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# Entanglement of Several Classical and Dynamic Estimates with Unified Approach on Time Scales

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**Abstract** In this research article, we present several generalizations of Qi's inequality on time scales. We establish dynamic versions of Callebaut's inequality and Cauchy-Schwarz's inequality on time scales. To establish our results, we apply the diamond-alpha integral and the time scale $\Delta$ or $\nabla$-Riemann-Liouville type fractional integrals. Our findings unify and extend discrete, continuous and quantum analogues.

**Keywords** Time scales, fractional Riemann-Liouville integrals, Qi's, Callebaut's and Cauchy-Schwarz's inequalities

**MSC(2010)** 26D15, 26E70, 34N05, 26A33.

## 1. Introduction

The calculus of time scales was initiated by Stefan Hilger [13]. A time scale is an arbitrary nonempty closed subset of the real numbers. This hybrid theory is also widely applied on dynamic inequalities, see [2, 15–20, 23, 24]. The basic ideas concerning the calculus of time scales are given in [7, 8].

The following Qi's inequality is proved in [12].

Let $r \geq 1$ and $\Phi$ be a nonnegative continuous function on $[\xi, \omega]$ such that $0 < \Phi(\lambda) \leq r(\omega - \xi)^{-1}$. Then we have the following inequality

$$\left(\int_{\xi}^{\omega} \Phi(\lambda) d\lambda\right)^{r} \leq \frac{r^r}{e^r} \exp\left(\int_{\xi}^{\omega} \Phi(\lambda) d\lambda\right) \leq \frac{r^{2r}}{(\omega-\xi)^{1+r}} \int_{\xi}^{\omega} \Phi^{-r}(\lambda) d\lambda. \tag{1.1}$$

The following Callebaut's inequality is given in [11].

Let $x_k > 0$, $y_k > 0$ and $w_k \geq 0$ for any $k \in \{1, 2, \dots, n\}$ with $\sum_{k=1}^{n} w_k = 1$. If there exist the constants $m$, $M > 0$ such that $0 < m \leq \frac{x_k}{y_k} \leq M < \infty$ for any $k \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}&\sum_{k=1}^{n} w_k x_k^{2(1-v)} y_k^{2v} \sum_{k=1}^{n} w_k x_k^{2v} y_k^{2(1-v)}\\ \leq &\sum_{k=1}^{n} w_k x_k^2 \sum_{k=1}^{n} w_k y_k^2\end{aligned}$$

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$$\leq K^{\delta}\left(\left(\frac{M}{m}\right)^{2}\right)\sum_{k=1}^{n} w_k x_k^{2(1-v)} y_k^{2v} \sum_{k=1}^{n} w_k x_k^{2v} y_k^{2(1-v)}, \tag{1.2}$$

for any $v \in [0,1]$ and $\delta = \max\{1-v, v\}$.

The following Qi's inequality is proved in [12].

Let $0 < p < q \leq 1$, $r > 0$ and $\Upsilon, \Phi$ be measurable nonnegative functions on $[\xi, \omega]$ such that $\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{q}(\gamma)d\gamma < \infty$. Then we have the following inequality

$$\left[\left(\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{p}(\gamma)d\gamma\right)^{\frac{1}{p}}\right]^{r} \leq \frac{r^r}{e^r}\left(\int_{\xi}^{\omega} \Upsilon(\gamma)d\gamma\right)^{\frac{r}{p}-\frac{r}{q}} \exp\left(\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{q}(\gamma)d\gamma\right)^{\frac{1}{q}}. \tag{1.3}$$

We shall unify and extend (1.1) and (1.2) in the calculus of time scales by applying the diamond-alpha integral. We shall also unify and extend (1.3) in the fractional calculus of time scales.

## 2. Preliminaries

Now we present a short introduction to the diamond-$\alpha$ derivative as given in [1,21].

Let $\mathbb{T}$ be a time scale and $\Phi(\lambda)$ be differentiable on $\mathbb{T}$ in the $\Delta$ and $\nabla$ senses. For $\lambda \in \mathbb{T}$, the diamond-$\alpha$ dynamic derivative $\Phi^{\diamond_\alpha}(\lambda)$ is defined by

$$\Phi^{\diamond_\alpha}(\lambda) = \alpha\Phi^{\Delta}(\lambda) + (1-\alpha)\Phi^{\nabla}(\lambda), \quad 0 \leq \alpha \leq 1.$$

Thus $\Phi$ is diamond-$\alpha$ differentiable if and only if $\Phi$ is $\Delta$ and $\nabla$ differentiable.

The following definition is given in [21].

Let $\xi, \kappa \in \mathbb{T}$ and $\Phi : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from $\xi$ to $\kappa$ of $\Phi$ is defined by

$$\int_{\xi}^{\kappa} \Phi(\lambda) \diamond_\alpha \lambda = \alpha \int_{\xi}^{\kappa} \Phi(\lambda)\Delta\lambda + (1-\alpha)\int_{\xi}^{\kappa} \Phi(\lambda)\nabla\lambda, \quad 0 \leq \alpha \leq 1, \tag{2.1}$$

provided that there exist delta and nabla integrals of $\Phi$ on $\mathbb{T}$.

The following inequality is given in [6,22].

Let $r > 0$ and $z > 0$. Then the following inequality is valid:

$$z^r \leq \frac{r^r}{e^r} e^z. \tag{2.2}$$

The following well-known Young's inequality holds:

For $\Omega, \chi > 0$ and $v \in [0,1]$, we have

$$\Omega^{1-v}\chi^{v} \leq (1-v)\Omega + v\chi. \tag{2.3}$$

Kantorovich's ratio is defined by

$$K(h) := \frac{(h+1)^2}{4h},$$

where $h > 0$.

The following inequality is given in [14].

For any $\Omega, \chi \in [m, M] \subset (0, \infty)$ and $v \in [0, 1]$, we have

$$(1-v)\Omega + v\chi \leq K^{\delta}\left(\frac{M}{m}\right)\Omega^{1-v}\chi^{v}, \tag{2.4}$$

where $\delta = \max\{1-v, v\}$.

The following definition concerning the time scale $\Delta$-Riemann–Liouville type fractional integral is given in [3, 5].

For $\alpha \geq 1$, the time scale $\Delta$-Riemann–Liouville type fractional integral for a function $\Phi \in C_{rd}$ is defined by

$$\mathcal{I}_{\xi}^{\alpha}\Phi(\kappa) = \int_{\xi}^{\kappa} h_{\alpha-1}(\kappa, \sigma(\gamma))\Phi(\gamma)\Delta\gamma, \tag{2.5}$$

which is an integral on $[\xi, \kappa)_{\mathbb{T}}$, see [9] and $h_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd-continuous functions, such that $h_0(\kappa, \zeta) = 1$,

$$h_{\alpha+1}(\kappa, \zeta) = \int_{\zeta}^{\kappa} h_{\alpha}(\gamma, \zeta)\Delta\gamma, \ \forall \zeta, \kappa \in \mathbb{T}. \tag{2.6}$$

Notice that

$$\mathcal{I}_{\xi}^{1}\Phi(\kappa) = \int_{\xi}^{\kappa} \Phi(\gamma)\Delta\gamma,$$

which is absolutely continuous in $\kappa \in [\xi, \omega]_{\mathbb{T}}$, see [9].

The following definition concerning the time scale $\nabla$-Riemann–Liouville type fractional integral is given in [4, 5].

For $\alpha \geq 1$, the time scale $\nabla$-Riemann–Liouville type fractional integral for a function $\Phi \in C_{ld}$ is defined by

$$\mathcal{J}_{\xi}^{\alpha}\Phi(\kappa) = \int_{\xi}^{\kappa} \hat{h}_{\alpha-1}(\kappa, \rho(\gamma))\Phi(\gamma)\nabla\gamma, \tag{2.7}$$

which is an integral on $(\xi, \kappa]_{\mathbb{T}}$, see [9] and $\hat{h}_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld-continuous functions, such that $\hat{h}_0(\kappa, \zeta) = 1$,

$$\hat{h}_{\alpha+1}(\kappa, \zeta) = \int_{\zeta}^{\kappa} \hat{h}_{\alpha}(\gamma, \zeta)\nabla\gamma, \ \forall \zeta, \kappa \in \mathbb{T}. \tag{2.8}$$

Notice that

$$\mathcal{J}_{\xi}^{1}\Phi(\kappa) = \int_{\xi}^{\kappa} \Phi(\gamma)\nabla\gamma,$$

which is absolutely continuous in $\kappa \in [\xi, \omega]_{\mathbb{T}}$, see [9].

**Theorem 2.1** ( [1]). *Let $\xi, \omega \in \mathbb{T}$ and $\eta_1, \eta_2 \in \mathbb{R}$. Suppose $\Psi \in C_{rd}([\xi, \omega]_{\mathbb{T}}, (\eta_1, \eta_2))$ and $\Upsilon \in C_{rd}([\xi, \omega]_{\mathbb{T}}, \mathbb{R})$ with $\int_{\xi}^{\omega} |\Upsilon(\lambda)|\Delta\lambda > 0$. If $\digamma \in C((\eta_1, \eta_2), \mathbb{R})$ is convex, then*

$$\digamma\left(\frac{\int_{\xi}^{\omega} |\Upsilon(\lambda)|\Psi(\lambda)\Delta\lambda}{\int_{\xi}^{\omega} |\Upsilon(\lambda)|\Delta\lambda}\right) \leq \frac{\int_{\xi}^{\omega} |\Upsilon(\lambda)|\digamma\left(\Psi(\lambda)\right)\Delta\lambda}{\int_{\xi}^{\omega} |\Upsilon(\lambda)|\Delta\lambda}. \tag{2.9}$$

*If $\digamma$ is strictly convex, then the inequality $\leq$ can be replaced by $<$.*

In this paper, it is assumed that all considerable integrals exist and are finite. Let $\mathbb{T}$ be a time scale, $\xi, \omega \in \mathbb{T}$ with $\xi < \omega$ and an interval $[\xi, \omega]_{\mathbb{T}}$ means the intersection of the real interval with the given time scale.

# 3. Integral inequalities

First, we give an extension of Feng Qi's inequality by using the diamond-alpha integral.

**Theorem 3.1.** *Let $r \geq 1$ and $\Upsilon, \Phi \in C([\xi,\omega]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions such that $0 < |\Upsilon(\lambda)\Phi(\lambda)| \leq r(\omega-\xi)^{-1}$ on the set $[\xi,\omega]_{\mathbb{T}}$. Then*

$$\left(\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda\right)^r \leq \frac{r^r}{e^r}\exp\left(\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda\right) \leq \frac{r^{2r}}{(\omega-\xi)^{1+r}}\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)|^{-r} \diamond_\alpha \lambda. \tag{3.1}$$

**Proof.** From the given condition, we have

$$\textstyle\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda \leq r \text{ and } r^{-r}(\omega-\xi)^r \leq |\Upsilon(\lambda)\Phi(\lambda)|^{-r}.$$

Applying (2.2) to $z = \int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda$, we have

$$\begin{aligned}
&\frac{e^r}{r^r}\left(\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda\right)^r \\
&\leq \exp\left(\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)| \diamond_\alpha \lambda\right) \\
&\leq e^r r^{-r}(\omega-\xi)^{-1-r} r^r(\omega-\xi)^{1+r} \\
&\leq \frac{e^r r^r}{(\omega-\xi)^{1+r}}\int_\xi^\omega |\Upsilon(\lambda)\Phi(\lambda)|^{-r} \diamond_\alpha \lambda.
\end{aligned}$$

The proof of Theorem 3.1 is completed. □

**Remark 3.1.** Let $\mathbb{T} = \mathbb{R}$, $\Upsilon \equiv 1$ and $\Phi > 0$. Then inequality (3.1) reduces to inequality (1.1).

**Remark 3.2.** Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = n+1$, $\Upsilon \equiv 1$ and $\Phi(k) = x_k > 0$, $k = 1, 2, \ldots, n$. Then inequality (3.1) reduces to

$$\left(\sum_{k=1}^n x_k\right)^r \leq \frac{r^r}{e^r}\exp\left(\sum_{k=1}^n x_k\right) \leq \frac{r^{2r}}{(\omega-\xi)^{1+r}}\sum_{k=1}^n x_k^{-r}. \tag{3.2}$$

Throughout this section, we will assume that neither $\Phi \equiv 0$ nor $\Psi \equiv 0$. Now, we present Callebaut's inequality [10] and reverse Callebaut's inequality on time scales by applying the diamond-alpha integral.

**Theorem 3.2.** *Let $\Upsilon, \Phi, \Psi \in C([\xi,\omega]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions. Assume further that $0 < m \leq \frac{|\Phi(\lambda)|}{|\Psi(\lambda)|} \leq M < \infty$ on the set $[\xi,\omega]_{\mathbb{T}}$. Let $v \in [0,1]$. Then*

$$\begin{aligned}
&\int_\xi^\omega |\Upsilon(\lambda)||\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v} \diamond_\alpha \lambda \int_\xi^\omega |\Upsilon(\lambda)||\Phi(\lambda)|^{2v}|\Psi(\lambda)|^{2(1-v)} \diamond_\alpha \lambda \\
\leq &\int_\xi^\omega |\Upsilon(\lambda)||\Phi(\lambda)|^2 \diamond_\alpha \lambda \int_\xi^\omega |\Upsilon(\lambda)||\Psi(\lambda)|^2 \diamond_\alpha \lambda \\
\leq &K^\delta\left(\left(\frac{M}{m}\right)^2\right)\int_\xi^\omega |\Upsilon(\lambda)||\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v} \diamond_\alpha \lambda
\end{aligned}$$

$$\times \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{2v}|\Psi(\lambda)|^{2(1-v)} \diamond_\alpha \lambda, \tag{3.3}$$

*where* $\delta = \max\{1-v, v\}$.

**Proof.** For $\lambda, \gamma \in [\xi, \omega]_{\mathbb{T}}$, it is clear that

$$m^2 \leq \frac{|\Phi(\lambda)|^2}{|\Psi(\lambda)|^2}, \frac{|\Phi(\gamma)|^2}{|\Psi(\gamma)|^2} \leq M^2. \tag{3.4}$$

Let $\Omega(\lambda) = \frac{|\Phi(\lambda)|^2}{|\Psi(\lambda)|^2}$ and $\chi(\gamma) = \frac{|\Phi(\gamma)|^2}{|\Psi(\gamma)|^2}$, $\lambda, \gamma \in [\xi, \omega]_{\mathbb{T}}$. Using (2.3) and (2.4), we have

$$\begin{aligned}\left(\frac{|\Phi(\lambda)|^2}{|\Psi(\lambda)|^2}\right)^{1-v}\left(\frac{|\Phi(\gamma)|^2}{|\Psi(\gamma)|^2}\right)^{v} &\leq (1-v)\frac{|\Phi(\lambda)|^2}{|\Psi(\lambda)|^2} + v\frac{|\Phi(\gamma)|^2}{|\Psi(\gamma)|^2} \\ &\leq K^{\delta}\left(\left(\frac{M}{m}\right)^2\right)\left(\frac{|\Phi(\lambda)|^2}{|\Psi(\lambda)|^2}\right)^{1-v}\left(\frac{|\Phi(\gamma)|^2}{|\Psi(\gamma)|^2}\right)^{v}. \end{aligned} \tag{3.5}$$

Multiplying by $|\Psi(\lambda)|^2|\Psi(\gamma)|^2$, $\lambda, \gamma \in [\xi, \omega]_{\mathbb{T}}$, (3.5) takes the form

$$\begin{aligned} &|\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v}|\Phi(\gamma)|^{2v}|\Psi(\gamma)|^{2(1-v)} \\ \leq &(1-v)|\Phi(\lambda)|^2|\Psi(\gamma)|^2 + v|\Psi(\lambda)|^2|\Phi(\gamma)|^2 \\ \leq &K^{\delta}\left(\left(\frac{M}{m}\right)^2\right)|\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v}|\Phi(\gamma)|^{2v}|\Psi(\gamma)|^{2(1-v)}. \end{aligned} \tag{3.6}$$

Multiplying by $|\Upsilon(\lambda)|$ and integrating (3.6) with respect to $\lambda$ from $\xi$ to $\omega$, we obtain

$$\begin{aligned} &\left(\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v} \diamond_\alpha \lambda\right)|\Phi(\gamma)|^{2v}|\Psi(\gamma)|^{2(1-v)} \\ \leq &(1-v)\left(\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^2 \diamond_\alpha \lambda\right)|\Psi(\gamma)|^2 \\ &+ v\left(\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Psi(\lambda)|^2 \diamond_\alpha \lambda\right)|\Phi(\gamma)|^2 \\ \leq &K^{\delta}\left(\left(\frac{M}{m}\right)^2\right)\left(\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{2(1-v)}|\Psi(\lambda)|^{2v} \diamond_\alpha \lambda\right)|\Phi(\gamma)|^{2v}|\Psi(\gamma)|^{2(1-v)}. \end{aligned} \tag{3.7}$$

Again, multiplying by $|\Upsilon(\gamma)|$ and integrating (3.7) with respect to $\gamma$ from $\xi$ to $\omega$, we obtain the desired inequality (3.3). □

**Remark 3.3.** Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $\xi = 1$, $\omega = n+1$, $\Phi(k) = x_k > 0$, $\Psi(k) = y_k > 0$ and $\Upsilon(k) = w_k \geq 0$ for any $k \in \{1, 2, \ldots, n\}$ with $\sum_{k=1}^{n} w_k = 1$. Then inequality (3.3) reduces to (1.2).

**Remark 3.4.** We have the following results:

(*i*) If we replace $v$ by $\frac{1}{2}(1-v)$ with $v \in [0,1]$ in (3.3), then we get

$$\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{1+v}|\Psi(\lambda)|^{1-v} \diamond_\alpha \lambda \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{1-v}|\Psi(\lambda)|^{1+v} \diamond_\alpha \lambda$$

$$\begin{aligned}
&\leq \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^2 \diamond_\alpha \lambda \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Psi(\lambda)|^2 \diamond_\alpha \lambda \\
&\leq K^{\frac{1+v}{2}} \left( \left( \frac{M}{m} \right)^2 \right) \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{1+v} |\Psi(\lambda)|^{1-v} \diamond_\alpha \lambda \\
&\quad \times \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{1-v} |\Psi(\lambda)|^{1+v} \diamond_\alpha \lambda. \qquad (3.8)
\end{aligned}$$

$(ii)$ Also, if we take $v = \frac{1}{2}u$ with $u \in [0, 2]$ in (3.3), then we get

$$\begin{aligned}
&\int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{2-u} |\Psi(\lambda)|^{u} \diamond_\alpha \lambda \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{u} |\Psi(\lambda)|^{2-u} \diamond_\alpha \lambda \\
&\leq \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^2 \diamond_\alpha \lambda \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Psi(\lambda)|^2 \diamond_\alpha \lambda \\
&\leq K^{\varsigma} \left( \left( \frac{M}{m} \right)^2 \right) \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{2-u} |\Psi(\lambda)|^{u} \diamond_\alpha \lambda \\
&\quad \times \int_{\xi}^{\omega} |\Upsilon(\lambda)||\Phi(\lambda)|^{u} |\Psi(\lambda)|^{2-u} \diamond_\alpha \lambda, \qquad (3.9)
\end{aligned}$$

where $\varsigma = \max\left\{\frac{1}{2}u, 1 - \frac{1}{2}u\right\}$.

# 4. Fractional inequalities

In this section, we give an extension of Qi's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 4.1.** *Let $0 < p < q \leq 1$, $r > 0$ and $\Upsilon, \Phi \in C_{rd}([\xi, \omega]_{\mathbb{T}}, \mathbb{R})$ be $\Delta$-integrable functions such that $\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^q) < \infty$, $\forall \kappa \in [\xi, \omega]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\left(\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^p)\right)^{\frac{r}{p}} \leq \frac{r^r}{e^r} \left(\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)|)\right)^{\frac{r}{p} - \frac{r}{q}} \exp\left(\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^q)\right)^{\frac{1}{q}}. \qquad (4.1)$$

**Proof.** Applying the inequality (2.2) for $z = \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^q \Delta\gamma\right)^{\frac{1}{q}}$, we have

$$\left[\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^q \Delta\gamma\right)^{\frac{1}{q}}\right]^r \leq \frac{r^r}{e^r} \exp\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^q \Delta\gamma\right)^{\frac{1}{q}}. \qquad (4.2)$$

Choosing $\digamma(\gamma) = \gamma^{\frac{q}{p}}$ in Theorem 2.1, which for $0 < p < q \leq 1$ is obviously a convex function on $[0, \infty)$, we have

$$\begin{aligned}
&\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^p \Delta\gamma\right)^{\frac{r}{p}} \\
&\leq \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)| \Delta\gamma\right)^{\frac{r}{p} - \frac{r}{q}} \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^q \Delta\gamma\right)^{\frac{r}{q}} \\
&\leq \frac{r^r}{e^r} \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)| \Delta\gamma\right)^{\frac{r}{p} - \frac{r}{q}} \exp\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^q \Delta\gamma\right)^{\frac{1}{q}}.
\end{aligned}$$

Replacing $|\Upsilon(\gamma)|$ by $h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)|$ in the last inequalities, we get the desired inequality. The proof of Theorem 4.1 is completed. □

Next, we give an extension of Qi's inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 4.2.** *Let $0 < p < q \leq 1$, $r > 0$ and $\Upsilon, \Phi \in C_{ld}([\xi, \omega]_{\mathbb{T}}, \mathbb{R})$ be $\nabla$-integrable functions such that $\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^q) < \infty$, $\forall \kappa \in [\xi, \omega]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\left(\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^p)\right)^{\frac{r}{p}} \leq \frac{r^r}{e^r}\left(\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)|)\right)^{\frac{r}{p}-\frac{r}{q}} \exp(\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^q))^{\frac{1}{q}}. \quad (4.3)$$

**Proof.** Similar to the proof of Theorem 4.1. □

**Remark 4.1.** Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $\kappa = \omega$ and $\Upsilon, \Phi \geq 0$. Then inequality (4.1) reduces to inequality (1.3).

Next, we give another extension of Qi's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 4.3.** *Let $\frac{1}{r} + \frac{1}{s} = 1$ for $r, s > 1$ and $\Upsilon, \Phi, \Psi \in C_{rd}([\xi, \omega]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\Delta$-integrable functions such that $0 < m \leq \frac{|\Phi(\gamma)|^r}{|\Psi(\gamma)|^s} \leq M < \infty$ on the set $[\xi, \kappa]_{\mathbb{T}}$, $\forall \kappa \in [\xi, \omega]_{\mathbb{T}}$. Let $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$. Then we have the following inequality*

$$\left(\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^r)\right)^{\frac{1}{r}}\left(\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Psi(\kappa)|^s)\right)^{\frac{1}{s}} \leq \left(\frac{M}{m}\right)^{\frac{1}{rs}} \mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)\Psi(\kappa)|), \quad (4.4)$$

*and hence deduce that*

$$\begin{aligned}\left(\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)|^r)\right)\left(\mathcal{I}_{\xi}^{\alpha}\left(|\Psi(\kappa)|^{\frac{r}{r-1}}\right)\right)^{r-1} &\leq \left(\frac{M}{m}\right)^{1-\frac{1}{r}}\left(\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|)\right)^r \\ &\leq \left(\frac{M}{m}\right)^{1-\frac{1}{r}}\frac{r^r}{e^r}\exp(\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|)). \quad (4.5)\end{aligned}$$

**Proof.** Using the given condition, for $\gamma \in [\xi, \kappa]_{\mathbb{T}}$, $\forall \kappa \in [\xi, \omega]_{\mathbb{T}}$, we have

$$|\Psi(\gamma)| \geq M^{-\frac{1}{s}}|\Phi(\gamma)|^{\frac{r}{s}}.$$

Multiplying both sides by $h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)|$ and integrating over $\gamma$ from $\xi$ to $\kappa$, we have

$$\begin{aligned}&\left(\int_{\xi}^{\kappa} h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)||\Phi(\gamma)|^r\Delta\gamma\right)^{\frac{1}{r}} \\ \leq &M^{\frac{1}{rs}}\left(\int_{\xi}^{\kappa} h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)||\Phi(\gamma)\Psi(\gamma)|\Delta\gamma\right)^{\frac{1}{r}}. \quad (4.6)\end{aligned}$$

On the other hand, we have

$$|\Phi(\gamma)| \geq m^{\frac{1}{r}}|\Psi(\gamma)|^{\frac{s}{r}}.$$

Multiplying both sides by $h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)|$ and integrating over $\gamma$ from $\xi$ to $\kappa$, we have

$$\left(\int_{\xi}^{\kappa} h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)||\Psi(\gamma)|^s\Delta\gamma\right)^{\frac{1}{s}}$$

$$\leq m^{-\frac{1}{rs}}\left(\int_{\xi}^{\kappa} h_{\alpha-1}(\kappa,\sigma(\gamma))|\Upsilon(\gamma)||\Phi(\gamma)\Psi(\gamma)|\Delta\gamma\right)^{\frac{1}{s}}. \tag{4.7}$$

Combining (4.6) and (4.7), we get (4.4).

When $\Upsilon\equiv 1$, then (4.4) takes the form

$$\left(\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)|^{r})\right)^{\frac{1}{r}}\left(\mathcal{I}_{\xi}^{\alpha}(|\Psi(\kappa)|^{s})\right)^{\frac{1}{s}}\leq\left(\frac{M}{m}\right)^{\frac{1}{rs}}\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|). \tag{4.8}$$

Applying (2.2) to $z=\mathcal{I}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|)$, $\forall\kappa\in[\xi,\omega]_{\mathbb{T}}$, we get (4.5) from (4.8). This completes the proof of Theorem 4.3. □

Next, we give another extension of Qi's inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 4.4.** *Let $\frac{1}{r}+\frac{1}{s}=1$ for $r,s>1$ and $\Upsilon,\Phi,\Psi\in C_{ld}\left([\xi,\omega]_{\mathbb{T}},\mathbb{R}-\{0\}\right)$ be $\nabla$-integrable functions such that $0<m\leq\frac{|\Phi(\gamma)|^{r}}{|\Psi(\gamma)|^{s}}\leq M<\infty$ on the set $[\xi,\kappa]_{\mathbb{T}}$, $\forall\kappa\in[\xi,\omega]_{\mathbb{T}}$. Let $\alpha\geq 1$ and $\hat{h}_{\alpha-1}(.,.)>0$. Then we have the following inequality*

$$\left(\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|^{r})\right)^{\frac{1}{r}}\left(\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Psi(\kappa)|^{s})\right)^{\frac{1}{s}}\leq\left(\frac{M}{m}\right)^{\frac{1}{rs}}\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)\Psi(\kappa)|), \tag{4.9}$$

*and hence deduce that*

$$\begin{aligned}\left(\mathcal{J}_{\xi}^{\alpha}(|\Phi(\kappa)|^{r})\right)\left(\mathcal{J}_{\xi}^{\alpha}\left(|\Psi(\kappa)|^{\frac{r}{r-1}}\right)\right)^{r-1}&\leq\left(\frac{M}{m}\right)^{1-\frac{1}{r}}\left(\mathcal{J}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|)\right)^{r}\\&\leq\left(\frac{M}{m}\right)^{1-\frac{1}{r}}\frac{r^{r}}{e^{r}}\exp(\mathcal{J}_{\xi}^{\alpha}(|\Phi(\kappa)\Psi(\kappa)|)).\end{aligned} \tag{4.10}$$

**Proof.** Similar to the proof of Theorem 4.3. □

Now, we give an extension of Cauchy–Schwarz's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 4.5.** *Let $\Upsilon,\Phi,\Psi\in C_{rd}\left([\xi,\omega]_{\mathbb{T}},\mathbb{R}\right)$ be $\Delta$-integrable functions. We assume that $m,n,M,N\in(0,+\infty)$ such that $(N|\Psi(\gamma)|-m|\Phi(\gamma)|)(M|\Phi(\gamma)|-n|\Psi(\gamma)|)\geq 0$ on the set $[\xi,\kappa]_{\mathbb{T}}$, $\forall\kappa\in[\xi,\omega]_{\mathbb{T}}$. Let $\alpha\geq 1$ and $h_{\alpha-1}(.,.)>0$. Then we have the following inequality*

$$\frac{\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|)\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Psi(\kappa)|)}{\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)|)\mathcal{I}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)\Psi(\kappa)|)}\leq\frac{1}{2}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right). \tag{4.11}$$

**Proof.** From the given condition, for $\gamma\in[\xi,\kappa]_{\mathbb{T}}$, $\forall\kappa\in[\xi,\omega]_{\mathbb{T}}$, we have

$$(MN+mn)|\Phi(\gamma)\Psi(\gamma)|\geq Mm|\Phi(\gamma)|^{2}+Nn|\Psi(\gamma)|^{2}. \tag{4.12}$$

Multiplying both sides of inequality (4.12) by $|\Upsilon(\gamma)|$ and integrating over $\gamma$ from $\xi$ to $\kappa$, we obtain

$$(MN+mn)\int_{\xi}^{\kappa}|\Upsilon(\gamma)||\Phi(\gamma)\Psi(\gamma)|\Delta\gamma$$

$$\geq Mm \int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^2 \Delta\gamma + Nn \int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Psi(\gamma)|^2 \Delta\gamma. \tag{4.13}$$

From Jensen's inequality (2.9), we have

$$\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|\Delta\gamma\right)^2 \leq \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)|\Delta\gamma\right)\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|^2\Delta\gamma\right), \tag{4.14}$$

and

$$\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Psi(\gamma)|\Delta\gamma\right)^2 \leq \left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)|\Delta\gamma\right)\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Psi(\gamma)|^2\Delta\gamma\right). \tag{4.15}$$

By using inequalities (4.14) and (4.15) and applying the AM-GM inequality, the inequality (4.13) takes the form

$$\begin{aligned}
&(MN + mn)\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)|\Delta\gamma\right)\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)\Psi(\gamma)|\Delta\gamma\right)\\
\geq &Mm\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|\Delta\gamma\right)^2 + Nn\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Psi(\gamma)|\Delta\gamma\right)^2\\
\geq &2\sqrt{MNmn}\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Phi(\gamma)|\Delta\gamma\right)\left(\int_{\xi}^{\kappa} |\Upsilon(\gamma)||\Psi(\gamma)|\Delta\gamma\right). 
\end{aligned} \tag{4.16}$$

Replacing $|\Upsilon(\gamma)|$ by $h_{\alpha-1}(\kappa, \sigma(\gamma))|\Upsilon(\gamma)|$ in (4.16), we obtain the desired claim. □

Next, we give an extension of Cauchy–Schwarz's inequality by using the time scale ∇-Riemann–Liouville type fractional integral.

**Theorem 4.6.** *Let $\Upsilon, \Phi, \Psi \in C_{ld}([\xi, \omega]_{\mathbb{T}}, \mathbb{R})$ be ∇-integrable functions. We assume that $m, n, M, N \in (0, +\infty)$ such that $(N|\Psi(\gamma)| - m|\Phi(\gamma)|)(M|\Phi(\gamma)| - n|\Psi(\gamma)|) \geq 0$ on the set $[\xi, \kappa]_{\mathbb{T}}$, $\forall \kappa \in [\xi, \omega]_{\mathbb{T}}$. Let $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)|)\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Psi(\kappa)|)}{\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)|)\mathcal{J}_{\xi}^{\alpha}(|\Upsilon(\kappa)||\Phi(\kappa)\Psi(\kappa)|)} \leq \frac{1}{2}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right). \tag{4.17}$$

**Proof.** Similar to the proof of Theorem 4.5. □

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