Journal of Nonlinear Modeling and Analysis
Volume 7, Number 3, June 2025, 938–945
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.938

Positive Periodic Solutions for First-Order Nonlinear Neutral Differential Equations

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Abstract We consider a first-order nonlinear neutral differential equation. By employing Krasnoselskii's fixed point theorem, we provide several new criteria for the existence of positive periodic solutions to this equation. The theorems we have formulated are exemplified through a specific example.

Keywords Fixed point, neutral equations, positive periodic solution, first-order

MSC(2010) 34C25, 34K13.

1. Introduction

In this current study, we explore the existence of positive ω-periodic solutions for a first-order neutral differential equation given by

$$(a(t)x(t))' = -b(t)x(t) + c(t)x'(t-h(t)) + f(t, x(t-h(t))), \tag{1.1}$$

where $a \in C^1(\mathbb{R}, (0,\infty))$, $b \in C(\mathbb{R}, (0,\infty))$, $c \in C^1(\mathbb{R}, \mathbb{R})$, $h \in C^2(\mathbb{R}, (0,\infty))$ with $h'(t) \neq 1$ for all $t \in [0, \omega]$, which are ω-periodic functions. Additionally, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is an ω-periodic function in t, and ω is a positive constant.

In fact, neutral differential equations and periodic phenomena appear in different models from real world applications; please see, e.g., [1, 7, 9, 13]. Our investigation builds upon the positive periodic solutions of the equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t-g(t))), \tag{1.2}$$

with $0 \leqslant \frac{c(t)}{1-g'(t)} < 1$, $-1 \leqslant \frac{c(t)}{1-g'(t)} \leqslant 0$, initiated in [18]. Our study extends and generalizes the results from [18] by considering the special case when $a(t) = 1$, leading to the equation above. This indicates that our findings not only encompass, but also offer broader insight compared to those obtained in [18], particularly for the more general equation.

In summary, our study provides generalizations and new criteria for positive periodic solutions in (1.1), complementing existing research in [2, 3, 5, 6, 8, 10–12, 14–17, 19] that explores positive periodic solutions in various types of first-order neutral differential equations. Additionally, the work in [4] focuses on positive periodic solutions to second-order neutral differential equations.

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2. Main results

Consider a function space Φ_ω consisting of ω-periodic continuous functions equipped with the supremum norm, denoted as $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is evident that the pair $(\Phi_\omega, \|\cdot\|)$ forms a Banach space. Let $a_0 = \min_{t\in[0,\omega]} a(t)$ and $a_1 = \max_{t\in[0,\omega]} a(t)$.

Theorem 2.1. *Let $0 \leqslant c_0 \leqslant \frac{c(t)}{a(t)(1-h'(t))} \leqslant c_1 < 1$. Furthermore, assume that there exist positive constants m_0 and m_1 with $m_0 < m_1$ such that*

$$(1-c_0)a_1 m_0 \leqslant \frac{a(t)}{b(t)}\Big(f(t,x) - r(t)x\Big) \leqslant (1-c_1)a_0 m_1 \tag{2.1}$$

for all $t \in [0,\omega]$, $x \in [m_0, m_1]$, where

$$r(t) = \frac{\Big(c'(t) + \frac{b(t)}{a(t)}c(t)\Big)\Big(1-h'(t)\Big) + h''(t)c(t)}{\Big(1-h'(t)\Big)^2}.$$

Then, (1.1) has at least one positive ω-periodic solution $x(t) \in [m_0, m_1]$.

Proof. Clearly, obtaining an ω-periodic solution of (1.1) is equivalent to finding an ω-periodic solution for the following integral equation

$$\begin{aligned} x(t) = \frac{1}{a(t)}\Big[&\frac{c(t)}{1-h'(t)}x(t-h(t)) \\ &+ \int_t^{t+\omega} G(t,s)\Big[f(s, x(s-h(s))) - r(s)x(s-h(s))\Big]ds\Big], \end{aligned}$$

where

$$G(t,s) = \frac{e^{\int_t^s \frac{b(u)}{a(u)}du}}{e^{\int_0^\omega \frac{b(u)}{a(u)}du} - 1}.$$

Consider the set $\Phi = \{x \in \Phi_\omega : m_0 \leqslant x(t) \leqslant m_1,\ t \in [0,\omega]\}$, which forms a bounded closed and convex subset of Φ_ω. Now, define the operators $\mathcal{T}, \mathcal{S} : \Phi \to \Phi_\omega$ as follows:

$$(\mathcal{T}x)(t) = \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) \tag{2.2}$$

and

$$(\mathcal{S}x)(t) = \frac{1}{a(t)}\int_t^{t+\omega} G(t,s)\Big[f(s, x(s-h(s))) - r(s)x(s-h(s))\Big]ds. \tag{2.3}$$

For every $x \in \Phi$ and $t \in \mathbb{R}$, deducing from (2.2) and (2.3), it becomes evident that

$$\begin{aligned} (\mathcal{T}x)(t+\omega) &= \frac{c(t+\omega)}{a(t+\omega)(1-h'(t+\omega))}x(t+\omega-h(t+\omega)) \\ &= \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) \\ &= (\mathcal{T}x)(t) \end{aligned}$$

and

$$\begin{aligned}(\mathcal{S}x)(t+\omega) &= \frac{1}{a(t+\omega)}\int_{t+\omega}^{t+2\omega} G(t+\omega,s)\Big[f(s,x(s-h(s))) - r(s)x(s-h(s))\Big]ds\\ &= \frac{1}{a(t+\omega)}\int_{t}^{t+\omega} G(t+\omega,v+\omega)\Big[f(v+\omega,x(v+\omega-h(v+\omega)))\\ &- r(v+\omega)x(v+\omega-h(v+\omega))\Big]dv\\ &= \frac{1}{a(t)}\int_{t}^{t+\omega} G(t,v)\Big[f(v,x(v-h(v))) - r(v)x(v-h(v))\Big]dv\\ &= (\mathcal{S}x)(t).\end{aligned}$$

This indicates that $\mathcal{T}(\Phi)\subset\Phi_\omega$ and $\mathcal{S}(\Phi)\subset\Phi_\omega$. Next, we demonstrate that $\mathcal{T}x+\mathcal{S}y\in\Phi$ for all $x,y\in\Phi$ and $t\in\mathbb{R}$. Utilizing (2.1), (2.2), and (2.3), we obtain

$$\begin{aligned}(\mathcal{T}x)(t)+(\mathcal{S}y)(t) &= \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t))\\ &+ \frac{1}{a(t)}\int_{t}^{t+\omega} G(t,s)\Big[f(s,y(s-h(s))) - r(s)y(s-h(s))\Big]ds\\ &\leqslant c_1m_1 + \frac{1}{a_0}\int_{t}^{t+\omega} G(t,s)\frac{b(s)}{a(s)}\left[\frac{a(s)}{b(s)}\Big(f(s,y(s-h(s))) - r(s)y(s-h(s))\Big)\right]ds\\ &\leqslant c_1m_1 + \frac{1}{a_0}(1-c_1)a_0m_1\int_{t}^{t+\omega} G(t,s)\frac{b(s)}{a(s)}ds\\ &= m_1\end{aligned}$$

and

$$\begin{aligned}(\mathcal{T}x)(t)+(\mathcal{S}y)(t) &= \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t))\\ &+ \frac{1}{a(t)}\int_{t}^{t+\omega} G(t,s)\Big[f(s,y(s-h(s))) - r(s)y(s-h(s))\Big]ds\\ &\geqslant c_0m_0 + \frac{1}{a_1}\int_{t}^{t+\omega} G(t,s)\frac{b(s)}{a(s)}\left[\frac{a(s)}{b(s)}\Big(f(s,y(s-h(s))) - r(s)y(s-h(s))\Big)\right]ds\\ &\geqslant c_0m_0 + \frac{1}{a_1}(1-c_0)a_1m_0\int_{t}^{t+\omega} G(t,s)\frac{b(s)}{a(s)}ds\\ &= m_0.\end{aligned}$$

This indicates that $\mathcal{T}x+\mathcal{S}y\in\Phi$ for all $x,y\in\Phi$. Now, we aim to demonstrate that $\mathcal{T}$ is a contraction mapping on Φ. For $x,y\in\Phi$, we observe

$$\begin{aligned}|(\mathcal{T}x)(t)-(\mathcal{T}y)(t))| &= \left|\frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) - \frac{c(t)}{a(t)(1-h'(t))}y(t-h(t))\right|\\ &\leqslant \frac{c(t)}{a(t)(1-h'(t))}\Big|x(t-h(t))-y(t-h(t))\Big|.\end{aligned}$$

By considering the supremum norm on both sides, it is evident that

$$\|\mathcal{T}x-\mathcal{T}y\|\leqslant c_1\|x-y\|.$$

This establishes that $\mathcal{T}$ is a contraction mapping. We will illustrate that $\mathcal{S}$ is continuous. Let $\{x_n\} \in \Phi$ be a convergent sequence of elements such that $x_n(t) \to x(t)$ as $n \to \infty$. As Φ is closed, it implies $x \in \Phi$. For $t \in [0, \omega]$, we have

$$\begin{aligned} |(\mathcal{S}x_n)(t) - (\mathcal{S}x)(t)| =& \bigg| \frac{1}{a(t)} \int_t^{t+\omega} G(t,s)\Big[f(s, x_n(s-h(s))) - r(s)x_n(s-h(s))\Big]ds \\ & - \frac{1}{a(t)} \int_t^{t+\omega} G(t,s)\Big[f(s, x(s-h(s))) - r(s)x(s-h(s))\Big]ds \bigg| \\ \leqslant & \frac{1}{a_0} \int_t^{t+\omega} G(t,s)|r(s)|\Big|x_n(s-h(s)) - x(s-h(s))\Big|ds \\ & + \frac{1}{a_0} \int_t^{t+\omega} G(t,s)\Big|f(s, x_n(s-h(s))) - f(s, x(s-h(s)))\Big|ds. \end{aligned}$$

By the continuity of f and by the Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \|(\mathcal{S}x_n) - (\mathcal{S}x)\| = 0.$$

This indicates that $\mathcal{S}$ is continuous. Now, we show that the family of functions $\{\mathcal{S}x : x \in \Phi\}$ is uniformly bounded and equicontinuous on $[0, \omega]$. We observe from (2.1) that

$$\begin{aligned} |(\mathcal{S}x)(t)| =& \bigg| \frac{1}{a(t)} \int_t^{t+\omega} G(t,s)\Big[f(s, x(s-h(s))) - r(s)x(s-h(s))\Big]ds \bigg| \\ \leqslant & \bigg| \frac{1}{a(t)} \int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}\left[\frac{a(s)}{b(s)}\Big(f(s, x(s-h(s))) - r(s)x(s-h(s))\Big)\right]ds \bigg| \\ \leqslant & \frac{1}{a_0}(1-c_1)a_0 m_1 \int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}ds \\ =& (1-c_1)m_1 \end{aligned}$$

and it follows that

$$\|\mathcal{S}x\| \leqslant (1-c_1)m_1.$$

Moreover, by using (2.1), we derive

$$\begin{aligned} |(\mathcal{S}x)'(t)| =& \bigg| \frac{d}{dt}\Big[\frac{1}{a(t)} \int_t^{t+\omega} G(t,s)\Big[f(s, x(s-h(s))) - r(s)x(s-h(s))\Big]ds \bigg| \\ \leqslant & \bigg| \frac{d}{dt}\Big[\frac{1}{a(t)}\Big]a(t)(\mathcal{S}x)(t) + \frac{1}{a(t)}\Big[G(t,t+\omega)\Big[f(t, x(t-h(t))) - r(t)x(t-h(t))\Big] \\ & - G(t,t)\Big[f(t, x(t-h(t))) - r(t)x(t-h(t))\Big]\Big] - b(t)(\mathcal{S}x)(t) \bigg| \\ \leqslant & \bigg| \frac{-a'(t)}{a(t)}(\mathcal{S}x)(t) \end{aligned}$$

$$+\frac{1}{a(t)}\left[\frac{b(t)}{a(t)}\left[\frac{a(t)}{b(t)}\Big(f(t,x(t-h(t)))-r(t)x(t-h(t))\Big)\right]-b(t)(\mathcal{S}x)(t)\right]\Bigg|$$
$$\leqslant \frac{1}{a_0}\left[\|a'\|\|\mathcal{S}x\|+\|b\|(1-c_1)m_1+\|b\|\|\mathcal{S}x\|\right]$$
$$\leqslant \frac{1}{a_0}\Big[\|a'\|+2\|b\|\Big](1-c_1)m_1.$$

Thus, $\{\mathcal{S}x : x \in \Phi\}$ is uniformly bounded and equicontinuous on $[0,\omega]$. Consequently, $\mathcal{S}\Phi$ is relatively compact. These observations lead us to the conclusion that $\mathcal{S}$ is completely continuous. Applying the fixed-point theorem of Krasnoselskii, we infer the existence of $x \in \Phi$ such that $\mathcal{T}x + \mathcal{S}x = x$. This implies that $x(t)$ is a positive ω-periodic solution of (1.1). □

Theorem 2.2. *Let $-1 < c_0 \leqslant \frac{c(t)}{a(t)(1-h'(t))} \leqslant c_1 \leqslant 0$. Furthermore, assume that there exist positive constants m_0 and m_1 with $m_0 < m_1$ such that*

$$(m_0 - c_0 m_1)a_1 \leqslant \frac{a(t)}{b(t)}\Big(f(t,x)-r(t)x\Big) \leqslant (m_1 - c_1 m_0)a_0 \tag{2.4}$$

for all $t \in [0,\omega]$, $x \in [m_0, m_1]$, where

$$r(t) = \frac{\Big(c'(t)+\frac{b(t)}{a(t)}c(t)\Big)\Big(1-h'(t)\Big)+h''(t)c(t)}{\Big(1-h'(t)\Big)^2}, \quad t \in [0,\omega].$$

Then, (1.1) has at least one positive ω-periodic solution $x(t) \in [m_0, m_1]$.

Proof. We define Φ, $G(t,s)$, $\mathcal{T}$, and $\mathcal{S}$ as outlined in the proof of Theorem 2.1. It is evident from the proof of Theorem 2.1 that $\mathcal{T}(\Phi) \subset \Phi_\omega$ and $\mathcal{S}(\Phi) \subset \Phi_\omega$. Next, we demonstrate that $\mathcal{T}x + \mathcal{S}y \in \Phi$ for all $x, y \in \Phi$ and $t \in \mathbb{R}$. By utilizing (2.2), (2.3), and (2.4), we obtain

$$(\mathcal{T}x)(t)+(\mathcal{S}y)(t) = \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t))$$
$$+\frac{1}{a(t)}\int_t^{t+\omega} G(t,s)\Big[f(s,y(s-h(s)))-r(s)y(s-h(s))\Big]ds$$
$$\leqslant c_1 m_0 + \frac{1}{a_0}\int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}\left[\frac{a(s)}{b(s)}\Big(f(s,y(s-h(s)))-r(s)y(s-h(s))\Big)\right]ds$$
$$\leqslant c_1 m_0 + \frac{1}{a_0}(m_1 - c_1 m_0)a_0\int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}ds$$
$$= m_1$$

and

$$(\mathcal{T}x)(t)+(\mathcal{S}y)(t) = \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t))$$
$$+\frac{1}{a(t)}\int_t^{t+\omega} G(t,s)\Big[f(s,y(s-h(s)))-r(s)y(s-h(s))\Big]ds$$

$$
\begin{aligned}
&\geqslant c_0 m_1 + \frac{1}{a_1}\int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}\left[\frac{a(s)}{b(s)}\Big(f(s,y(s-h(s))) - r(s)y(s-h(s))\Big)\right]ds \\
&\geqslant c_0 m_1 + \frac{1}{a_1}(m_0 - c_0 m_1)a_1\int_t^{t+\omega} G(t,s)\frac{b(s)}{a(s)}ds \\
&= m_0.
\end{aligned}
$$

This results in $\mathcal{T}x + \mathcal{S}y \in \Phi$ for all $x, y \in \Phi$. Now, let's establish that $\mathcal{T}$ is a contraction mapping on Φ. For $x, y \in \Phi$, we observe

$$
\begin{aligned}
|(\mathcal{T}x)(t) - (\mathcal{T}y)(t))| &= \left|\frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) - \frac{c(t)}{a(t)(1-h'(t))}y(t-h(t))\right| \\
&\leqslant \frac{-c(t)}{a(t)(1-h'(t))}\Big|x(t-h(t)) - y(t-h(t))\Big|.
\end{aligned}
$$

Taking the supremum norm on both sides, we get

$$\|\mathcal{T}x - \mathcal{T}y\| \leqslant -c_0\|x-y\|.$$

Therefore, $\mathcal{T}$ is a contraction mapping. As the remainder of the proof parallels that of Theorem 2.1, it will be omitted to avoid redundancy. □

Example 2.1. Consider the first-order neutral differential equation

$$
\begin{aligned}
(2e^{\frac{\cos(t)}{10}}x(t))' = {}& -10e^{\frac{11\cos(t)}{10}}x(t) + e^{-\frac{\cos(t)}{5}}x'(t - e^{\frac{\cos(t)}{5}}) \\
&+ e^{1-0.16\sin(t)+0.8\cos(t)}\Big(12 + 2x(t - e^{\frac{\cos(t)}{5}})\Big).
\end{aligned}
\tag{2.5}
$$

It can be seen that (2.5) is of the form (1.1) with $a(t) = 2e^{\frac{\cos(t)}{10}}$, $b(t) = 10e^{\frac{11\cos(t)}{10}}$, $c(t) = e^{-\frac{\cos(t)}{5}}$, $h(t) = e^{\frac{\cos(t)}{5}}$, $\omega = 2\pi$, $f(t,x) = e^{1-0.16\sin(t)+0.8\cos(t)}(12+2x)$, $a_0 = \min\limits_{t\in[0,\omega]} a(t) = 1.8090$, and $a_1 = \max\limits_{t\in[0,\omega]} a(t) = 2.2103$. After straightforward calculation, we can obtain

$$
c'(t) = \frac{\sin(t)}{5}e^{-\frac{\cos(t)}{5}}, \qquad h'(t) = -\frac{\sin(t)}{5}e^{\frac{\cos(t)}{5}},
$$

$$
h''(t) = e^{\frac{\cos(t)}{5}}\Big[\frac{-\cos(t)}{5} + \frac{\sin^2(t)}{25}\Big],
$$

$$
\begin{aligned}
r(t) &= \frac{\Big(c'(t) + \frac{b(t)}{a(t)}c(t)\Big)\Big(1-h'(t)\Big) + h''(t)c(t)}{\Big(1-h'(t)\Big)^2} \\
&= \frac{\Big(\frac{\sin(t)}{5} + 5e^{\cos(t)}\Big)\Big(e^{-\frac{\cos(t)}{5}} + \frac{\sin(t)}{5}\Big) + \Big(\frac{-\cos(t)}{5} + \frac{\sin^2(t)}{25}\Big)}{\Big(1 + \frac{\sin(t)}{5}e^{\frac{\cos(t)}{5}}\Big)^2},
\end{aligned}
$$

$$
0 \leqslant 0.3444 = c_0 \leqslant \frac{c(t)}{a(t)(1-h'(t))} = \frac{e^{-\frac{\cos(t)}{5}}}{2e^{\frac{\cos(t)}{10}}\Big(1 + \frac{\sin(t)}{5}e^{\frac{\cos(t)}{5}}\Big)} \leqslant c_1 = 0.7083 < 1,
$$

$(1-c_0)a_1m_0 = (1-0.3444)(2.2103)2 = 2.8981$ and
$(1-c_1)a_0m_1 = (1-0.7083)(1.8090)21 = 11.0814$. Therefore,

$$2.8981 = (1-c_0)a_1m_0 < 5.1345 \leqslant \frac{a(t)}{b(t)}\Big(f(t,x) - r(t)x\Big) \leqslant 9.6196$$
$$< (1-c_1)a_0m_1 = 11.0814.$$

It shows that the conditions of Theorem 2.1 are satisfied when $m_0 = 2$ and $m_1 = 21$. Thus (2.5) has at least one 2π-periodic positive solution $x(t)$ satisfying $2 \leqslant x(t) \leqslant 21$.

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