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A Fixed Point Results for Multivalued Mappings in Hausdorff Fuzzy $b-$Metric Spaces and Applications

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Abstract In this paper, we are interested in proving a general fixed point theorem for multivalued mappings in fuzzy $b-$metric spaces. The results presented in this paper not only generalize the findings from [23], but also yield additional specific outcomes. We present an application to establish the existence of a solution to the integral equation, demonstrating the significance of our result.

Keywords Fuzyy metric space, fuzzy $b-$metric space, t-norm, fixed point, implicit relation

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1. Introduction

Zadeh [26] first introduced the concept of fuzzy sets in 1965. A fuzzy set M in X is a function with domain X and values in $[0,1]$. Heilpern [16] introduced the notion of fuzzy maps and established some fixed-point theorems for them.

In 1975, Kramosil and Michalek [17] proposed the idea of a fuzzy distance between two elements of a nonempty set, using the concepts of a fuzzy set and a t-norm.

A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions: T is continuous, associative and commutative, $T(a,1) = a$ for all $a \in [0,1]$ and for all $a,b,c,d \in [0,1]$ if $a \le c$ and $b \le d$ then $T(a,b) \le T(c,d)$.

Typical examples of a continuous t-norm are $T_p(a,b) = a.b, T_{min}(a,b) = \min\{a,b\}$ and $T_L(a,b) = \max\{a+b-1,0\}$. George and Veeramani [11] generalized the concept of fuzzy metric spaces introduced by Kramosil and Michalek [17]. Given a non-empty set X, and T is a contnuous t-norm, the 3-tuple (X,M,T) is said to be a fuzzy metric space [11], [12] if M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ $t,u > 0$:

$$\begin{cases} 1)\ M(x,y,t) > 0, \\ 2)\ M(x,y,t) = M(y,x,t) = 1 \quad \text{iff} \quad x = y, \\ 3)\ M(x,z,t+u) \ge T(M(x,y,t), M(y,z,u)), \\ 4)\ M(x,y,.) \quad \text{is left continuous function from} \quad (0,\infty) \to [0,1]. \end{cases}$$

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In mathematics, the study of fixed point theory in metric spaces has several applications, especially in solving differential equations. Many authors have studied the new class of generalized metric space, known as b-metric space, introduced by Bakhtin [5] in 1989. For example, see [1]- [4], [6]- [9], [20]. The relationship between b-metric and fuzzy metric spaces is considered in [15]. Conversely, [24] introduced the concept of a fuzzy b-metric space, substituting the triangle inequality with a weaker one.

In this paper, we prove the existence and uniqueness of the fixed point in fuzzy b-metric spaces. We present an application to determine the existence and uniqueness of a solution to an integral equation, demonstrating the significance of our result.

Throughout this paper, $C(X)$ will denote the family of nonempty compact subsets of X. For all $A, B \in C(X)$ and for all $t > 0$, we define a function on $C(X) \times C(X) \times (0, \infty)$ by

$$H_M(A, B, t) = \min\left\{ \inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t) \right\},$$

$$\text{where } M(C, y, t) = \sup_{z \in C} M(z, y, t).$$

The fuzzy b-metric induces H_M, which we refer to as the Hausdorff fuzzy b-metric. The triplet $(C(X), H_M, T)$ is referred to as the Hausdorff fuzzy b-metric space.

We define also $\delta_M(A, B, t)$ as follows:

$$\delta_M(A, B, t) = \inf\{M(a, b, t), \quad a \in A \quad b \in B\}, \quad t > 0.$$

It follows immediately from the definition of δ_M that

$$\delta_M(A, B, t) = 1 \iff A = B = \{.\} \text{ and}$$

$$M(a, b, t) \geq \delta_M(A, B, t) \quad \forall a \in A \quad \forall b \in B, \quad t > 0.$$

2. Preliminary

Definition 2.1 ([24]). A 3-tuple (X, M, T) is called a fuzzy b-metric space if X is an arbitrary nonempty set, T is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the conditions for all $x, y, z \in X$, $t, u > 0$ and a given real number $s \geq 1$:

(b_1) $M(x, y, t) > 0$,

(b_2) $M(x, y, t) = 1$ if and only if $x = y$,

(b_3) $M(x, y, t) = M(y, x, t)$,

(b_4) $M(x, z, s(t+u)) \geq T(M(x, y, t), M(y, z, u))$,

(b_5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Remark 2.1. In this paper we will further use a fuzzy b-metric space in the sense of definition 2.1 with an additional condition $\lim_{t \to \infty} M(x, y, t) = 1$.

Note that every fuzzy metric space is a fuzzy b-metric space with $s = 1$. However, the following example illustrates that the converse need not hold true.

Example 2.1 ([10]). Let $M(x,y,t) = e^{-\frac{|x-y|^p}{t}}$, where $p > 1$ is a real number, and $T(a,b) = a.b$ for all $a, b \in [0,1]$. Then (X, M, T) is a fuzzy b-metric space with $s = 2^{p-1}$.

Definition 2.2 ([24]). We say that a t-norm T is of H-type if the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$, that is,

$$\forall\, \varepsilon \in (0,1)\, \exists \alpha \in (0,1): \, x > 1-\alpha \Rightarrow T_n(x) > 1-\varepsilon, \forall\, n \in \mathbb{N},$$
$$\text{where } T_1(x) = T(x,x), \quad T_{n+1}(x) = T(T_n x, x), \text{ for every } n \geq 1.$$

The t-norm T_{min} is a trivial example of t-norm of H-type.

Proposition 2.1 ([23]). *Let (x_n) be a sequence in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$, and let T be a t-norm of H-type. Then $\lim_{n\to\infty} T_{i=n}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{\infty} x_{n+1} = 1$,*

$$\text{where } T_{i=1}^{1} x_i = x_1, \quad T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, ..., x_n).$$

Definition 2.3 ([10]). A function $f : \mathbb{R} \to \mathbb{R}$ is called s-nondecreasing, if $x > sy$ implies $fx \geq fy$ for each $x, y \in \mathbb{R}$.

Lemma 2.1 ([10]). *Let (X, M, T) be a fuzzy b-metric space with constant s. Then $M(x,y,t)$ is $s-$nondecreasing with respect to t, for all $x, y \in X$. Also,*

$$M(x,y,s^n t) \geq M(x,y,t), \quad \forall n \in \mathbb{N}.$$

Definition 2.4 ([24], [25]). Let (X, M, T) be a fuzzy b-metric space.

(i) A sequence (x_n) converges to x if $M(x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence (x_n) is called a Cauchy sequence if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

(iii) The fuzzy b-metric space (X, M, T) is said to be complete if every Cauchy sequence is convergent.

(iv) A subset $A \subset X$ is said to be closed if every sequence $x_n \in A$ such that $x_n \longrightarrow x$ we have $x \in A$.

(v) A subset $A \subset X$ is said to be compact if every sequence $x_n \in A$ has a convergent subsequence.

Lemma 2.2 ([24], [25]). *In a fuzzy $b-$metric space (X, M, T) we have*

(i) If a sequence (x_n) in X converges to x, then x is unique.

(ii) If a sequence (x_n) in X converges to x, then it is a Cauchy sequence.

Proposition 2.2 ([25] Prop 1.10). *Let (x_n) be a sequence in a fuzzy b-metric space (X, M, T) with constant s, and suppose that (x_n) converges to x. Then we have*

$$M\left(x, y, \frac{t}{s}\right) \leq \lim\sup_{n\to\infty} M(x_n, y, t) \leq M(x, y, st),$$
$$M\left(x, y, \frac{t}{s}\right) \leq \lim\inf_{n\to\infty} M(x_n, y, t) \leq M(x, y, st).$$

Lemma 2.3 ([23]). *Let (x_n) be a sequence in a fuzzy b-metric space (X, M, T) with constant s. Suppose that there exists $\lambda \in (0, \frac{1}{s})$ such that*

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), \qquad n \in \mathbb{N},\ t > 0,$$

and $\upsilon \in (0,1)$ such that $\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, x_1, \frac{t}{\upsilon^i}) = 1, \quad t > 0$. Then (x_n) is a Cauchy sequence.

Lemma 2.4 ([19]). *In a fuzzy b-metric space (X, M, T), if the function M is continuous with respect to one of its variable, then it is continuous with respect to the other.*

Lemma 2.5 ([19]). *Let (X, M, T) be a fuzzy b-metric space and let $A \subset C(X)$. If M is continuous with respect to one of its variables, then for all $x \in X$, there exists $y_0 \in A$ such that*

$$M(x, A, t) = \sup_{y\in A} M(x, y, t) = M(x, y_0, t), \quad t > 0.$$

Proposition 2.3 ([19]). *Let (X, M, T) be a fuzzy b-metric space with constant s. Then H_M is a fuzzy set on $C(X) \times C(X) \times (0, \infty)$ satisfying the conditions for all $A, B, C \in C(X)$, $t, u > 0$:*

(H_1) $H_M(A, B, t) > 0$,

(H_2) $H_M(A, B, t) = 1$ *if and only if* $A = B$,

(H_3) $H_M(A, C, s(t + u)) \geq T(H_M(A, B, t), H_M(B, C, u))$,

(H_4) $H_M(A, B, .) : (0, \infty) \to [0, 1]$ *is continuous,*

(H_5) $\lim_{t\to\infty} H_M(A, B, t) = 1$ *if and only if* $\lim_{t\to\infty} M(x, y, t) = 1$.

3. Main results

Lemma 3.1. *Let (X, M, T) be a fuzzy b-metric space with constant s.*
For all $k \in (0, 1)$, $x \in X$, $t > 0$ and $A, B \in C(X)$.

(L1) If ,

$$H_M(A, B, kt) \geq H_M(A, B, t), \qquad \text{then } A = B. \tag{3.1}$$

(L2) If ,

$$\delta_M(A, B, kt) \geq \delta_M(A, B, t), \qquad \text{then } A = B = \{.\}. \tag{3.2}$$

(L3) If ,

$$M(A, x, kt) \geq M(A, x, t), \qquad \text{then } x \in A. \tag{3.3}$$

Proof. $(L1)$ By (3.1) we have

$$H_M(A, B, t) \geq H_M\left(A, B, \frac{t}{k^n}\right), \quad n \in \mathbb{N}.$$

Now

$$H_M(A,B,t) \geq \lim_{n\to\infty} H_M\left(A,B,\frac{t}{k^n}\right) = 1.$$

By using the Proposition 2.3 (H_2) it follows that $A = B$.

$(L2)$ Similarly by (3.2) we have

$$\delta_M(A,B,t) \geq \lim_{n\to\infty} \delta_M\left(A,B,\frac{t}{k^n}\right) = 1.$$

Then, $\delta_M(A,B,t) = 1$, hence $A = B = \{.\}$.

$(L3)$ Similarly by (3.3) we have

$$M(A,x,t) \geq \lim_{n\to\infty} M\left(A,x,\frac{t}{k^n}\right) = 1.$$

So, $x \in A$. □

Definition 3.1. Let T be a t-norm, and Φ_T be the set of all continuous functions $\phi_T(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \longrightarrow \mathbb{R}$ such that:
(ϕ_{T1}) : ϕ_T is nondecreasing in variable t_1 and non-increasing in variable t_3,t_4,t_5,t_6.
(ϕ_{T2}) : $\forall u,v,w \in [0,1]$ $\phi_T(u,v,v,w,T(v,w),1) \geq 0 \Longrightarrow \quad u \geq \min\{v,w\}$.
(ϕ_{T3}) : $\forall u,v, \in [0,1]$ $\phi_T(u,1,1,v,v,1) \geq 0$ or $\phi_T(u,v,1,1,v,v) \geq 0 \Longrightarrow \quad u \geq v$.

Example 3.1. $\phi_T(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - t_2$.

Example 3.2. $\phi_T(t_1,t_2,t_3,t_4,t_5,t_6) = \beta(t_2) - \beta(t_1)$, with $\beta : (0,1] \to [0,\infty)$ and $\beta(1) = 0$ is a continuous function strictly decreasing.

Example 3.3. $\phi_T(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4\}$.

Example 3.4. $\phi_T(t_1,t_2,t_3,t_4,t_5,t_6) = 2t_1 - t_3 - t_4 + |t_3 - t_4|$.

Example 3.5. $\phi_{T_{min}}(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4,t_5,t_6\}$.

Example 3.6. $\phi_{T_p}(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4,\sqrt{t_5},\sqrt{t_6}\}$.

Theorem 3.1. *Let (X,M,T) be a complete fuzzy b-metric space with constant s. We suppose that M is continuous with respect to one of its variables, and let $F : X \longrightarrow C(X)$.*

Suppose that there exist $k \in]0,\frac{1}{s}[$, $x_0 \in X$ and $\upsilon \in (0,1)$.
$\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0,x_1,\frac{t}{\upsilon^i}) = 1$, $x_1 \in Fx_0$. *Let $\phi_T \in \Phi_T$ such that:* $\forall x,y \in X, \quad t > 0,$

$$\phi_T\begin{pmatrix} H_M(Fx,Fy,kt), M(x,y,t), M(Fx,x,t), M(Fy,y,t), \\ M(x,Fy,2st), M(Fx,y,t) \end{pmatrix} \geq 0. \tag{3.4}$$

Then F has a fixed point $x \in X$.

Moreover, if x is absolutely fixed for F (which means that $F(x) = \{x\}$), then the fixed point is unique.

Proof. <u>**Existence.**</u> Let $x_0 \in X$, and $x_1 \in Fx_0$. For $x = x_0$ and $y = x_1$ in (3.4) we have:

$$\phi_T\begin{pmatrix} H_M(Fx_0,Fx_1,kt), M(x_0,x_1,t), M(Fx_0,x_0,t), M(Fx_1,x_1,t), \\ M(x_0,Fx_1,2st), M(Fx_0,x_1,t) \end{pmatrix} \geq 0.$$

Since $x_1 \in Fx_0$, then

$$M(Fx_0, x_0, t) \geq M(x_1, x_0, t) \text{ and } M(Fx_1, x_1, t) \geq H_M(Fx_1, Fx_0, t), \quad \forall t > 0.$$

According to (ϕ_{T1}) we have

$$\phi_T \begin{pmatrix} H_M\left(Fx_0, Fx_1, kt\right), M\left(x_0, x_1, t\right), M\left(x_1, x_0, t\right), H_M\left(Fx_0, Fx_1, t\right), \\ T(M\left(x_0, x_1, t\right), M\left(x_1, Fx_1, t\right)), 1 \end{pmatrix} \geq 0.$$

This implies

$$\phi_T \begin{pmatrix} H_M\left(Fx_0, Fx_1, kt\right), M\left(x_0, x_1, t\right), M\left(x_0, x_1, t\right), H_M\left(Fx_0, Fx_1, t\right), \\ T(M\left(x_0, x_1, t\right), H_M\left(Fx_0, Fx_1, t\right)), 1 \end{pmatrix} \geq 0.$$

By (ϕ_{T2}) we have

$$H_M\left(Fx_0, Fx_1, kt\right) \geq \min\left\{M\left(x_0, x_1, t\right), H_M\left(Fx_0, Fx_1, t\right)\right\}.$$

If $H_M\left(Fx_0, Fx_1, kt\right) \geq H_M\left(Fx_0, Fx_1, t\right),\ t > 0$, then by Lemma 3.1 it follows that $x_1 \in Fx_0 = Fx_1$. So,

$$H_M\left(Fx_0, Fx_1, kt\right) \geq M\left(x_0, x_1, t\right), \qquad t > 0.$$

Since $x_1 \in Fx_0$, we get

$$M\left(x_1, Fx_1, kt\right) \geq H_M\left(Fx_0, Fx_1, kt\right) \geq M\left(x_0, x_1, t\right).$$

By Lemma 2.5 there exists $x_2 \in Fx_1$ such that:

$$M\left(x_1, x_2, kt\right) = M\left(x_1, Fx_1, kt\right) \geq M\left(x_0, x_1, t\right).$$

By recurrence, we construct a sequence (x_n) such that $x_{n+1} \in Fx_n$, which satisfies:

$$M\left(x_{n+1}, x_n, kt\right) \geq M\left(x_n, x_{n-1}, t\right), \quad n \in \mathbb{N}^*,\ t > 0.$$

By Lemma 2.3, (x_n) is a Cauchy sequence in X. Since (X,M,T) is complete, hence there exists $x \in X$ such that $\lim\limits_{n\to\infty} M(x_n, x, t) = 1, \quad t > 0.$

Next we show that $x \in Fx$, indeed, by (3.4) we have:

$$\phi_T \begin{pmatrix} H_M\left(Fx_n, Fx, kt\right), M\left(x_n, x, t\right), M\left(Fx_n, x, t\right), M\left(Fx, x, t\right), \\ M\left(x_n, Fx, 2st\right), M\left(Fx_n, x, t\right) \end{pmatrix} \geq 0.$$

According to (ϕ_{T1}) we have

$$\phi_T \begin{pmatrix} M\left(x_{n+1}, Fx, kt\right), M\left(x_n, x, t\right), M\left(x_{n+1}, x, t\right), M\left(Fx, x, t\right), \\ T(M\left(x_n, x, t\right), M\left(x, Fx, t\right)), M\left(x_{n+1}, x, t\right) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$,

$$\phi_T\left(M\left(x, Fx, kt\right), 1, 1, M\left(Fx, x, t\right), M\left(x, Fx, t\right), 1\right) \geq 0.$$

By (ϕ_{T3}) we get

$$M\left(Fx,x,kt\right)\geq M\left(Fx,x,t\right),\quad t>0.$$

By Lemma 3.1, $x\in Fx$.
Unicity. Suppose that $y\in X$ is another fixed point of f, then by (3.4) we have

$$\phi_T\left(\begin{array}{c}H_M\left(Fx,Fy,kt\right),M\left(x,y,t\right),M\left(Fx,x,t\right),M\left(Fy,y,t\right),\\ M\left(x,Fy,2st\right),M\left(Fx,y,t\right)\end{array}\right)\geq 0$$
$$\Rightarrow\phi_T\left(\begin{array}{c}H_M\left(\{x\},Fy,kt\right),M\left(x,y,t\right),M\left(x,x,t\right),M\left(Fy,y,t\right),\\ T(M\left(x,y,t\right),M\left(y,Fy,t\right)),M\left(x,y,t\right)\end{array}\right)\geq 0.$$

Since $H_M\left(\{x\},Fy,kt\right)=\inf\limits_{z\in Fy}M(\{x\},z,kt)\leq M(x,y,kt)$, by (ϕ_{T1}) we get

$$\phi_T\Big(M\left(x,y,kt\right),M\left(x,y,t\right),1,1,M\left(x,y,t\right),M\left(x,y,t\right)\Big)\geq 0.$$

By (ϕ_{T3}) we get

$$M\left(x,y,kt\right)\geq M\left(x,y,t\right),\quad t>0.$$

By Lemma 3.1, $x=y$. □

Theorem 3.2. *Let (X,M,T) be a complete fuzzy b-metric space with constant s and let $F:X\longrightarrow CL\left(X\right)$.*

Suppose that there exist $k\in]0,\frac{1}{s}[$, $x_0\in X$ and $\upsilon\in(0,1)$, $\lim\limits_{n\to\infty}T_{i=n}^{\infty}M(x_0,x_1,\frac{t}{\upsilon^i})=1$, $x_1\in Fx_0$. Let $\phi_T\in\Phi_T$ such that: $\forall x,y\in X,\quad t>0$,

$$\phi_T\left(\begin{array}{c}\delta_M\left(Fx,Fy,kt\right),M\left(x,y,t\right),M\left(Fx,x,t\right),M\left(Fy,y,t\right),\\ M\left(x,Fy,2st\right),M\left(Fx,y,t\right)\end{array}\right)\geq 0.\qquad(3.5)$$

Then F has a fixed point $x\in X$.

Moreover, if x is absolutely fixed for F (which means that $F(x)=\{x\}$), then the fixed point is unique.

$CL(X)$ will denote the family of nonempty closed subsets of X.
Proof. **Existence.** Let $x_0\in X$, and define the sequence (x_n) of elements from X such that: $x_{n+1}\in Fx_n$ for every $n\in\mathbb{N}$.

According to (3.5), with $x=x_{n-1}$ and $y=x_n$ we have

$$\phi_T\left(\begin{array}{c}\delta_M\left(Fx_{n-1},Fx_n,kt\right),M\left(x_{n-1},x_n,t\right),M\left(Fx_{n-1},x_{n-1},t\right),M\left(Fx_n,x_n,t\right),\\ M\left(x_{n-1},Fx_n,2st\right),M\left(Fx_{n-1},x_n,t\right)\end{array}\right)\geq 0.$$

Since $x_n\in Fx_{n-1}$ and $x_{n+1}\in Fx_n$ we get

$$M\left(Fx_{n-1},x_{n-1},t\right)\geq M\left(x_n,x_{n-1},t\right),\quad M\left(Fx_n,x_n,t\right)\geq M\left(x_{n+1},x_n,t\right)$$
$$\text{and }\delta_M\left(Fx_{n-1},Fx_n,kt\right)\leq M\left(x_n,x_{n+1},kt\right).$$

By (ϕ_{T1}) we have

$$\phi_T\begin{pmatrix} M(x_n,x_{n+1},kt), M(x_{n-1},x_n,t), M(x_n,x_{n-1},t), M(x_{n+1},x_n,t), \\ T(M(x_{n-1},x_n,t), M(x_n,Fx_n,t)), M(x_n,x_n,t) \end{pmatrix} \geq 0$$

$$\Rightarrow \phi_T\begin{pmatrix} M(x_n,x_{n+1},kt), M(x_{n-1},x_n,t), M(x_n,x_{n-1},t), M(x_{n+1},x_n,t), \\ T(M(x_{n-1},x_n,t), M(x_n,x_{n+1},t)), 1 \end{pmatrix} \geq 0.$$

According to (ϕ_{T2}) we get

$$M(x_n,x_{n+1},kt) \geq \min\{M(x_{n-1},x_n,t), M(x_n,x_{n+1},t)\}.$$

If $M(x_n,x_{n+1},kt) \geq M(x_n,x_{n+1},t)$ then by Lemma 3.1 it follows that $x_n = x_{n+1} \in Fx_n$. So

$$M(x_n,x_{n+1},kt) \geq M(x_{n-1},x_n,t), \quad n \in \mathbb{N}^*,\ t>0.$$

By lemma 2.3, (x_n) is a Cauchy sequence in X. Since (X,M,T) is complete, hence there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n,x,t) = 1, \quad t>0$. Next we show that $x \in Fx$, indeed, by (3.5) we have:

$$\phi_T\begin{pmatrix} \delta_M(Fx_n,Fx,kt), M(x_n,x,t), M(Fx_n,x,t), M(Fx,x,t), \\ M(x_n,Fx,2st), M(Fx_n,x,t) \end{pmatrix} \geq 0.$$

According to (ϕ_{T1}) we have

$$\phi_T\begin{pmatrix} M(x_{n+1},Fx,kt), M(x_n,x,t), M(x_{n+1},x,t), M(Fx,x,t), \\ T(M(x_n,x,t), M(x,Fx,t)), M(x_{n+1},x,t) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$,

$$\phi_T\left(\limsup_{n\to\infty} M(x_{n+1},Fx,kt), 1, 1, M(Fx,x,t), M(x,Fx,t), 1\right) \geq 0.$$

By (ϕ_{T3}) we get

$$\limsup_{n\to\infty} M(x_{n+1},Fx,kt) \geq M(Fx,x,t), \quad t>0.$$

By proposition 2.2 we get

$$\limsup_{n\to\infty} M(x_{n+1},Fx,kt) \leq M(Fx,x,skt), \quad t>0.$$

Then

$$M(Fx,x,\alpha t) \geq M(Fx,x,t), \quad \alpha = sk \in (0,1), \quad t>0.$$

By Lemma 3.1, $x \in Fx$.
Unicity. Suppose that $y \in X$ is another fixed point of f, with by (3.5) we have

$$\phi_T\begin{pmatrix} \delta_M(Fx,Fy,kt), M(x,y,t), M(Fx,x,t), M(Fy,y,t), \\ M(x,Fy,2st), M(Fx,y,t) \end{pmatrix} \geq 0$$

$$\Rightarrow \phi_T\Big(M(x,y,kt), M(x,y,t), 1, 1, M(x,y,t), M(x,y,t)\Big) \geq 0.$$

By (ϕ_{T3}) we get

$$M(x,y,kt) \geq M(x,y,t), \quad t > 0.$$

By Lemma 3.1, $x = y$. □

As a consequence of Theorem 3.2, if $F = \Im$ is single-valued mapping, then we obtain the following.

Corollary 3.1. *Let (X, M, T) be a complete fuzzy b-metric space with constant s, and let $\Im : X \longrightarrow X$. Suppose that there exist $k \in (0, \frac{1}{s})$, $x_0 \in X$ and $\upsilon \in (0,1)$ such that $\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, \Im x_0, \frac{t}{\upsilon^i}) = 1$. Let $\phi_T \in \Phi_T$, for all $x, y \in X$, $t > 0$, such that*

$$\phi_T\begin{pmatrix} M(\Im x, \Im y, kt), M(x,y,t), M(\Im x, x, t), M(\Im y, y, t), \\ M(x, \Im y, 2st), M(\Im x, y, t)\end{pmatrix} \geq 0. \qquad (3.6)$$

Then $\Im$ has a unique fixed point $x \in X$.

Example 3.7. Let X be the subset of $\mathbb{R}^3$ defined by $X = \{A, B, C, D\}$, where $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ and $D = (2,2,2)$. $T(c,d) = c.d$ for all $c, d \in [0,1]$ and (X, M, T) is a complete fuzzy $b-$metric space such that:

$$M(x,y,t) = e^{\frac{-d(x,y)}{t}}, \quad x, y \in X,\ t > 0,$$

where $d(x,y)$ denotes the Euclidean distance of $\mathbb{R}^3$.

Let $\Im : X \to X$ be given by $\Im(A) = \Im(B) = \Im(C) = A, \Im(D) = B$.

To show that for all $x, y \in X$, $k \in (\frac{\sqrt{2}}{3}, 1)$.

$$\phi_T\begin{pmatrix} M(\Im x, \Im y, kt), M(x,y,t), M(\Im x, x, t), M(\Im y, y, t) \\ M(x, \Im y, 2t), M(\Im x, y, t)\end{pmatrix} \leq 0,$$

with ϕ_T as in Example 3.2, and $\beta(t) = -ln(t)$.

Indeed: If $x, y \in \{A, B, C\}$ we have $M(\Im x, \Im y, kt) = M(A, A, kt) = 1$.

So $\beta(M(x,y,t)) - \beta(M(\Im x, \Im y, kt)) = \beta(M(x,y,t)) \geq 0$.

If $x \in \{A, B, C\}$ and $y = D$ we find

$$M(\Im x, \Im y, kt) = e^{\frac{-\sqrt{2}}{kt}} \text{ and } M(x,y,t) = e^{\frac{-3}{t}}.$$

So $\beta(M(x,y,t)) - \beta(M(\Im x, \Im y, kt)) = \dfrac{3k - \sqrt{2}}{kt} \geq 0$.

For $x_0 \in X$, we have $\lim_{n\to\infty} M(x_0, \Im x_0, \frac{t}{k^n}) = 1$, then

$\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, \Im x_0, \sqrt{3}^i t) = 1$, with $k = \dfrac{\sqrt{3}}{3}$.

Now, all the hypotheses of Corollary 3.1 are satisfied and thus $\Im$ has a unique fixed point, that is $x = A$.

Remark 3.1. From Corollary 3.1 and Example 3.1 we obtain Theorem 2.4 [23].

Remark 3.2. From Corollary 3.1 and Example 3.3 we obtain Theorem 2.5 [23].

4. Application

Let $X = C([a,b],\mathbb{R})$ be the set of real continuous functions defined on $[a,b]$, and $T(c,d) = c.d$ for all $c,d \in [0,1]$ and let (X,M,T) be a complete fuzzy $b-$metric space with $s = 2$ and fuzzy $b-$metric

$$M(x,y,t) = e^{-\frac{\sup_{u\in[a,b]} |x(u)-y(u)|^2}{t}}, \quad x,y \in X,\ t > 0.$$

Consider the following integral equation

$$x(u) = g(u) + \int_a^b G(u,v)f(v,x(v))dv, \quad u \in [a,b], \tag{4.1}$$

where $f : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [a,b] \longrightarrow \mathbb{R}$ are two continuous functions and $G : [a,b] \times [a,b] \longrightarrow \mathbb{R}^+$ is a function such that $G(u,.) \in L^1([a,b])$ for all $u \in [a,b]$.

Consider the operator F: $X \longrightarrow X$ defined by

$$Fx(u) = g(u) + \int_a^b G(u,v)f(v,x(v))dv, \quad u \in [a,b]. \tag{4.2}$$

Theorem 4.1. *Suppose that the following conditions are satisfied:*
(H_1) *there exists* $\theta : X \times X \longrightarrow \mathbb{R}^+$ *for all* $v \in [a,b]$,

$$| f(v,x(v)) - f(v,y(v)) | \leq \theta(x,y) \ | x(v) - y(v) | \quad \forall x,y \in X,$$

(H_2) *there exists* $\lambda \in [0,\frac{1}{2})$, *such that*

$$\sup_{u\in[a,b]} \int_a^b G(u,v)\theta(x,y)dv \leq \sqrt{\lambda}.$$

Then the integral equation (4.1) *has a unique solution in* X.

Proof. It is clear that any fixed point of (4.2) is a solution of (4.1). By conditions (H_1) and (H_2), we have

$$\begin{aligned}
\sup_{u\in[a,b]} |Fx(u) - Fy(u)|^2 &= \sup_{u\in[a,b]} \left| \int_a^b G(u,v)f(v,x(v))dv - \int_a^b G(u,v)f(v,y(v))dv \right|^2 \\
&= \sup_{u\in[a,b]} \left| \int_a^b G(u,v)[f(v,x(v)) - f(v,y(v))]dv \right|^2 \\
&\leq \sup_{u\in[a,b]} \left(\int_a^b G(u,v)\theta(x,y) \, |x(v) - y(v)| \, dv \right)^2 \\
&\leq \sup_{u\in[a,b]} |x(u) - y(u)|^2 \times \left(\sup_{u\in[a,b]} \int_a^b G(u,v)\theta(x,y))dv \right)^2 \\
&\leq \lambda \sup_{u\in[a,b]} |x(u) - y(u)|^2 .
\end{aligned}$$

This implies

$$e^{-\frac{\sup\limits_{u\in[a,b]}|Fx(u)-Fy(u)|^2}{t}} \geq e^{\frac{-\lambda\sup\limits_{u\in[a,b]}|x(u)-y(u)|^2}{t}}, \quad x,y\in X,\ t>0.$$

Therefore,

$$\begin{aligned} M(Fx,Fy,\lambda t) &\geq M(x,y,t), \\ &\geq \min\{M(x,y,t),M(Fx,x,t),M(Fy,y,t)\} \quad x,y\in X,\ t>0. \end{aligned}$$

For $x_0\in X$, we have $\lim\limits_{n\to\infty} M(x_0,Fx_0,2^nt)=1$, then $\lim\limits_{n\to\infty} T_{i=n}^{\infty}M(x_0,Fx_0,2^it)=1$. Then all conditions of Corollary 3.1 are satisfied with ϕ_T as in Example 3.3. Thus the operator F has a unique fixed point, which means the integral has a unique solution in X. □

Example 4.1. The following integral equation has a solution in $X=(C[1,e],\mathbb{R})$.

$$x(u)=\frac{1}{1+u^2}+\int_1^e \frac{\ln(u.v)}{e^2}x(v)dv, \quad u\in[1,e]. \tag{4.3}$$

Proof. Let F: $X\longrightarrow X$ defined by

$$Fx(u)=\frac{1}{1+u^2}+\int_1^e \frac{\ln(u.v)}{e^2}x(v)dv, \quad u\in[1,e].$$

By specifying $G(u,v)=\dfrac{\ln(u.v)}{e^2}$, $f(v,x)=x$ and $g(t)=\frac{1}{1+u^2}$ in Theorem 4.1, we get :
(1) For all $x(.),y(.)\in X$, it is clear that the condition (H_1) in Theorem 4.1 is satisfied with $\theta=1$.
(2)

$$\begin{aligned} \sup_{u\in[1,e]}\int_1^e \frac{\ln(u.v)}{e^2}dv &= \frac{1}{e^2}\sup_{u\in[1,e]}\int_1^e(\ln(v)+\ln(u))dv \\ &= \frac{1}{e^2}\sup_{u\in[1,e]}[v\ln(v)-v+v\ln(u)]_1^e \\ &= \frac{1}{e^2}\sup_{u\in[1,e]}(\ln(u)(e-1)+1) \\ &= \frac{1}{e}\leq\sqrt{\lambda}, \quad \lambda\in\left(\frac{1}{e^2},\frac{1}{2}\right). \end{aligned}$$

Therefore, all conditions of Theorem 4.1 are satisfied, hence the mapping F has a fixed point in X, which is a solution to equation (4.3). □

5. Conclusion

In this paper, we are interested in proving a fixed point theorem for multivalued mappings in fuzzy $b-$metric spaces. Some examples in this space are presented. Additionally, some new fixed-point results in this space are formulated and proven, which extend the results of [23], and the existence and uniqueness of the fixed point in such a space are demonstrated.

The approach proposed may pave the way for new developments in generalized metrical structures and fixed-point theory.

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