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Two Minimal Residual NHSS Iteration Methods for Complex Symmetric Linear Systems

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Abstract For the large sparse complex symmetric linear systems, by applying the minimal residual technique to accelerate a preconditioned variant of new Hermitian and skew-Hermitian splitting (P*NHSS) method and efficient parameterized P*NHSS (PPNHSS) method, we construct the minimal residual P*NHSS (MRP*NHSS) method and the minimal residual PPNHSS (MRPPNHSS) method. The convergence properties of the two iteration methods are studied. Theoretical analyses imply that the MRP*NHSS method and the MRPPNHSS method converge unconditionally to the unique solution. In addition, we also give the inexact versions of MRP*NHSS method and MRPPNHSS method and their convergence proofs. Finally, numerical experiments show the high efficiency and robustness of our methods.

Keywords Complex symmetric linear systems, minimal residual technique, inexact versions, convergence properties

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1. Introduction

In this paper, we consider iterative methods for solving the complex symmetric linear systems of the form:

$$Ax \equiv (W + iT)x = b, \tag{1.1}$$

where $i = \sqrt{-1}$ denotes the imaginary unit, and W, $T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite. Furthermore, we assume that W is a symmetric positive definite matrix. In addition, $b \in \mathbb{C}^n$ is given and $x \in \mathbb{C}^n$ is what we need to get. The linear system (1.1) appears in many applications, such as fast Fourier transform-based solution of certain time-dependent PDEs [1], structural dynamics [2], diffuse optical tomography [3], molecular scattering [4], parabolic and hyperbolic problems [5], and so on.

In recent years, based on the Hermitian and skew-Hermitian splitting (HSS) of matrix A, i.e., $A = H(A) + S(A)$, with

$$H(A) = \frac{1}{2}(A + A^*) = W, \quad S(A) = \frac{1}{2}(A - A^*) = iT,$$

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Bai, Golub and Ng [6] introduced an HSS iteration method to approximate the solution of the system of linear equations. Nevertheless, a shifted skew-Hermitian linear system needs to be solved at each iteration step of the HSS iteration method. To improve this situation, Bai, Benzi and Chen [7] came up with a modified HSS (MHSS) method. If sparse triangular factorizations are used to solve the linear sub-systems involved at each step, the MHSS iteration method is likely to require considerably less storage than the HSS iteration method since only two triangular factors rather than three have to be computed and stored. To further increase efficiency of MHSS method, Bai et al. proposed the preconditioned MHSS (PMHSS) iteration method and applied it to the distributed control problems in [8, 9]. Since an arbitrary matrix V is positive definite, $\alpha V + W$ and $\alpha V + T$ are both real symmetric positive definite, the two sub-systems involved in each step of the PMHSS iteration can be effectively solved either exactly by a sparse Cholesky factorization, or inexactly by PCG method. Moreover, the PMHSS iteration method converges to the unique solution of the system of linear equations (1.1) for any positive constant α and any initial guess $x^{(0)}$.

Subsequently, the authors of [10] raised a new HSS (NHSS) method to solve the non-Hermitian positive definite linear systems. The convergence analysis showed that the NHSS method converges to the unique solution if $\sigma_{\max} \leq \lambda_{\min}$, where $\sigma_{\max}$ is the maximum singular value of the matrix $S(A)$ and $\lambda_{\min}$ is the minimum eigenvalue of the matrix $H(A)$. Besides, numerical examples showed the NHSS method performs very well when the Hermitian part of the coefficient matrix is dominant. In 2018, based on the NHSS iteration method, Xiao, Wang and Yin [11] introduced a preconditioned variant of NHSS (P*NHSS) and an efficient parameterized P*NHSS (PPNHSS) iteration methods for solving a class of complex symmetric linear systems. They proved that these iterative sequences are convergent to the unique solution of the linear system for any initial guess under a loose restriction on the parameters α and ω. Numerical results showed that the PPNHSS iteration method outperforms PMHSS, NHSS and a preconditioned variant of the generalized successive over-relaxation (PGSOR) [12] methods from the point of view of iterations and CPU times whether the experimental optimal parameters are used or not. More iteration methods for solving a class of complex symmetric linear systems, see [13–20].

Recently, Yang, Cao and Wu [21] proposed a minimum residual HSS (MRHSS) method to improve the efficiency of the HSS method by making use of the minimum residual technique to HSS iteration scheme. Numerical results revealed that the MRHSS method is much more effective than the HSS method. Then, Yang [22] improved the problem of inconvenient verification of the convergence of MRHSS. In order to avoid shifted skew-Hermitian linear system, Zhang, Yang and Wu [23, 24] further used the minimum residual technique to the MHSS iteration scheme and proposed the minimum residual MHSS (MRMHSS) method. Inspired by the above idea, we apply the minimum residual technique on P*NHSS and PPNHSS methods and develop the minimum residual P*NHSS (MRP*NHSS) method and the minimum residual parameterized P*NHSS (MRPPNHSS) method to improve the efficiency of the two methods of [11]. The corresponding convergence theories are also established.

Throughout this paper, we denote by $(x, y) = y^*x$ the Euclidean inner product for any complex vectors $x, y \in \mathbb{C}^n$, and denote by $\|x\| = \sqrt{(x, x)}$ the Euclidean norm for any complex vector $x \in \mathbb{C}^n$. For an arbitrary matrix $X \in \mathbb{C}^{n\times n}$, $\|X\|$

denotes the spectral norm of X. In addition, we denote $\mathcal{F}(M)$ as the field values of the complex matrix $M \in \mathbb{C}^{n\times n}$, i.e., $\mathcal{F}(M) = \{(My, y)/(y, y)|0 \neq y \in \mathbb{C}^n\}$. What is more, $\Re(\cdot)$ represents the real part of the complex matrix or the complex number.

The paper is organized as follows. In Section 2, we show the derivation process of the MRP*NHSS method and its inexact version. Their convergence properties are also considered. In Section 3, the convergence properties of the MRPPNHSS iteration method and its inexact version are discussed. In Section 4, some numerical examples are given to evaluate the effectiveness of the MRP*NHSS and MRPPNHSS iteration methods. Finally, a brief conclusion is drawn in Section 5.

2. The minimal residual P*NHSS algorithm

The iteration scheme of the P*NHSS iteration method [11] has the form of

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -iTx^{(k)} + b, \\ (\alpha V + W)x^{(k+1)} = (\alpha V - iT)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

Using

$$W^{-1}(-iT) = I - W^{-1}A$$

and

$$(\alpha V + W)^{-1}(\alpha V - iT) = I - (\alpha V + W)^{-1}A,$$

the P*NHSS iteration scheme can be equivalently rewritten as

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + W^{-1}(b - Ax^{(k)}), \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + (\alpha V + W)^{-1}(b - Ax^{(k+\frac{1}{2})}). \end{cases} \tag{2.1}$$

To get the minimal residual P*NHSS algorithm, we will modify the above iteration scheme. Denoting $r^{(k)} = b - Ax^{(k)}$ and $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$, the P*NHSS iteration scheme (2.1) can be rewritten as

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + W^{-1}r^{(k)}, \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + (\alpha V + W)^{-1}r^{(k+\frac{1}{2})}. \end{cases}$$

Then we introduce two parameters β_k and γ_k to control the step sizes, which lead to a new iteration scheme:

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + \beta_k W^{-1}r^{(k)}, \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + \gamma_k(\alpha V + W)^{-1}r^{(k+\frac{1}{2})}. \end{cases} \tag{2.2}$$

Note that $A \in \mathbb{C}^{n\times n}$ and x, $b \in \mathbb{C}^n$. It should be better to choose β_k and γ_k in the complex field $\mathbb{C}$. At the same time, we denote $M_1 := AW^{-1}$ and $M_2 := A(\alpha V + W)^{-1}$. After that, the residual form of the iteration scheme (2.2) can be written as

$$\begin{cases} r^{(k+\frac{1}{2})} = r^{(k)} - \beta_k M_1 r^{(k)}, \\ r^{(k+1)} = r^{(k+\frac{1}{2})} - \gamma_k M_2 r^{(k+\frac{1}{2})}. \end{cases}$$

Similar to [21], we can obtain the values of β_k and γ_k by minimizing the residual norms $\|r^{(k+\frac{1}{2})}\|$ and $\|r^{(k+1)}\|$,

$$\begin{aligned}\|r^{(k+\frac{1}{2})}\|^2 =&\|r^{(k)}\|^2 - 2Re(\beta_k)(H(M_1)r^{(k)}, r^{(k)}) - 2Im(\beta_k)(iS(M_1)r^{(k)}, r^{(k)})\\ &+ (Re(\beta_k)^2 + Im(\beta_k)^2)\|M_1 r^{(k)}\|^2\end{aligned}$$

and

$$\begin{aligned}&\|r^{(k+1)}\|^2\\ =&\|r^{(k+\frac{1}{2})}\|^2 - 2Re(\gamma_k)(H(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) - 2Im(\gamma_k)(iS(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})\\ &+ (Re(\gamma_k)^2 + Im(\gamma_k)^2)\|M_2 r^{(k+\frac{1}{2})}\|^2,\end{aligned}$$

where $Re(\cdot)$ and $Im(\cdot)$ denote the real and the imaginary parts of a complex number, respectively. It is easily observed that these two norms can be viewed as four real valued convex functions of two variables $Re(\beta_k)$ and $Im(\beta_k)$, $Re(\gamma_k)$ and $Im(\gamma_k)$, respectively. So, the minimum point of each function can be directly derived as

$$Re(\beta_k) = \frac{(H(M_1)r^{(k)}, r^{(k)})}{\|M_1 r^{(k)}\|^2},\ Im(\beta_k) = \frac{(iS(M_1)r^{(k)}, r^{(k)})}{\|M_1 r^{(k)}\|^2} \tag{2.3}$$

and

$$Re(\gamma_k) = \frac{(H(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})}{\|M_2 r^{(k+\frac{1}{2})}\|^2},\ Im(\gamma_k) = \frac{(iS(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})}{\|M_2 r^{(k+\frac{1}{2})}\|^2}. \tag{2.4}$$

For easy computation, we can represent above form as

$$\begin{aligned}\beta_k = Re(\beta_k) + iIm(\beta_k) &= \frac{(H(M_1)r^{(k)}, r^{(k)})}{\|M_1 r^{(k)}\|^2} + i\frac{(iS(M_1)r^{(k)}, r^{(k)})}{\|M_1 r^{(k)}\|^2}\\ &= \frac{(r^{(k)}, M_1 r^{(k)})}{\|M_1 r^{(k)}\|^2}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\gamma_k = Re(\gamma_k) + iIm(\gamma_k) &= \frac{(H(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})}{\|M_2 r^{(k+\frac{1}{2})}\|^2} + i\frac{(iS(M_2)r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})}{\|M_2 r^{(k+\frac{1}{2})}\|^2}\\ &= \frac{(r^{(k+\frac{1}{2})}, M_2 r^{(k+\frac{1}{2})})}{\|M_2 r^{(k+\frac{1}{2})}\|^2}.\end{aligned} \tag{2.6}$$

Thus, we can complete the form of MRP*NHSS in Algorithm 1:

Algorithm 1 The MRP*NHSS method

1. Input: A, b, positive constants α, the stop tolerance ϵ, evaluate $r^{(0)} = b$ and initial estimate $x^{(0)}$.
2. Output: approximate $x^{(k)}$ solving $Ax = b$.
3. **for** $k = 0, 1, 2...$ until satisfying the stopping criteria $\frac{||b-Ax^{(k)}||}{||b||} \leq \varepsilon$
   (a) Solve $Wt_1 = r^{(k)}$;
   (b) Compute $t_2 = At_1$;
   (c) Compute $\beta_k = \frac{(r^{(k)}, t_2)}{\|t_2\|^2}$;
   (d) Set $x^{(k+\frac{1}{2})} \leftarrow x^{(k)} + \beta_k t_1$ and evaluate $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$;
   (e) Solve $(\alpha V + W)t_3 = r^{(k+\frac{1}{2})}$;
   (f) Compute $t_4 = At_3$;
   (g) Compute $\gamma_k = \frac{(r^{(k+\frac{1}{2})}, t_4)}{\|t_4\|^2}$;
   (h) Set $x^{(k+1)} \leftarrow x^{(k+\frac{1}{2})} + \gamma_k t_3$ and evaluate $r^{(k+1)} = b - Ax^{(k+1)}$;
   (i) $k = k + 1$.
4. **end for**.

Along the same lines as in Theorem 1 of [23], we can easily get

Theorem 2.1. *The quadruple* $(Re(\beta_k), Im(\beta_k), Re(\gamma_k), Im(\gamma_k))$ *obtained by (2.3) and (2.4) is a global minimum point of* $\|r^{(k+1)}\|$*, which implies that the values of* β_k *and* γ_k *defined by (2.5) and (2.6) are optimal in the complex field* $\mathbb{C}$*.*

Remark 2.1. When we choose $\beta_k = \gamma_k = 1$ at each iteration step of above method, the MRP*NHSS iteration method immediately reduces to the P*NHSS iteration method.

Next, we give the following convergence property of the MRP*NHSS iteration method.

Theorem 2.2. *Let* W *and* T *be symmetric positive definite and symmetric positive semi-definite, respectively. The* MRP*NHSS *iteration method used for solving the complex symmetric linear system (1.1) is convergent unconditionally for any initial guess* $x^{(0)} \in \mathbb{C}^n$*. Furthermore, the residuals satisfy*

$$\|r^{(k+1)}\| \leq \frac{\sqrt{\|AW^{-1}\|^2 - \xi_1^2}}{\|AW^{-1}\|} \frac{\sqrt{\|A(\alpha V + W)^{-1}\|^2 - \xi_2^2}}{\|A(\alpha V + W)^{-1}\|} \|r^{(k)}\|, \tag{2.7}$$

where ξ_1 *and* ξ_2 *represent the distances from zero to* $\mathcal{F}(AW^{-1})$ *and* $\mathcal{F}(A(\alpha V + W)^{-1})$*, respectively.*

Proof. At the beginning, we consider the first half of the iteration method.

Note that

$$\begin{aligned}(r^{(k+\frac{1}{2})}, M_1 r^{(k)}) =& ((I - \beta_k M_1) r^{(k)}, M_1 r^{(k)}) \\ =& (r^{(k)}, M_1 r^{(k)}) - \frac{(r^{(k)}, M_1 r^{(k)})}{\|M_1 r^{(k)}\|^2} (M_1 r^{(k)}, M_1 r^{(k)}) = 0,\end{aligned}$$

i.e.,

$$r^{(k+\frac{1}{2})} \perp M_1 r^{(k)}.$$

Subsequently, by using above condition, we can obtain the relation between $r^{(k)}$ and $r^{(k+\frac{1}{2})}$:

$$\begin{aligned}
(r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) =& (r^{(k)} - \beta_k M_1 r^{(k)}, r^{(k+\frac{1}{2})}) \\
=& (r^{(k)}, r^{(k+\frac{1}{2})}) - \beta_k (M_1 r^{(k)}, r^{(k+\frac{1}{2})}) \\
=& (r^{(k)}, r^{(k+\frac{1}{2})}) \\
=& (r^{(k)} - \beta_k M_1 r^{(k)}, r^{(k)}) \\
=& (r^{(k)}, r^{(k)}) - \beta_k (M_1 r^{(k)}, r^{(k)}) \\
=& (r^{(k)}, r^{(k)}) - \frac{(M_1 r^{(k)}, r^{(k)})}{\|M_1 r^{(k)}\|^2} (M_1 r^{(k)}, r^{(k)}) \\
=& (r^{(k)}, r^{(k)}) - \frac{(M_1 r^{(k)}, r^{(k)})^2}{\|M_1 r^{(k)}\|^2 \|r^{(k)}\|^2} (r^{(k)}, r^{(k)}) \\
=& (r^{(k)}, r^{(k)})(1 - \cos^2 \angle_k) \\
=& (r^{(k)}, r^{(k)}) \sin^2 \angle_k,
\end{aligned} \tag{2.8}$$

where $\angle_k$ is the angle between $r^{(k)}$ and $M_1 r^{(k)}$. In the same way, we can also get

$$r^{(k+1)} \perp M_2 r^{(k+\frac{1}{2})}$$

and the relation between $r^{(k+\frac{1}{2})}$ and $r^{(k+1)}$:

$$(r^{(k+1)}, r^{(k+1)}) = (r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) \sin^2 \angle_{k+\frac{1}{2}},$$

where $\angle_{k+\frac{1}{2}}$ is the angle between $r^{(k+\frac{1}{2})}$ and $M_2 r^{(k+\frac{1}{2})}$. Thus, we have

$$\|r^{(k+1)}\|^2 = \|r^{(k)}\|^2 \cdot \sin^2 \angle_k \cdot \sin^2 \angle_{k+\frac{1}{2}},$$

that is to say,

$$\|r^{(k+1)}\| = \|r^{(k)}\| \cdot |\sin \angle_k| \cdot |\sin \angle_{k+\frac{1}{2}}|.$$

To ensure that Algorithm 1 is convergent, we should guarantee $|\sin \angle_k|$ and $|\sin \angle_{k+\frac{1}{2}}|$ are not equal to one in the meantime. In other words, Algorithm 1 is convergent if and only if

$$0 \notin \mathcal{F}(M_1) \cap \mathcal{F}(M_2).$$

As a matter of fact, since W is symmetric positive definite and T is symmetric positive semi-definite, we have

$$\Re(\mathcal{F}(M_1)) > 0.$$

Thus,

$$0 \notin \mathcal{F}(M_1).$$

Therefore, the MRP*NHSS iteration method converges unconditionally to the exact solution of the system of linear equation (1.1).

On the other hand, we define

$$\xi_1 = \min_{\mathbf{0}\neq y\in\mathbb{C}^n}\left|\frac{(M_1y,y)}{(y,y)}-0\right| \text{ and } \xi_2 = \min_{\mathbf{0}\neq y\in\mathbb{C}^n}\left|\frac{(M_2y,y)}{(y,y)}-0\right|.$$

From (2.8), we can obtain

$$\begin{aligned}\|r^{(k+\frac{1}{2})}\|^2 =&\|r^{(k)}\|^2-\frac{(M_1r^{(k)},r^{(k)})}{\|M_1r^{(k)}\|^2}(M_1r^{(k)},r^{(k)})\\ =&\|r^{(k)}\|^2\left(1-\left|\frac{(M_1r^{(k)},r^{(k)})}{\|r^{(k)}\|^2}\right|^2\frac{\|r^{(k)}\|^2}{\|M_1r^{(k)}\|^2}\right)\\ \leq&\|r^{(k)}\|^2(1-\frac{\xi_1^2}{\|M_1\|^2}).\end{aligned}$$

In the similar way, we can also obtain

$$\|r^{(k+1)}\|^2 \leq \|r^{(k+\frac{1}{2})}\|^2\left(1-\frac{\xi_2^2}{\|M_2\|^2}\right).$$

Combining the above two inequalities, the proof can be completed. □

Next, we will solve $Wt = r^{(k)}$ and $(\alpha V + W)t = r^{(k+\frac{1}{2})}$ by iterative method and then give the inexact version of the MRP*NHSS method in Algorithm 2.

Algorithm 2 The IMRP*NHSS method

1. Input: A, b, positive constants α, the stop tolerance ε, evaluate $\bar{r}^{(0)} = b$, initial estimate $\bar{x}^{(0)}$, δ_k and η_k, where $k = 1, 2\ldots$.
2. Output: Approximate $\bar{x}^{(k)}$ solving $Ax = b$.
3. **for** $k = 0, 1, 2, ...$ until satisfying the stopping criteria $\frac{||b-A\bar{x}^{(k)}||}{||b||} \leq \varepsilon$
   (a) Solve $Wg^{(k+\frac{1}{2})} = \bar{r}^{(k)}$ inexactly by using CG(PCG) method until the approximate solution $g^{(k+\frac{1}{2})}$ satisfying $\|u^{(k)}\| \leq \delta_k\|\bar{r}^{(k)}\|$, where $u^{(k)} = Wg^{(k+\frac{1}{2})} - \bar{r}^{(k)}$;
   (b) Compute $h^{(k+\frac{1}{2})} = Ag^{(k+\frac{1}{2})}$;
   (c) Compute $\bar{\beta}_k = \frac{(\bar{r}^{(k)},h^{(k+\frac{1}{2})})}{\|h^{(k+\frac{1}{2})}\|^2}$;
   (d) Set $\bar{x}^{(k+\frac{1}{2})} \leftarrow \bar{x}^{(k)} + \bar{\beta}_k h^{(k+\frac{1}{2})}$ and evaluate $\bar{r}^{(k+\frac{1}{2})} = b - A\bar{x}^{(k+\frac{1}{2})}$;
   (e) Solve $(\alpha V + W)g^{(k+1)} = \bar{r}^{(k+\frac{1}{2})}$ inexactly by using CG(PCG) method until the approximate solution $g^{(k+1)}$ satisfying $\|u^{(k+\frac{1}{2})}\| \leq \eta_k\|\bar{r}^{(k+\frac{1}{2})}\|$, where $u^{(k+\frac{1}{2})} = (\alpha V + W)g^{(k+1)} - \bar{r}^{(k+\frac{1}{2})}$;
   (f) Compute $h^{(k+1)} = Ag^{(k+1)}$;
   (g) Compute $\bar{\gamma}_k = \frac{(\bar{r}^{(k)},h^{(k+1)})}{\|h^{(k+1)}\|^2}$;
   (h) Set $\bar{x}^{(k+1)} \leftarrow \bar{x}^{(k+\frac{1}{2})} + \bar{\gamma}_k h^{(k+1)}$ and evaluate $\bar{r}^{(k+1)} = b - A\bar{x}^{(k+1)}$;
   (i) $k = k + 1$.
4. **end for**.

The convergence theory of the IMRP*NHSS method is given as follows.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. If* $\{\bar{x}^{(k)}\}$ *is an iterative sequence obtained from the* IMRP*NHSS *iteration method, then it holds that*

$$\|\bar{r}^{(k+1)}\| \leq \Upsilon(\delta_k, \eta_k)\|\bar{r}^{(k)}\|$$

where

$$\begin{aligned}\Upsilon(\delta_k, \eta_k) =&\|I - \gamma_k M_2\|\|I - \beta_k M_1\| + \eta_k\|\gamma_k M_2\|\|I - \beta_k M_1\| \\ &+ \delta_k\|I - \gamma_k M_2\|\|\beta_k M_1\| + \eta_k\delta_k\|\gamma_k M_2\|\|\beta_k M_1\|.\end{aligned}$$

Furthermore, if $\delta_{\max} = \max\{\delta_k\}$ *and* $\eta_{\max} = \max\{\eta_k\}$ *satisfy*

$$\Upsilon(\delta_{\max}, \eta_{\max}) < 1, \tag{2.9}$$

then the sequence $\{\bar{x}^{(k)}\}$ *converges to the exact solution.*

Proof. By applying the conditions $\|u^{(k)}\| \leq \delta_k\|\bar{r}^{(k)}\|$ and $\|u^{(k+\frac{1}{2})}\| \leq \eta_k\|\bar{r}^{(k+\frac{1}{2})}\|$, we are going to get the relation between $\|\bar{r}^{(k+\frac{1}{2})}\|$ and $\|\bar{r}^{(k+1)}\|$:

$$\begin{aligned}\|\bar{r}^{(k+1)}\| =&\|\bar{r}^{(k+\frac{1}{2})} - \bar{\gamma}_k A g^{(k+1)}\| \\ \leq&\|\bar{r}^{(k+\frac{1}{2})} - \frac{(\bar{r}^{(k+\frac{1}{2})}, M_2\bar{r}^{(k+\frac{1}{2})})}{\|M_2\bar{r}^{(k+\frac{1}{2})}\|^2} A g^{(k+1)}\| \\ =&\|\bar{r}^{(k+\frac{1}{2})} - \frac{(\bar{r}^{(k+\frac{1}{2})}, M_2\bar{r}^{(k+\frac{1}{2})})}{\|M_2\bar{r}^{(k+\frac{1}{2})}\|^2} M_2(\bar{r}^{(k+\frac{1}{2})} + u^{(k+\frac{1}{2})})\| \\ \leq&\|\bar{r}^{(k+\frac{1}{2})} - \gamma_k M_2\bar{r}^{(k+\frac{1}{2})}\| + \|\gamma_k M_2 u^{(k+\frac{1}{2})}\| \\ \leq&\|I - \gamma_k M_2\|\|\bar{r}^{(k+\frac{1}{2})}\| + \|\gamma_k M_2\|\|u^{(k+\frac{1}{2})}\| \\ \leq&\|I - \gamma_k M_2\|\|\bar{r}^{(k+\frac{1}{2})}\| + \eta_k\|\gamma_k M_2\|\|\bar{r}^{(k+\frac{1}{2})}\| \\ =&(\|I - \gamma_k M_2\| + \eta_k\|\gamma_k M_2\|)\|\bar{r}^{(k+\frac{1}{2})}\|.\end{aligned} \tag{2.10}$$

The first inequality is because of the definition of $\bar{\gamma}_k$. In the similar way, we can also obtain the relation between $\|\bar{r}^{(k+\frac{1}{2})}\|$ and $\|\bar{r}^{(k)}\|$:

$$\|\bar{r}^{(k+\frac{1}{2})}\| \leq (\|I - \beta_k M_1\| + \delta_k\|\beta_k M_1\|)\|\bar{r}^{(k)}\|. \tag{2.11}$$

Thus, by combining (2.10) and (2.11) and simple calculations, we can obtain

$$\begin{aligned}\|\bar{r}^{(k+1)}\| \leq&(\|I - \gamma_k M_2\|\|I - \beta_k M_1\| + \eta_k\|\gamma_k M_2\|\|I - \beta_k M_1\| \\ &+\delta_k\|I - \gamma_k M_2\|\|\beta_k M_1\| + \eta_k\delta_k\|\gamma_k M_2\|\|\beta_k M_1\|)\|\bar{r}^{(k)}\|.\end{aligned}$$

It can be seen from the proof procedure of the MRP*NHSS that $\|I - \gamma_k M_2\|\|I - \beta_k M_1\| < 1$. Besides, notice that $\|\gamma_k M_2\|\|I - \beta_k M_1\|$, $\|I - \gamma_k M_2\|\|\beta_k M_1\|$ and $\|\gamma_k M_2\|\|\beta_k M_1\|$ are all positive, let $\delta_{\max} = \max\{\delta_k\}$ and $\eta_{\max} = \max\{\eta_k\}$, we have

$$\Upsilon(\delta_k, \eta_k) \leq \Upsilon(\delta_{\max}, \eta_{\max}).$$

This implies that the IMRP*NHSS iteration method is convergent if (2.9) holds. □

Remark 2.2. In fact, the IMRP*NHSS iteration method is going to be very close to the MRP*NHSS iteration method when $\delta_k \to 0$ and $\eta_k \to 0 (k \to +\infty)$. Thus, the IMRP*NHSS iteration method is sure to be convergent when $\delta_{\max}$ and $\eta_{\max}$ are small enough.

3. The minimal residual PPNHSS algorithm

The iteration scheme of the PPNHSS iteration method has the form of

$$\begin{cases}(\omega W+T)x^{(k+\frac{1}{2})}=-i(\omega T-W)x^{(k)}+(\omega-i)b,\\(\alpha V+\omega W+T)x^{(k+1)}=[\alpha V-i(\omega T-W)]x^{(k+\frac{1}{2})}+(\omega-i)b,\end{cases}\tag{3.1}$$

where ω and α are given positive constants and $V\in\mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

Similar to (2.2), we can rewrite (3.1) as its residual form:

$$\begin{cases}x^{(k+\frac{1}{2})}=x^{(k)}+(\omega-i)(\omega W+T)^{-1}r^{(k)},\\x^{(k+1)}=x^{(k+\frac{1}{2})}+(\omega-i)(\alpha V+\omega W+T)^{-1}r^{(k+\frac{1}{2})}.\end{cases}\tag{3.2}$$

Subsequently, we also bring in two parameters λ_k and θ_k:

$$\begin{cases}x^{(k+\frac{1}{2})}=x^{(k)}+\lambda_k(\omega-i)(\omega W+T)^{-1}r^{(k)},\\x^{(k+1)}=x^{(k+\frac{1}{2})}+\theta_k(\omega-i)(\alpha V+\omega W+T)^{-1}r^{(k+\frac{1}{2})}.\end{cases}\tag{3.3}$$

Further, we have

$$\begin{cases}r^{(k+\frac{1}{2})}=r^{(k)}-\lambda_k(\omega-i)(\omega W+T)^{-1}r^{(k)},\\r^{(k+1)}=r^{(k+\frac{1}{2})}-\theta_k(\omega-i)(\alpha V+\omega W+T)^{-1}r^{(k+\frac{1}{2})}.\end{cases}\tag{3.4}$$

What's more, let $G_1:=A(\omega W+T)^{-1}$ and $G_2:=A(\alpha V+\omega W+T)^{-1}$, we calculate λ_k and θ_k similarly:

$$(Re(\lambda_k),Im(\lambda_k))=(\frac{(H((\omega-i)G_1)r^{(k)},r^{(k)})}{\|(\omega-i)G_1r^{(k)}\|^2},\frac{(iS((\omega-i)G_1)r^{(k)},r^{(k)})}{\|(\omega-i)G_1r^{(k)}\|^2})\tag{3.5}$$

and

$$(Re(\theta_k),Im(\theta_k))=(\frac{(H((\omega-i)G_2)r^{(k)},r^{(k)})}{\|(\omega-i)G_2r^{(k)}\|^2},\frac{(iS((\omega-i)G_2)r^{(k)},r^{(k)})}{\|(\omega-i)G_2r^{(k)}\|^2}).\tag{3.6}$$

The expressions of λ_k and θ_k can be given as follows:

$$\lambda_k=Re(\lambda_k)+iIm(\lambda_k)=\frac{(r^{(k)},(\omega-i)G_1r^{(k)})}{\|(\omega-i)G_1r^{(k)}\|^2}\tag{3.7}$$

and

$$\theta_k=Re(\theta_k)+iIm(\theta_k)=\frac{(r^{(k+\frac{1}{2})},(\omega-i)G_2r^{(k+\frac{1}{2})})}{\|(\omega-i)G_2r^{(k+\frac{1}{2})}\|^2}.\tag{3.8}$$

Therefore, the iteration scheme of the MRPPNHSS method is

$$\begin{cases}x^{(k+\frac{1}{2})}&=x^{(k)}+\frac{(r^{(k)},(\omega-i)G_1r^{(k)})}{\|(\omega-i)G_1r^{(k)}\|^2}(\omega-i)(\omega W+T)^{-1}r^{(k)}\\&=x^{(k)}+\frac{(r^{(k)},(G_1r^{(k)})}{\|G_1r^{(k)}\|^2}(\omega W+T)^{-1}r^{(k)},\\x^{(k+1)}&=x^{(k+\frac{1}{2})}+\frac{(r^{(k+\frac{1}{2})},(\omega-i)G_2r^{(k+\frac{1}{2})})}{\|(\omega-i)G_2r^{(k+\frac{1}{2})}\|^2}(\omega-i)(\alpha V+\omega W+T)^{-1}r^{(k+\frac{1}{2})}\\&=x^{(k+\frac{1}{2})}+\frac{(r^{(k+\frac{1}{2})},G_2r^{(k+\frac{1}{2})})}{\|G_2r^{(k+\frac{1}{2})}\|^2}(\alpha V+\omega W+T)^{-1}r^{(k+\frac{1}{2})}.\end{cases}$$

We write the complete form of MRPPNHSS in Algorithm 3.

Algorithm 3 The MRPPNHSS method

1. Input: A, b, positive constants α and ω, symmetric positive definite matrix V, the stop tolerance ϵ, evaluate $r^{(0)} = b$ and initial estimate $x^{(0)}$.
2. Output: approximate $x^{(k)}$ solving $Ax = b$.
3. **for** $k = 0, 1, 2...$ until satisfying the stopping criteria $\frac{||b-Ax^{(k)}||}{||b||} \leq \varepsilon$
   (a) Solve $(\omega W + T)t_1 = r^{(k)}$;
   (b) Compute $t_2 = At_1$;
   (c) Compute $\lambda_k^{'} = \frac{(r^{(k)}, t_2)}{\|t_2\|^2}$;
   (d) Set $x^{(k+\frac{1}{2})} \leftarrow x^{(k)} + \lambda_k^{'} t_1$ and evaluate $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$;
   (e) Solve $(\alpha V + \omega W + T)t_3 = r^{(k+\frac{1}{2})}$;
   (f) Compute $t_4 = At_3$;
   (g) Compute $\theta_k^{'} = \frac{(r^{(k+\frac{1}{2})}, t_4)}{\|t_4\|^2}$;
   (h) Set $x^{(k+1)} \leftarrow x^{(k+\frac{1}{2})} + \theta_k^{'} t_3$ and evaluate $r^{(k+1)} = b - Ax^{(k+1)}$;
   (i) $k = k + 1$.
4. **end for**.

Remark 3.1. When we choose $\lambda_k' = \theta_k' = \omega - i$ at each iteration step of above method, then the MRPPNHSS iteration method reduces to the PPNHSS iteration method.

Similar to the MRP*NHSS method, we give the following theorem. The proof is similar to Theorem 1 of [23] and we omit it.

Theorem 3.1. *The quadruple* $(Re(\lambda_k), Im(\lambda_k), Re(\theta_k), Im(\theta_k))$ *obtained by (3.5) and (3.6) is a global minimum point of* $\|r^{(k+1)}\|$ *of* MRPPNHSS*, which implies that the values of* λ_k *and* θ_k *defined by (3.7) and (3.8) are optimal in the complex field* $\mathbb{C}$*.*

The convergence properties of the MRPPNHSS iteration method are derived as follows.

Theorem 3.2. *Let the assumptions of Theorem 2.2 hold. The* MRPPNHSS *iteration method used for solving the complex symmetric system of (1.1) is convergent unconditionally for any initial guess* $x^{(0)} \in \mathbb{C}^n$*. Furthermore, the residuals satisfy*

$$\|r^{(k+1)}\| \leq \frac{\sqrt{\|A(\omega W + T)^{-1}\|^2 - \rho_1^2}}{\|A(\omega W + T)^{-1}\|} \frac{\sqrt{\|A(\alpha V + \omega W + T)^{-1}\|^2 - \rho_2^2}}{\|A(\alpha V + \omega W + T)^{-1}\|} \|r^{(k)}\|, \quad (3.9)$$

where ρ_1 *and* ρ_2 *represent the distances from zero to* $\mathcal{F}(A(\omega W + T)^{-1})$ *and* $\mathcal{F}(A(\alpha V + \omega W + T)^{-1})$*, respectively.*

Proof. First, we consider the first half of the iteration method.

Note that

$$\begin{aligned}(r^{(k+\frac{1}{2})}, G_1 r^{(k)}) =&((I-\lambda_k(\omega-i)G_1)r^{(k)}, G_1 r^{(k)})\\ =&(r^{(k)}, G_1 r^{(k)}) - \frac{(r^{(k)}, (\omega-i)G_1 r^{(k)})}{\|(\omega-i)G_1 r^{(k)}\|^2}((\omega-i)G_1 r^{(k)}, G_1 r^{(k)})\\ =&0,\end{aligned}$$

i.e.,

$$r^{(k+\frac{1}{2})} \perp G_1 r^{(k)}.$$

Subsequently, by using above condition, we can obtain the relation between $r^{(k)}$ and $r^{(k+\frac{1}{2})}$:

$$\begin{aligned}(r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) =&(r^{(k)} - \lambda_k(\omega-i)G_1 r^{(k)}, r^{(k+\frac{1}{2})})\\ =&(r^{(k)}, r^{(k+\frac{1}{2})}) - \lambda_k((\omega-i)G_1 r^{(k)}, r^{(k+\frac{1}{2})})\\ =&(r^{(k)}, r^{(k+\frac{1}{2})})\\ =&(r^{(k)} - \lambda_k(\omega-i)G_1 r^{(k)}, r^{(k)})\\ =&(r^{(k)}, r^{(k)}) - \lambda_k((\omega-i)G_1 r^{(k)}, r^{(k)})\\ =&(r^{(k)}, r^{(k)}) - \frac{((\omega-i)G_1 r^{(k)}, r^{(k)})}{\|(\omega-i)G_1 r^{(k)}\|^2}((\omega-i)G_1 r^{(k)}, r^{(k)})\\ =&(r^{(k)}, r^{(k)}) - \frac{((\omega-i)G_1 r^{(k)}, r^{(k)})^2}{\|(\omega-i)G_1 r^{(k)}\|^2\|r^{(k)}\|^2}(r^{(k)}, r^{(k)})\\ =&(r^{(k)}, r^{(k)})(1-\cos^2\angle'_k)\\ =&(r^{(k)}, r^{(k)})\sin^2\angle'_k,\end{aligned} \tag{3.10}$$

where $\angle'_k$ is the angle between $r^{(k)}$ and $G_1 r^{(k)}$. In the same way, we can also get

$$r^{(k+1)} \perp G_2 r^{(k+\frac{1}{2})}$$

and the relation between $r^{(k+\frac{1}{2})}$ and $r^{(k+1)}$:

$$(r^{(k+1)}, r^{(k+1)}) = (r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})})\sin^2\angle'_{k+\frac{1}{2}},$$

where $\angle'_{k+\frac{1}{2}}$ is the angle between $r^{(k+\frac{1}{2})}$ and $G_1 r^{(k+\frac{1}{2})}$. The rest of proof is similar to that of Theorem 2.2 and we omit it. □

Next, we will solve $Wt = r^{(k)}$ and $(\alpha V + W)t = r^{(k+\frac{1}{2})}$ systems by iterative method and then give the inexact version of the MRPPNHSS method in Algorithm 4.

Algorithm 4 The IMRPPNHSS method

1. Input: A, b, positive constants α and ω, the stop tolerance ε, evaluate $\bar{r}^{(0)} = b$, initial estimate $\bar{x}^{(0)}$, τ_k and υ_k, where $k = 1,\ 2\ldots$.
2. Output: Approximate $\bar{x}^{(k)}$ solving $Ax = b$.
3. **for** $k = 0, 1, 2...$ until satisfying the stopping criteria $\frac{||b-A\bar{x}^{(k)}||}{||b||} \leq \varepsilon$
   (a) Solve $(\omega W + T)g^{(k+\frac{1}{2})} = \bar{r}^{(k)}$ inexactly by using CG(PCG) method until the approximate solution $g^{(k+\frac{1}{2})}$ satisfying $\|u^{(k)}\| \leq \tau_k\|\bar{r}^{(k)}\|$, where $u^{(k)} = (\omega W + T)g^{(k+\frac{1}{2})} - \bar{r}^{(k)}$;
   (b) Compute $h^{(k+\frac{1}{2})} = Ag^{(k+\frac{1}{2})}$;
   (c) Compute $\bar{\lambda}_k' = \frac{(\bar{r}^{(k)}, h^{(k+\frac{1}{2})})}{\|h^{(k+\frac{1}{2})}\|^2}$;
   (d) Set $\bar{x}^{(k+\frac{1}{2})} \leftarrow \bar{x}^{(k)} + \bar{\lambda}_k' h^{(k+\frac{1}{2})}$ and evaluate $\bar{r}^{(k+\frac{1}{2})} = b - A\bar{x}^{(k+\frac{1}{2})}$;
   (e) Solve $(\alpha V + \omega W + T)g^{(k+1)} = \bar{r}^{(k+\frac{1}{2})}$ inexactly by using CG(PCG) method until the approximate solution $g^{(k+1)}$ satisfying $\|u^{(k+\frac{1}{2})}\| \leq \upsilon_k\|\bar{r}^{(k+\frac{1}{2})}\|$, where $u^{(k+\frac{1}{2})} = (\alpha V + \omega W + T)g^{(k+1)} - \bar{r}^{(k+\frac{1}{2})}$;
   (f) Compute $h^{(k+1)} = Ag^{(k+1)}$;
   (g) Compute $\bar{\theta}_k' = \frac{(\bar{r}^{(k)}, h^{(k+1)})}{\|h^{(k+1)}\|^2}$;
   (h) Set $\bar{x}^{(k+1)} \leftarrow \bar{x}^{(k+\frac{1}{2})} + \bar{\theta}_k' h^{(k+1)}$ and evaluate $\bar{r}^{(k+1)} = b - A\bar{x}^{(k+1)}$;
   (i) $k = k + 1$.
4. **end for**.

Next, the convergence of IMRPPNHSS method is given as follows.

Theorem 3.3. *Let the assumptions of Theorem 2.2 hold. If* $\{\bar{x}^{(k)}\}$ *is an iterative sequence obtained from the* IMRPPNHSS *iteration method, then it holds that*

$$\|\bar{r}^{(k+1)}\| \leq \Gamma(\tau_k, \upsilon_k)\|\bar{r}^{(k)}\|,$$

where

$$\begin{aligned}&\Gamma(\tau_k, \upsilon_k)\\ =&(\|I - \theta_k(\omega - i)G_2\|\|I - \beta_k(\omega - i)G_1\| + \upsilon_k\|\theta_k(\omega - i)G_2\|\|I - \lambda_k(\omega - i)G_1\|\\ &+ \tau_k\|I - \theta_k(\omega - i)G_2\|\|\lambda_k(\omega - i)G_1\| + \upsilon_k\tau_k\|\theta_k(\omega - i)G_2\|\|\lambda_k(\omega - i)G_1\|)\|\bar{r}^{(k)}\|.\end{aligned}$$

Furthermore, if $\tau_{\max} = \max\{\tau_k\}$ *and* $\upsilon_{\max} = \max\{\upsilon_k\}$ *satisfy*

$$\Gamma(\tau_k, \upsilon_k) < 1,$$

then the sequence $\{\bar{x}^{(k)}\}$ *converges to the exact solution.*

Proof. Similar to the proof of Theorem 2.3, we first obtain the relation between $\|\bar{r}^{(k+1)}\|$ and $\|\bar{r}^{(k+\frac{1}{2})}\|$:

$$\|\bar{r}^{(k+1)}\| = \|\bar{r}^{(k+\frac{1}{2})} - \frac{(\bar{r}^{(k+\frac{1}{2})}, (\omega - i)G_2 g^{(k+1)})}{\|(\omega - i)G_2 g^{(k+1)}\|^2}(\omega - i)Ag^{(k+1)}\|$$

$$
\begin{aligned}
&\leq \|\bar{r}^{(k+\frac{1}{2})} - \frac{(\bar{r}^{(k+\frac{1}{2})}, (\omega-i)G_2\bar{r}^{(k+\frac{1}{2})})}{\|(\omega-i)G_2\bar{r}^{(k+\frac{1}{2})}\|^2}(\omega-i)Ag^{(k+1)}\| \\
&= \|\bar{r}^{(k+\frac{1}{2})} - \frac{(\bar{r}^{(k+\frac{1}{2})}, (\omega-i)G_2\bar{r}^{(k+\frac{1}{2})})}{\|(\omega-i)G_2\bar{r}^{(k+\frac{1}{2})}\|^2}(\omega-i)G_2(\bar{r}^{(k+\frac{1}{2})} + u^{(k+\frac{1}{2})})\| \\
&\leq \|\bar{r}^{(k+\frac{1}{2})} - \theta_k(\omega-i)G_2\bar{r}^{(k+\frac{1}{2})}\| + \|\theta_k(\omega-i)G_2u^{(k+\frac{1}{2})}\| \\
&\leq \|I - \theta_k(\omega-i)G_2\|\|\bar{r}^{(k+\frac{1}{2})}\| + \|\theta_k(\omega-i)G_2\|\|u^{(k+\frac{1}{2})}\| \\
&\leq \|I - \theta_k(\omega-i)G_2\|\|\bar{r}^{(k+\frac{1}{2})}\| + \upsilon_k\|\theta_k(\omega-i)G_2\|\|\bar{r}^{(k+\frac{1}{2})}\| \\
&= (\|I - \theta_kG_2\| + \upsilon_k\|\theta_kG_2\|)\|\bar{r}^{(k+\frac{1}{2})}\|.
\end{aligned}
$$

The rest proof is similar to Theorem 2.3, so we omit it here. □

Remark 3.2. In fact, the IMRPPNHSS iteration method is going to be very close to the MRPPNHSS iteration method when $\tau_k \to 0$ and $\upsilon_k \to 0 (k \to +\infty)$. Thus, the IMRPPNHSS iteration method is sure to be convergent when $\tau_{\max}$ and $\upsilon_{\max}$ are small enough.

4. Numerical experiments

In this section, we will provide some numerical experiments on the sparse complex symmetric linear system (1.1) with different examples for comparing our proposed methods, including the MRP*NHSS and the MRPPNHSS iteration methods, with the following methods:

- The NHSS method [10].
- The P*NHSS and PPNHSS methods [11].
- The MRMHSS method [23].

In addition, the results of the inexact versions of these methods are also given.

We show the number of iteration steps (denoted as 'IT') and the computing time (denoted as 'CPU') in seconds of the above iterative algorithms for solving different linear systems. All experiments were performed by using MATLAB (R2018b) on a personal computer with 2.00 GHz central processing unit (Intel(R) Core(TM) i5-1038NG7 CPU), 16.00GB memory, and Windows operating system (Windows 10).

We compute $x^{(k+1)}$ in the system $Bx^{(k+1)} = e^{(k)}$ by using $x^{(k+1)} = B \setminus e^{(k)}$ for the sparse matrix B and vector $e^{(k)}$. Besides, for all tested inexact iteration methods, the subsystems are solved by the PCG method with tolerance $\eta_k = \delta_k = \tau_k = \upsilon_k = 10^{-3}(k = 1, 2\ldots)$, for which are preconditioned by the corresponding modified incomplete Cholesky factorization, and the related command of computing preconditioner is $ichol(\cdot, struct(type, ict, droptol, 1e{-}3, michol, on))$. The CPU time is recorded by the command "tic-toc". In our experiments, we always take the prescribed symmetric positive definite matrix $V = W$ for P*NHSS, PPNHSS, MRP*NHSS and MRPPNHSS iteration methods. The initial value is always chosen to be zero vector and the stopping criteria for all the methods are

$$RES = \frac{||b - Ax^{(k)}||}{||b||} \leq 10^{-6}$$

Table 1. The numerical results of Example 4.1 for different iteration methods when $(\varpi, \mu) = (0.01, 5)$.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
P*NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
PPNHSS	α_*	2.2	2.0	3.4	2.4	2.1
	ω_*	1	1	1	1	1
	IT	14	14	14	14	14
	CPU	0.0050	0.0225	0.1062	0.6370	3.0805
MRMHSS	α_*	0.1	0.2	0.2	0.4	0.6
	IT	1	1	1	1	1
	CPU	0.0067	0.0026	0.0088	0.0409	0.2070
MRP*NHSS	α_*	7.7	6.7	9.7	8.3	9.1
	IT	1	1	1	1	1
	CPU	0.0001	0.0015	0.0072	0.0401	0.1904
MRPPNHSS	α_*	6.5	9.8	3.4	1.1	0.7
	ω_*	20	15	13	18	15
	IT	1	1	1	1	1
	CPU	0.0001	0.0015	0.0078	0.0410	0.1958

and we use "-" to denote the case where the IT of the algorithms does not converge within 500 iterations.

For the choice of the parameter values of our test methods, we use the experimentally found optimal ones, which lead to the least number of iteration steps. If the optimal parameter values form an interval, then we use the one that belongs to this interval and leads to the least CPU time.

Example 4.1. (see [25, 26]) Consider the following complex symmetric linear system:

$$[(K - \varpi^2 M) + i(C_H + \varpi C_V)]x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ϖ is the driving circular frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = 1/(m + 1)$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set the right-hand side vector b to be

$$b_j = \frac{(1 + i)j}{h^2(j + 1)^2}, \ j = 1, 2, \ldots, n.$$

Furthermore, we normalize the system by multiplying both sides through by h^2.

Example 4.2. (see [25, 26]) Consider the following complex Helmholtz equation:

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1, σ_2 are real coefficient functions, and u satisfies Dirichlet boundary conditions in the square $D = [0,1] \times [0,1]$. We discretize this complex Helmholtz equation with finite differences on an $m \times m$ grid with mesh size $h = 1/(m+1)$. Therefore, we obtain a system of linear equations

$$[(K + \sigma_1 I) + i\sigma_2 I]x = b,$$

where the matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes V_m + V_m \otimes I \ with \ V_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In fact, K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$. In our tests, we take the right-hand side vector $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. In addition, we normalize the system by multiplying both sides by h^2.

Example 4.3. (see [7, 23]) Consider the complex symmetric linear systems of the form (1.1):

$$[(K + \frac{3-\sqrt{3}}{\tau} I) + i(K + \frac{3+\sqrt{3}}{\tau} I)x = b$$

where τ is the time step-size and K is the matrix of a standard five point centered difference formula, approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions on an uniform mesh in the two dimensional unit square $[0,1] \times [0,1]$, and we set $h = 1/(m+1)$, $\tau = h$, where m is a number of inner grid-points in one direction. Then, we can know that the matrix $K \in R^{n\times n}$ is with the tensor product form

$$K = I \otimes V_m + V_m \otimes I \ with \ B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

What's more, it is obvious that K is an $n \times n$ block tridiagonal matrix and $n = m^2$. In our tests, we take $W = K + \frac{3-\sqrt{3}}{\tau} I$ and $T = K + \frac{3+\sqrt{3}}{\tau} I$, and the right-hand side vector b with the jth entry b_j being the form as follows

$$b_j = \frac{(1-i)j}{h^2(j+1)^2}, \ j = 1, 2, \ldots, n.$$

In addition, we normalize the system by multiplying two sides by h^2. And for other details, we can refer to [23].

Table 2. The numerical results of Example 4.1 for different iteration methods when $(\varpi, \mu) = (0.01, 5)$.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
INHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
IP*NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
IPPNHSS	α_*	1.8	1.5	0.9	0.7	0.1
	ω_*	20	20	20	20	20
	IT	364	360	360	358	358
	CPU	0.1481	0.2619	0.9465	4.5885	25.6811
IMRMHSS	α_*	5.3	9.5	0.2	5.5	8.7
	IT	2	2	2	3	3
	CPU	0.0012	0.0021	0.0244	0.0512	0.2649
IMRP*NHSS	α_*	4.9	8.1	5.9	7.0	6.0
	IT	2	3	2	3	3
	CPU	0.0011	0.0030	0.0081	0.0511	0.2752
IMRPPNHSS	α_*	5.2	8.5	2.2	9.8	9.3
	ω_*	13	20	9	13	14
	IT	2	2	2	2	3
	CPU	0.0011	0.0031	0.0084	0.0569	0.2915

Table 3. The numerical results of Example 4.2 for different iteration methods when $(\sigma_1, \sigma_2) = (1, 10)$.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
NHSS	α_*	1.8	5.8	6.7	5.7	4.7
	IT	8	7	7	6	5
	CPU	0.0023	0.0091	0.0488	0.2358	0.5931
P*NHSS	α_*	0.4	0.3	0.4	0.6	0.2
	IT	8	7	7	6	5
	CPU	0.0026	0.0110	0.0494	0.2412	0.9612
PPNHSS	α_*	0.8	2.5	3.7	1.0	3.2
	ω_*	4	7	10	7	10
	IT	5	5	5	4	4
	CPU	0.0020	0.0080	0.0384	0.1930	0.9332
MRMHSS	α_*	2.3	4.9	1.8	1.4	1.6
	IT	6	7	7	6	6
	CPU	0.0012	0.0056	0.0262	0.1270	0.6446
MRP*NHSS	α_*	2.8	8.8	2.4	9.7	3.5
	IT	4	4	4	3	3
	CPU	0.0013	0.0061	0.0301	0.1272	0.6237
MRPPNHSS	α_*	6	1.5	6.6	9.2	1.2
	ω_*	19	18	20	17	19
	IT	4	4	4	3	3
	CPU	0.0013	0.0060	0.0293	0.1225	0.6027

Table 4. The numerical results of Example 4.2 for different iteration methods when $(\sigma_1, \sigma_2) = (1, 10)$.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
INHSS	α_*	4.8	4.4	10	4.0	7.3
	IT	8	7	7	6	6
	CPU	0.0041	0.0060	0.0223	0.1149	0.6846
IP*NHSS	α_*	0.4	0.2	0.4	0.4	0.6
	IT	8	7	7	6	6
	CPU	0.0054	0.0137	0.0270	0.1277	0.7554
IPPNHSS	α_*	0.2	0.4	0.3	0.1	0.3
	ω_*	1	1	1	1	1
	IT	8	8	7	6	6
	CPU	0.0066	0.0074	0.0285	0.1892	0.6123
IMRMHSS	α_*	2.9	8.4	3.7	10.0	3.3
	IT	5	4	4	4	4
	CPU	0.0042	0.0078	0.0360	0.2157	2.0062
IMRP*NHSS	α_*	9.0	6.0	3.6	6.9	3.3
	IT	4	4	4	4	4
	CPU	0.0021	0.0038	0.0159	0.0798	0.4777
IMRPPNHSS	α_*	9.9	9.6	6.9	7.4	5.0
	ω_*	15	17	10	3	2
	IT	4	4	4	4	4
	CPU	0.0022	0.0038	0.0162	0.0833	0.4817

Table 5. The numerical results of Example 4.3 for different iteration methods.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
P*NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
PPNHSS	α_*	0.7	1.5	1.1	1.5	0.9
	ω_*	1	1	1	1	1
	IT	7	8	8	8	8
	CPU	0.0027	0.0130	0.0634	0.3547	1.7075
MRMHSS	α_*	7.1	5.0	8.9	9.4	7.2
	IT	4	5	5	5	5
	CPU	0.0013	0.0078	0.0366	0.2199	1.0838
MRP*NHSS	α_*	1.5	9.9	4.3	3.6	0.9
	IT	6	7	8	8	8
	CPU	0.0019	0.0106	0.0596	0.3552	1.5589
MRPPNHSS	α_*	9.5	4.9	7.0	1.0	1.7
	ω_*	1	3	2	2	1
	IT	4	5	5	5	4
	CPU	0.0013	0.0075	0.0370	0.2130	0.8385

Table 6. The numerical results of Example 4.3 for different iteration methods.

Method		*Grid*				
		16×16	32×32	64×64	128×128	256×256
INHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
IP*NHSS	α_*	-	-	-	-	-
	IT	-	-	-	-	-
	CPU	-	-	-	-	-
IPPNHSS	α_*	0.6	0.4	1.1	0.6	0.1
	ω_*	4	6	7	8	10
	IT	80	101	118	128	129
	CPU	0.0377	0.0693	0.2085	0.7160	3.0958
IMRMHSS	α_*	5.9	6.4	8.0	7.6	3.3
	IT	6	7	8	8	8
	CPU	0.0031	0.0057	0.0213	0.0757	0.3579
IMRP*NHSS	α_*	6.8	9.5	9.8	2.8	8.0
	IT	6	8	8	8	8
	CPU	0.0028	0.0054	0.0175	0.0599	0.2932
IMRPPNHSS	α_*	2.4	8.3	3.9	9.6	2.4
	ω_*	1	3	2	2	3
	IT	4	5	5	5	5
	CPU	0.0025	0.0037	0.0141	0.0464	0.2259

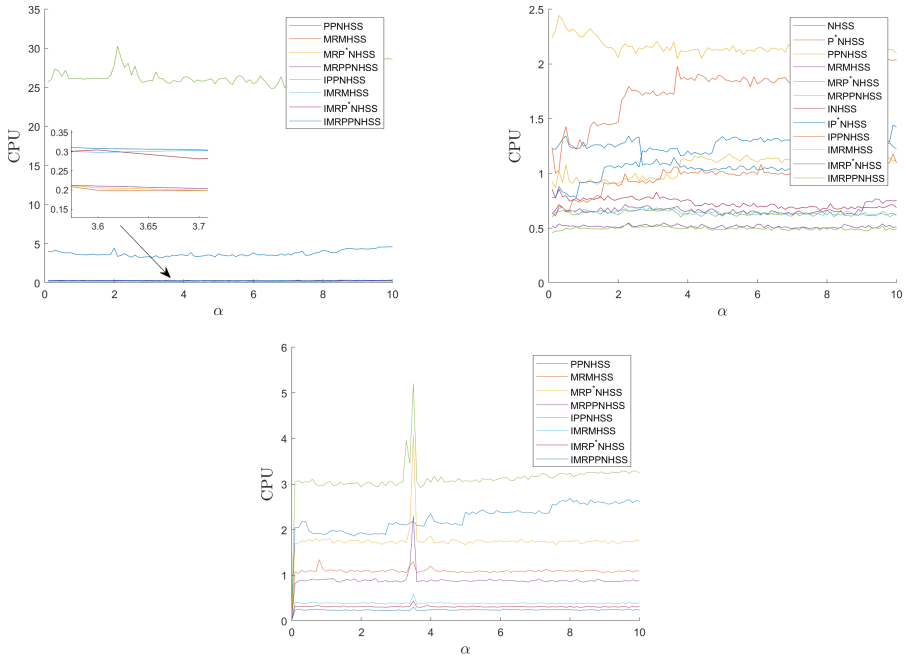

Figure 1. CPU of the convergent iteration methods with respect to the values of iteration parameter α when $m = 256$.

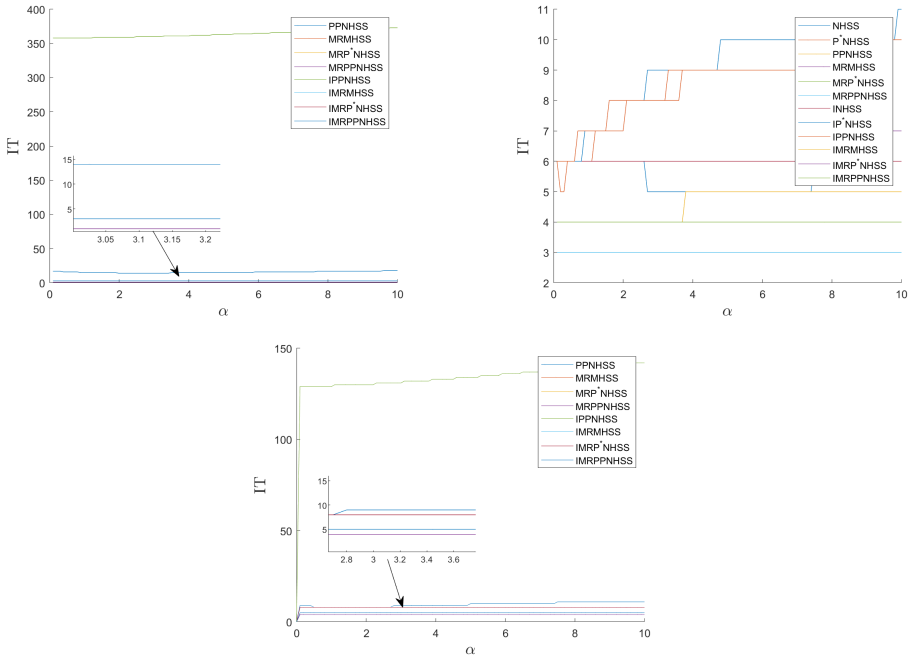

Figure 2. IT of the convergent iteration methods with respect to the values of iteration parameter α when $m = 256$.

We show the results of our numerical examples under the exact version of these five methods in Tables 1, 3, 5 and the inexact versions in Tables 2, 4, 6. It can be observed that the MRP*NHSS and MRPPNHSS methods are always superior to the NHSS, P*NHSS, PPNHSS and MRMHSS methods in terms of both the IT and CPU time. In the inexact situation, the IMRP*NHSS and IMRPPNHSS methods also have good performance compared with the INHSS, IP*NHSS, IPPNHSS and IMRMHSS methods. Furthermore, even though the P*NHSS and IP*NHSS method are not convergent in the Example 4.1, the MRP*NHSS and IMRP*NHSS methods are still convergent to the exact point. Moreover, the inexact versions perform better in terms of CPU than the exact versions in Examples 4.2 and 4.3, but a little worse in Example 4.1. As we know, the inexact version generally has more IT than the exact version. However, in our experiment, the IT of IMRP*NHSS and IMRPPNHSS methods are a little more than that of MRP*NHSS and MRPPNHSS methods. In addition, for our examples, the IT of our test methods is nearly h-independent.

We plot the CPU and IT of the convergent iterative methods with respect to the value of parameter α in figures 1 and 2 for $m = 256$, respectively. The results show that whether accurate or inaccurate versions, we approach IT is hardly affected by α and CPU only within the scope of the small fluctuations.

Therefore, we can conclude that the MRP*NHSS and MRPPNHSS methods discussed in this work are very efficient and robust when applied to solve the complex symmetric system (1.1).

5. Conclusion

In our work, we obtain MRP*NHSS and MRPPNHSS methods by using the minimum residual technique to the P*NHSS and PPNHSS methods for solving a class of complex symmetric linear systems. In addition, we also give their inexact versions. The convergence properties of our methods show that the MRP*NHSS and MRPPNHSS methods converge unconditionally. Finally, the numerical results illustrate that our methods are all very robust and powerful.

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