $\nu \in \mathfrak{X}$, such that

$$\mathcal{F}_i(\nu) \preceq_{LU} \mathcal{F}_i(\vartheta), i \in \theta_r.$$

That is,

$$\mathcal{F}_i^L(\nu) \leq \mathcal{F}_i^L(\vartheta) \text{ and } \mathcal{F}_i^U(\nu) \leq \mathcal{F}_i^U(\vartheta), i \in \theta_r.$$

The above inequalities together with the LU-μ-invexity of $\mathcal{F}_i, i \in \theta_r$ at ϑ, yield

$$\left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_i^L(\vartheta) \right\rangle \leq 0 \text{ and } \left\langle \mu(\nu,\vartheta)^{\mathcal{T}}, \nabla\mathcal{F}_i^U(\vartheta) \right\rangle \leq 0, i \in \theta_r. \tag{4.3}$$

Combining the above two inequalities, we have

$$\begin{cases} \left(\left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_1^L)(\vartheta) \right\rangle, ..., \left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_r^L)(\vartheta) \right\rangle \right) \leq 0, \\ \left(\left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_1^U)(\vartheta) \right\rangle, ..., \left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_r^U)(\vartheta) \right\rangle \right) \leq 0, \end{cases}$$

which shows that ϑ cannot be a solution to interval-valued variational-like inequality problem (SVVIP). This contradiction leads to the result. □

Algorithm 1. A LU-efficient solution identifying algorithm for (VVIP).

Step 1: Given input $\mathcal{F}_i^L, \mathcal{F}_i^U$ for $i \in \theta_r$.
Step 2: Check the functions $\mathcal{F}_i^L, \mathcal{F}_i^U$ are continuously gH-differentiable function on $\mathfrak{X}$.

Step 3: Define invexity function $\mu(\nu, \vartheta)$ on $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re$.
Step 4: Check the invexity of $\mathcal{F}_i^L, \mathcal{F}_i^U$ w.r.t define μ at ϑ.
Step 5: Combining the inequalities (4.1) and (4.2), we get

$$\left(\left\langle \mu(\nu, \vartheta),\ \nabla(\mathcal{F}_1^L + \mathcal{F}_1^U)(\vartheta) \right\rangle, ..., \left\langle \mu(\nu, \vartheta),\ \nabla(\mathcal{F}_r^L + \mathcal{F}_r^U)(y) \right\rangle\right) \leq 0.$$

Step 6: This contradiction leads to the result.

In the next part, we cite an example to validate the above result.

Example 4.1. (Verification of the Theorem 4.1) Let $\mathfrak{X} = \{\nu : 1 \leq \nu \leq 2\}$ and consider the following interval-valued problem:

$$\begin{aligned} \text{(IVP1)} \quad \min \quad & \mathcal{F}(\nu) = [\mathcal{F}^L(\nu), \mathcal{F}^U(\nu)] \\ & = [2\ \ln\ (\nu^2 - \nu + 5) + 5,\ 3\ \ln\ (\nu^2 - \nu + 5) + 7] \\ \text{subject to} \quad & \nu \in \mathfrak{X}, \end{aligned}$$

where $\mathcal{F} : \mathfrak{X} \to \Im$ is an interval-valued function.

We have to find a point $\vartheta \in \mathfrak{X}$, such that it solves (IVWVLIP).

Let $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re$ be defined by $\mu(\nu, \vartheta) = \nu - 1$.

At $\vartheta = 1$, we have

$$\begin{aligned} & \mathcal{F}^L(\nu) - \mathcal{F}^L(\vartheta) - \mu(\nu, \vartheta)\nabla\mathcal{F}^L(\vartheta) \\ & = \Big(2\ \ln\ (\nu^2 - \nu + 5) + 5\Big) - \Big(2\ \ln\ 5 + 5\Big) - \tfrac{2}{5}(\nu - 1) \\ & = 2\ \ln\ (\nu^2 - \nu + 5) - 2\ \ln\ 5 - \tfrac{2}{5}(\nu - 1) \\ & = 2\big(\ln\ (\nu^2 - \nu + 5) - \ln\ 5 - \tfrac{1}{5}(\nu - 1)\big)\ \geq 0, \quad \forall\ \nu \in \mathfrak{X}, \end{aligned}$$

which shows that $\mathcal{F}^L$ is an invex with respect to μ at $\vartheta = 1$. Similarly, we can show that $\mathcal{F}^U$ is an invex with respect to same μ at $\vartheta = 1$. Thus, we conclude that $\mathcal{F}$ is an invex with respect to μ at $\vartheta = 1$.

On the other hand, for $\vartheta = 1$ ($\nu \neq \vartheta$) we have

$$\begin{cases} \mu(\nu, \vartheta)\nabla\mathcal{F}^L(\vartheta) \\ = (\nu - 1)\ \frac{2(2\vartheta - 1)}{\vartheta^2 - \vartheta + 5} \\ = \frac{2}{5}(\nu - 1) > 0,\ \forall\ \nu \in \mathfrak{X} \end{cases} \quad \text{and} \quad \begin{cases} \mu(\nu, \vartheta)\nabla\mathcal{F}^U(\vartheta) \\ = (\nu - 1)\ \frac{3(2\vartheta - 1)}{\vartheta^2 - \vartheta + 5} \\ = \frac{3}{5}(\nu - 1) > 0,\ \forall\ \nu \in \mathfrak{X} \end{cases}.$$

Therefore, $\langle \mu(\nu, \vartheta),\ \nabla\mathcal{F}(\vartheta) \rangle >_{LU} \mathbf{0}$, at $\vartheta = 1$ and so $\vartheta = 1$ solves (IVWVLIP).

Furthermore, it is easy to verify that $\vartheta = 1$ is also a LU optimal solution to (IVP1). Hence all the conditions of the Theorem are contended.

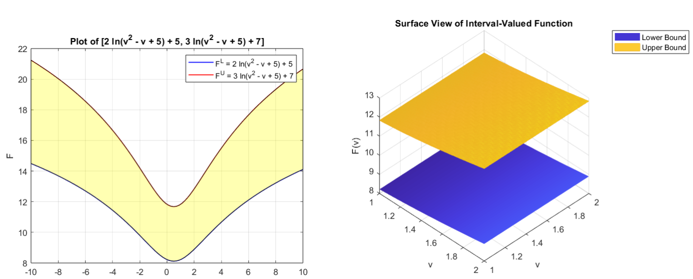

Figure 1. Surface view of $\mathcal{F}^L(\nu)$ and $\mathcal{F}^U(\nu)$ for Exa. 4.1

Theorem 4.2. *Let $\mathcal{F}_i : \mathfrak{X} \to \Im, i \in \theta_r$ be continuously gH-differentiable functions on $\mathfrak{X}$. Suppose that $-\mathcal{F}_i, i \in \theta_r$ are strictly LU-μ-invex at $\vartheta \in \mathfrak{X}$ and for all $\nu \in \mathfrak{X}, \nu \neq \vartheta$ such that $\mathfrak{f}_i(\nu)$ and $\mathfrak{f}_i(\vartheta)$ are comparable for all $i \in \theta_r$. If ϑ is a weak LU efficient solution to (IVOP), then ϑ solves interval-valued variational-like inequality problem (SVVIP).*

Proof. Suppose contrary to the result that ϑ does not solve (SVVIP). Then there exists a point $\nu \in \mathfrak{X}$, such that

$$\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_i(\vartheta) \rangle \preceq_{LU} \mathbf{0}, i \in \theta_r$$

and

$$\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h(\vartheta) \rangle \prec_{LU} \mathbf{0}$$

for at least one index h.

It implies that

$$\left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_i^L(\vartheta) \right\rangle \leq 0 \quad \text{and} \quad \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_i^U(\vartheta) \right\rangle \leq 0, i \in \theta_r$$

and

$$\begin{cases} \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^L(\vartheta) \right\rangle < 0 \\ \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^U(\vartheta) \right\rangle < 0 \end{cases}, \quad \text{or} \begin{cases} \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^L(\vartheta) \right\rangle \leq 0 \\ \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^U(\vartheta) \right\rangle < 0 \end{cases},$$

$$\text{or} \begin{cases} \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^L(\vartheta) \right\rangle < 0 \\ \left\langle \mu(\nu, \vartheta),\ \nabla \mathcal{F}_h^U(\vartheta) \right\rangle \leq 0 \end{cases},$$

for at least one index h.

The above inequalities together with the strict LU-μ-invexity of $-F_i, i \in \theta_r$ at ϑ, yield

$$\mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) \leq 0 \quad \text{and} \quad \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) \leq 0, i \in \theta_r$$

and

$$\begin{cases} \mathcal{F}_h^L(\nu) - \mathcal{F}_h^L(\vartheta) < 0 \\ \mathcal{F}_h^U(\nu) - \mathcal{F}_h^U(\vartheta) < 0 \end{cases}, \ \text{or} \ \begin{cases} \mathcal{F}_h^L(\nu) - \mathcal{F}_h^L(\vartheta) \leq 0 \\ \mathcal{F}_h^U(\nu) - \mathcal{F}_h^U(\vartheta) < 0 \end{cases}, \ \text{or} \ \begin{cases} \mathcal{F}_h^L(\nu) - \mathcal{F}_h^L(\vartheta) < 0 \\ \mathcal{F}_h^U(\nu) - \mathcal{F}_h^U(\vartheta) \leq 0 \end{cases}, \tag{4.4}$$

for at least one index h.

Since for all $\nu \in \mathfrak{X}, \nu \neq \vartheta$ such that the two interval-valued functions $\mathfrak{f}_i(\nu)$ and $\mathfrak{f}_i(\vartheta)$ are comparable for all $i \in \theta_r$, then

$$\mathcal{F}_i(\nu) \preceq_{LU} \mathcal{F}_i(\vartheta) \quad \text{or} \quad \mathcal{F}_i(\nu) \succeq_{LU} \mathcal{F}_i(\vartheta) \quad \text{for some} \quad \nu \in \mathfrak{X}. \tag{4.5}$$

From the inequality $\mathcal{F}_i(\nu) \preceq_{LU} \mathcal{F}_i(\vartheta)$ of (4.5), we see that the following is satisfied for some $\nu \in \mathfrak{X}$,

$$\mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) \leq 0 \quad \text{and} \quad \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) \leq 0 \quad i \in \theta_r, \tag{4.6}$$

or from the inequality $\mathfrak{f}_i(\nu) \prec_{LU} \mathfrak{f}_i(\vartheta)$ of (4.5), we have the following formulas which are satisfied for some $\nu \in \mathfrak{X}$,

$$\mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) \geq 0 \quad \text{and} \quad \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) \geq 0 \quad i \in \theta_r. \tag{4.7}$$

Combining (4.4) and (4.5)–(4.6), there exists $\nu \in \mathfrak{X}$ for some index h such that

$$\mathcal{F}_h(\nu) <_{LU} \mathcal{F}_h(\vartheta),$$

which shows that ϑ is not a weak LU efficient optimal solution to (IVOP). This contradiction leads to the result. □

Remark 4.1. Since every efficient solution is also a weak efficient solution to (IVOP), the following result is trivial to prove.

Let $\mathcal{F}_i : \mathfrak{X} \to \Im, i \in \theta_r$ be continuously gH-differentiable functions on $\mathfrak{X}$. Suppose that $-\mathcal{F}_i, i \in \theta_r$ are strictly LU-μ-invex at $\vartheta \in \mathfrak{X}$ and for all $\nu \in \mathfrak{X}, \nu \neq \vartheta$ such that $\mathfrak{f}_i(\nu)$ and $\mathfrak{f}_i(\vartheta)$ are comparable for all $i \in \theta_r$. If ϑ is a LU efficient solution to (IVOP), then ϑ solves the interval-valued variational-like inequality problem (SVVIP).

Algorithm 2. A weak LU-efficient solution identifying algorithm to (IVOP).

Step 1: Given input $\mathcal{F}_i^L, \mathcal{F}_i^U$ for $i \in \theta_r$.
Step 2: Check the functions $\mathcal{F}_i^L, \mathcal{F}_i^U$ are continuously gH-differentiable function on $\mathfrak{X}$.
Step 3: Define invexity function $\mu(\nu, \vartheta)$ on $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re$.
Step 4: Check the strictly invexity of $-\mathcal{F}_i^L, -\mathcal{F}_i^U$ w.r.t define μ at ϑ such that $\mathfrak{f}_i(\nu)$ and $\mathfrak{f}_i(\vartheta)$ are comparable for all $i \in \theta_r$.
Step 5: Combining (4.4) and (4.5)–(4.6), there exists $\nu \in \mathfrak{X}$,

$$\mathcal{F}_h(\nu) <_{LU} \mathcal{F}_h(\vartheta).$$

Step 6: This contradiction shows that ϑ is not a weak LU efficient optimal solution to (IVOP), which leads to the result.

Now, we present an example which verifies the above result.

Example 4.2. (Verification of the Theorem 4.2) Let $\mathfrak{X} = \{\nu : 0 \leq \nu \leq \pi\}$ and consider the following interval-valued problem:

$$\begin{aligned} (\text{IVP2}) \qquad \min \qquad & \mathcal{F}(\nu) = [\mathcal{F}^L(\nu),\ \mathcal{F}^U(\nu)] \\ & \qquad\;\; = [2\ \sin\ \nu + 1,\ 5\ \sin\ \nu + 3] \\ \text{subject to} \qquad & \nu \in \mathfrak{X}. \end{aligned}$$

Let $\mu : \mathfrak{X} \times \mathfrak{X} \to \Re$ be defined by $\mu(\nu, \vartheta) = 8\nu - \vartheta$.

Similar to Example 4.1, it is easy to verify that $-\mathcal{F}^L$ and $-\mathcal{F}^U$ are strictly invex with respect to μ at $\vartheta = 0$ and therefore $-F$ is strictly invex with respect to μ at $\vartheta = 0$.

Clearly, $\vartheta = 0$ is a weakly LU optimal solution to (IVP2).

To illustrate Theorem 4.2, we have to show that $\vartheta = 0$ solves (IVVLIP).

Now, at $\vartheta = 0$ we have

$$\begin{cases} \mu(\nu, \vartheta)\nabla\mathcal{F}^L(\vartheta) \\ = 2(8\nu - \vartheta)\ \cos\ \vartheta \\ = 16\nu \geq 0,\ \forall\ \nu \in \mathfrak{X} \end{cases} \quad \text{and} \quad \begin{cases} \mu(\nu, \vartheta)\nabla\mathcal{F}^U(\vartheta) \\ = 5(8\nu - \vartheta)\ \cos\ \vartheta \\ = 40\nu \geq 0,\ \forall\ \nu \in \mathfrak{X} \end{cases}.$$

Therefore, $\langle \mu(\nu, \vartheta),\ \nabla\mathcal{F}(\vartheta)\rangle \geq_{LU} \mathbf{0}$, at $\vartheta = 0$ and so $\vartheta = 0$ solves (IVVLIP).

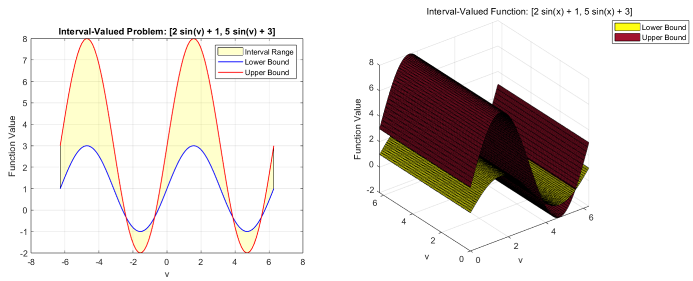

Figure 2. Surface view of $\mathcal{F}^L(\nu)$ and $\mathcal{F}^U(\nu)$ for Exa. 4.2

Theorem 4.3. *Let $\mathcal{F}_i : \mathfrak{X} \to I, i \in \theta_r$ be continuously gH-differentiable functions on $\mathfrak{X}$. Suppose that $\mathcal{F}_i, i \in \theta_r$ are strictly LU-μ-pseudo-invex at $\vartheta \in \mathfrak{X}$. If ϑ solves the interval-valued variational-like inequality problem (WVVIP), then ϑ is a weak LU efficient solution to (IVOP).*

Proof. Suppose contrary to the result that ϑ is not a weak LU efficient solution to (IVOP), then there exists a point $\nu \in \mathfrak{X}$, such that

$$\mathcal{F}_i(\nu) \prec_{LU} \mathcal{F}_i(\vartheta),\ i \in \theta_r.$$

That is for all $i \in \theta_r$ we have

$$\mathcal{F}_i^L(\nu) < \mathcal{F}_i^L(\vartheta) \ \text{ and } \ \mathcal{F}_i^U(x) < \mathcal{F}_i^U(\vartheta),$$

or

$$\mathcal{F}_i^L(\nu) < \mathcal{F}_i^L(\vartheta) \ \text{ and } \ \mathcal{F}_i^U(x) \leq \mathcal{F}_i^U(\vartheta),$$

or

$$\mathcal{F}_i^L(\nu) \leq \mathcal{F}_i^L(\vartheta) \ \text{ and } \ \mathcal{F}_i^U(x) < \mathcal{F}_i^U(\vartheta).$$

It implies that

$$\mathcal{F}_i^L(\nu) + \mathcal{F}_i^U(\nu) < \mathcal{F}_i^L(\vartheta) + \mathcal{F}_i^U(\vartheta).$$

The above inequalities together with the LU-μ-pseudo-invexity of $\mathcal{F}_i, i \in \theta_r$ at ϑ, yield

$$\left(\left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_1^L + \mathcal{F}_1^U)(\vartheta)\right\rangle, ..., \left\langle \mu(\nu,\vartheta),\ \nabla(\mathcal{F}_r^L + F_r^U)(\vartheta)\right\rangle\right) \leq 0,$$

which shows that ϑ cannot be a solution to interval-valued weak variational-like inequality problem (WVVIP). This contradiction leads to the result. □

Remark 4.2. When it comes to real-valued vector optimization, Theorem 4.3 in Ruiz-Garzron et al. [26] states that for the case of $\mu(\nu,\vartheta) = \nu - \vartheta$, the solution of (WVVIP) is also a weak efficient solution to the real-valued vector optimization problem, which is a necessary and sufficient optimality condition under certain conditions.

Even so, in the context of interval-valued vector optimization, (WVVIP) simply provides a sufficient condition for a weak efficient solution to the (IVOP). The vector variational inequality problems can be effectively solved using this method, as shown by the example that follows (IVOP).

Theorem 4.4. *Let $\mathfrak{X}$ be an invex set. If $\vartheta \in \mathfrak{X}$ is a LU optimal solution to (IVOP). Then ϑ solves interval-valued weak variational-like inequality problem (WVVIP).*

Proof. Let ϑ be a LU optimal solution to (IVOP). Then there exists no $\nu \in \mathfrak{X}$, such that for all $i \in \theta_r$ and $0 < \aleph < 1$

$$\mathcal{F}_i(\vartheta + \aleph\ \mu(\nu, \vartheta)) - \mathcal{F}_i(\vartheta) \leq_{LU} \mathbf{0}.$$

That is for all $i \in \theta_r$ and $0 < \aleph < 1$

$$\mathcal{F}_i^L(\vartheta + \aleph\ \mu(\nu, \vartheta)) - \mathcal{F}_i^L(\vartheta) \leq 0 \ \text{ and } \ \mathcal{F}_i^U(\vartheta + \aleph\ \mu(\nu, \vartheta)) - \mathcal{F}_i^U(\vartheta) \leq 0.$$

Now for $\mathcal{F}_i^L(\vartheta + \aleph\ \mu(\nu, \vartheta)) - \mathcal{F}_i^L(\vartheta) \leq 0, i \in \theta_r$ we see that

$$\left[\mathcal{F}_i^L(\vartheta) + \aleph\ \mu(\nu, \vartheta)^{\mathcal{T}}\ \nabla \mathcal{F}_i^L(\vartheta) + ...\right] - \mathcal{F}_i^L(\vartheta) \leq 0, i \in \theta_r.$$

Dividing the above inequality by $\aleph$ and taking the limit as $\aleph$ tends to zero, we have for all $i \in \theta_r$

$$\mu(\nu, \vartheta)^{\mathcal{T}} \nabla \mathcal{F}_i^L(\vartheta) \leq 0. \tag{4.8}$$

Similarly, we get

$$\mu(\nu, \vartheta)^{\mathcal{T}} \nabla \mathcal{F}_i^U(\vartheta) \leq 0, i \in \theta_r. \tag{4.9}$$

Combining the inequalities (4.8) and (4.9), we obtain that there exists no $\nu \in \mathfrak{X}$, such that for all $i \in \theta_r$

$$\left\langle \mu(\nu, \vartheta),\ \nabla_g \mathcal{F}_i(\vartheta) \right\rangle \leq_{LU} \mathbf{0}.$$

That is

$$\left(\left\langle \mu(\nu, \vartheta),\ \nabla_g F_1(\vartheta) \right\rangle, ..., \left\langle \mu(\nu, \vartheta),\ \nabla_g F_i(\vartheta) \right\rangle \right) \leq_{LU} \mathbf{0}.$$

Equivalently there exists a point ϑ, such that

$$\left(\left\langle \mu(\nu, \vartheta),\ \nabla_g \mathcal{F}_1(\vartheta) \right\rangle, ..., \left\langle \mu(\nu, \vartheta),\ \nabla_g \mathcal{F}_i(\vartheta) \right\rangle \right) >_{LU} \mathbf{0},\ \forall\ \nu \in \mathfrak{X},$$

which shows that ϑ solves interval-valued weak variational-like inequality problem (WVVIP). This completes the proof. □

Next, we construct an example to validate the above theorem 4.4.

Example 4.3. (Verification of the Theorem 4.4) Let $\mathfrak{X} = \{\nu : 1 \leq \nu \leq 3\}$ and consider the following interval-valued problem:

$$\begin{aligned} \text{(IVP3)} \quad \min \quad & \mathcal{F}(\nu) = [\mathcal{F}^L(\nu),\ \mathcal{F}^U(\nu)] \\ & \quad = [3 \ln (2\nu^2 + \nu + 3) + 1,\ 7 \ln (2\nu^2 + \nu + 3) + 3] \\ \text{subject to} \quad & \nu \in \mathfrak{X}. \end{aligned}$$

It is easy to verify that $\vartheta = 1$ is a LU optimal solution to (IVP3).

Now, we have

$$\begin{aligned} & \vartheta + \aleph\ \mu(\nu, \vartheta) \\ & = \vartheta + \aleph\ \tfrac{\nu - 2\vartheta + 1}{3} \in \mathfrak{X},\ \forall\ \nu, \vartheta \in \mathfrak{X} \text{ and } \aleph \in [0, 1]. \end{aligned}$$

It follows that $\mathfrak{X}$ is an invex set.

To illuminate Theorem 4.4, we have to show that $\vartheta = 1$ solves (IVWVLIP).

For $\vartheta = 1$ $(\nu \neq \vartheta)$, we have

$$\begin{cases} \mu(\nu,\vartheta)\nabla\mathcal{F}^L(\vartheta) \\ = \frac{1}{3}(\nu - 2\vartheta + 1)\ \frac{3(4\vartheta+1)}{2\vartheta^2+\vartheta+3} \\ = \frac{5}{6}(\nu - 1) > 0,\ \forall\ \nu \in \mathfrak{X} \end{cases} \quad \text{and} \quad \begin{cases} \mu(\nu,\vartheta)\nabla\mathcal{F}^U(\vartheta) \\ = \frac{1}{3}(\nu - 2\vartheta + 1)\ \frac{7(4\vartheta+1)}{2\vartheta^2+\vartheta+3} \\ = \frac{35}{18}(\nu - 1) > 0,\ \forall\ \nu \in \mathfrak{X} \end{cases}.$$

Therefore, $\langle \mu(\nu,\vartheta),\ \nabla\mathcal{F}(\vartheta)\rangle >_{LU} \mathbf{0}$, at $\vartheta = 1$ and so $\vartheta = 1$ solves (IVWVLIP). Hence all the conditions of the Theorem are satisfied.

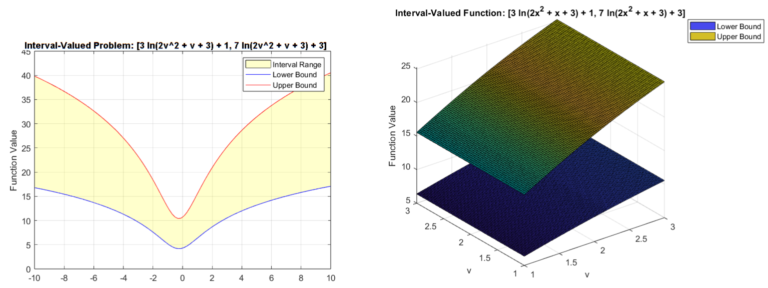

Figure 3. Surface view of $\mathcal{F}^L(\nu)$ and $\mathcal{F}^U(\nu)$ for Exa.4.3

Theorem 4.5. *Let $\mathcal{F}_i : \mathfrak{X} \to \Im, i \in \theta_r$ be gH-differentiable functions on $\mathfrak{X}$. Suppose that $\mathcal{F}$ is strictly LU-μ-invex at $\vartheta \in \mathfrak{X}$. If ϑ solves interval-valued variational-like inequality problem (VVIP) with respect to the same μ, then ϑ is a weakly LU optimal solution to (IVOP).*

Proof. Suppose contrary to the result thar ϑ is not a weakly LU optimal solution to (IVOP), then there exists a point $\nu \in \mathfrak{X}$, such that

$$\mathcal{F}_i(\nu) \prec_{LU} \mathcal{F}_i(\vartheta), i \in \theta_r.$$

That is for all $i \in \theta_r$, we have

$$\begin{cases} \mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) < 0 \\ \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) \leq 0 \\ \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mathcal{F}_i^L(\nu) - \mathcal{F}_i^L(\vartheta) < 0 \\ \mathcal{F}_i^U(\nu) - \mathcal{F}_i^U(\vartheta) \leq 0 \end{cases}.$$

The above inequalities together with the strict LU-μ-invexity of $\mathcal{F}_i, i \in \theta_r$ at ϑ, yield

$$\begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^L(\vartheta) < 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^L(\vartheta) \leq 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^L(\vartheta) < 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}_i^U(\vartheta)\vartheta \leq 0 \end{cases}.$$

That is for all $i \in \theta_r$ we have

$$\mu(\nu,\vartheta)^{\mathcal{T}}\left(\nabla\mathcal{F}_i^L(\vartheta) + \nabla\mathcal{F}_i^U(\vartheta)\right) < 0.$$

It follows from the above relations

$$\left(\left\langle \mu(\nu,\vartheta),\ \nabla\mathcal{F}_i^L(\vartheta)+\nabla\mathcal{F}_i^U(\vartheta)\right\rangle,...,\left\langle \mu(\nu,\vartheta),\ \nabla\mathcal{F}_i^L(\vartheta)+\nabla\mathcal{F}_i^U(\vartheta)\right\rangle\right) \prec_{LU} \mathbf{0},$$

which shows that ϑ does not solve interval-valued variational-like inequality problem (VVIP). This contradiction leads to the result. □

We provide the following examples that authenticates Theorem 4.5.

Example 4.4. (Verification of the Theorem 4.5) Let $\mathfrak{X} = \{\nu : 1 \leq \nu \leq 4\}$ and consider the following interval-valued problem:

$$\begin{array}{lll} \text{(IVP4)} & \min & \mathcal{F}(\nu) = [\mathcal{F}^L(\nu),\ \mathcal{F}^U(\nu)] \\ & & \quad = [3\nu^2+\nu+2,\ 4\nu^2+3\nu+5] \\ & \text{subject to} & \nu \in \mathfrak{X}. \end{array}$$

Let $\mu : \mathfrak{X}\times\mathfrak{X} \to \Re$ be defined by $\mu(\nu,\vartheta) = \frac{1}{12}\ \ln \nu$.

Now, it is easy to verify that $\mathcal{F}^L$ and $\mathcal{F}^U$ are strictly invex *with respect to* μ at $\vartheta = 1$ and so $\mathcal{F}$ is strictly invex *with respect to* μ at $\vartheta = 1$.

On the other hand, for $\vartheta = 1$

$$\begin{cases} \mu(\nu,\vartheta)\nabla\mathcal{F}^L(\vartheta) \\ = \frac{1}{12}(6\vartheta+1)\ \ln \nu \\ = \frac{7}{12}\ \ln \nu \geq 0,\ \forall\ \nu \in \mathfrak{X} \end{cases} \quad \text{and} \quad \begin{cases} \mu(\nu,\vartheta)\nabla\mathcal{F}^U(\vartheta) \\ = \frac{1}{12}(8\vartheta+3)\ \ln \nu \\ = \frac{11}{12}\ \ln \nu \geq 0,\ \forall\ \nu \in \mathfrak{X} \end{cases}.$$

Therefore, $\langle \mu(\nu,\vartheta),\ \nabla\mathcal{F}(\vartheta)\rangle \geq_{LU} \mathbf{0}$, at $\vartheta = 1$ and so $\vartheta = 1$ solves (IVVLIP).

Furthermore, it is easy to verify that $\vartheta = 1$ is also a weakly LU optimal solution to (IVP4).

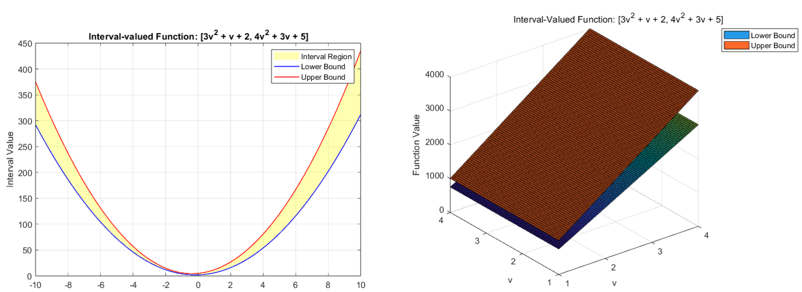

Figure 4. Surface view of $\mathcal{F}^L(\nu)$ and $\mathcal{F}^U(\nu)$ for Exa. 4.4

Theorem 4.6. *Let $\mathcal{F}_i : \mathfrak{X} \to \Im, i \in \theta_r$ be gH-differentiable functions on $\mathfrak{X}$. Suppose that $\mathcal{F}_i, i \in \theta_r$ are strictly LU-μ-invex at $\vartheta \in \mathfrak{X}$. If ϑ is a weakly LU optimal solution to (IVOP), then ϑ is a LU optimal solution to (IVOP).*

Proof. Suppose contrary to the result that ϑ is not a LU optimal solution to (IVOP). Then there exists a point $\nu \in \mathfrak{X}$ such that

$$\mathcal{F}(\nu) \leq_{LU} \mathcal{F}(\vartheta).$$

That is,

$$i.e \qquad \mathcal{F}^L(\nu) \leq \mathcal{F}^L(\vartheta) \text{ and } \mathcal{F}^U(\nu) \leq \mathcal{F}^U(\vartheta).$$

The above inequalities together with the strict LU-μ-invexity of $\mathcal{F}$ *w.r.t.* μ at ϑ, yield that

along with the previous inequalities, we have $\mathcal{F}$ strict invexity with respect to μ at ϑ, we get

$$\begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^L(\vartheta) < 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^L(\vartheta) \leq 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^U(\vartheta) < 0 \end{cases}, \text{ or } \begin{cases} \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^L(\vartheta) < 0 \\ \mu(\nu,\vartheta)^{\mathcal{T}}\nabla\mathcal{F}^U(\vartheta) \leq 0 \end{cases}.$$

It follows from the above relations that

$$\langle \mu(\nu,\vartheta),\ \nabla\mathcal{F}(\vartheta)\rangle <_{LU} \mathbf{0},$$

which shows that ϑ does not solve interval-valued variational-like inequality problem (IVVLIP). Hence, ϑ is not a weakly LU optimal solution to (IVP) according to the previous Theorem 4.2, which goes contradictory to the presumption that it is. □

5. Conclusions

Our paper presents a novel category of variational-like inequality problems that are interval-valued. Additionally, we establish certain connections between variational-like inequality problems and interval-valued optimization. Illustrative examples are provided to clarify the inferred connections. It will be fascinating to discover the outcomes shown in this research under generalized invexity assumptions for non-smooth interval-valued optimization problems. By connecting IVOP with VVLI, researchers can develop better strategies for optimizing under uncertain conditions, leading to more robust and resilient solutions. Furthermore, upcoming research aims to explore the uncertain environment of a similar nature to investigate the optimality conditions that involve fuzzy parameters.

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