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On Fractional Hybrid Integral Inequalities via Extended s-Convexity

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Abstract In this study, we introduce a novel hybrid identity that successfully combines Newton-Cotes and Gauss quadratures, enabling us to recover both Simpson's second formula and the left and right Radau 2 point rules, among others. Based on this versatile foundation, we establish some new biparametric fractional integral inequalities for functions whose first derivatives are extended s-convex in the second sense. To support our findings, we present illustrative examples featuring graphical representations and conclude with several practical applications to demonstrate the effectiveness of our results.

Keywords Newton-Cotes inequalities, extended s-convex functions, Gauss-Radau formula, P-functions, hypergeometric function

MSC(2010) 26D10, 26D15, 26A51

1. Introduction

Integral inequalities find extensive applications in diverse fields, including mathematical disciplines such as approximation theory, numerical analysis, and differential equations, as well as other scientific domains such as physics, economics, biology, and engineering.

Furthermore, the theory of convex functions serves as a potent analytical tool, particularly within optimization and inequality theories, which share a close relationship. Various integral inequalities, including Hadamard's, Ostrowski's, Simpson's, and Newton's inequalities, utilize different classes of functions to encompass a wide range of function spaces. Among the widely studied and applied classes, the category of convex functions, along with its variants and generalizations, stands out. It's worth noting that a function ξ is deemed to be extended s-convex in the

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second sense if

$$\xi(\mu f+(1-\mu)g)\ \leq \mu^s\xi(f)+(1-\mu)^s\xi(g)$$

holds, for all $f,g\in I,\mu\in[0,1]$ and $s\in[-1,1]$ (see [33]). This class recaptures several other classes of convex functions. For $s=1$ we obtain the classical convex ([22]) functions; for $s=0$ we obtain the P-function ([6]); for $s=-1$ we obtain Godunova-Levin function ([12]); for $s\in(0,1]$ we obtain the s-convex functions ([4]); for $s\in[-1,0)$ s-Godunova-Levin function ([7]).

Regarding some papers dealing with integral inequalities via convexity, we refer the reader to [1–3, 5, 8, 18–20, 25–27].

The following inequality is known in the literature as Simpson's second formula inequality (see [11]).

$$\left|\tfrac{1}{8}\left(\xi(f)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi(g)\right)-\tfrac{1}{g-f}\int\limits_f^g\xi(u)\,du\right|\leq\tfrac{(g-f)^4}{6480}\left\|\xi^{(4)}\right\|_\infty,$$

where ξ is four times continuously differentiable mapping on the interval $[f,g]$ and $\left\|\xi^{(4)}\right\|_\infty=\sup\limits_{u\in[f,g]}\left|\xi^{(4)}(u)\right|$.

In [21], Noor et al., established the following results regarding the Simpson's second formula

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\xi(f)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi(g)\right)-\tfrac{1}{g-f}\int\limits_f^g\xi(u)\,du\right|\\&\leq(g-f)\left(\tfrac{17}{756}\left(\tfrac{973|\xi'(f)|^q+251|\xi'(g)|^q}{1224}\right)^{\frac{1}{q}}\right.\\&\quad\left.+\tfrac{1}{36}\left(\tfrac{|\xi'(f)|^q+|\xi'(g)|^q}{2}\right)^{\frac{1}{q}}+\tfrac{17}{756}\left(\tfrac{251|\xi'(f)|^q+973|\xi'(g)|^q}{1224}\right)^{\frac{1}{q}}\right),\end{aligned}$$

where $|\xi'|^q$ is convex with $q\geq1$. Furthermore, they demonstrated the following estimations for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\xi(f)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi(g)\right)-\tfrac{1}{g-f}\int\limits_f^g\xi(u)\,du\right|\\&\leq(g-f)\left(\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{|\xi'(f)|^q+|\xi'(\frac{2f+g}{3})|^q}{6}\right)^{\frac{1}{q}}\right.\\&\quad+\left(\tfrac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{|\xi'(\frac{2f+g}{3})|^q+|\xi'(\frac{f+2g}{3})|^q}{6}\right)^{\frac{1}{q}}\\&\quad\left.+\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{|\xi'(\frac{f+2g}{3})|^q+|\xi'(g)|^q}{6}\right)^{\frac{1}{q}}\right)\end{aligned}$$

and

$$\left|\frac{1}{8}\left(\xi\left(f\right)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi\left(g\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$
$$\leq\left(g-f\right)\left(\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5\left|\xi'(f)\right|^{q}+\left|\xi'(g)\right|^{q}}{18}\right)^{\frac{1}{q}}+\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^{q}+\left|\xi'(g)\right|^{q}}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^{q}+5\left|\xi'(g)\right|^{q}}{18}\right)^{\frac{1}{q}}\right).$$

In [16], Laribi and Meftah, gave several Simpson's second formula type inequalities for s-convex derivatives as follows:

$$\left|\frac{1}{8}\left(\xi\left(f\right)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi\left(g\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9(s+1)(s+2)}\left(\left(\frac{3s-2}{8}+2\left(\frac{5}{8}\right)^{s+2}\right)\left(\left|\xi'\left(f\right)\right|+\left|\xi'\left(g\right)\right|\right)\right.$$
$$\left.+\left(\frac{9s+2}{8}+2\left(\frac{3}{8}\right)^{s+2}+\left(\frac{1}{2}\right)^{s+1}\right)\left(\left|\xi'\left(\frac{2f+g}{3}\right)\right|+\left|\xi'\left(\frac{f+2g}{3}\right)\right|\right)\right),$$

and for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\frac{1}{8}\left(\xi\left(f\right)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi\left(g\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9(p+1)^{\frac{1}{p}}}\left(\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^{q}+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{2}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}+\left|\xi'(g)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right).$$

Recently, Meftah et al. [19], investigated the following 2-point left-Radau type inequalities.

$$\left|\frac{1}{4}\left(\xi\left(f\right)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{3s-2}{(s+1)(s+2)}+\frac{10}{(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)\left|\xi'\left(f\right)\right|\right.$$
$$\left.+\left(\frac{7s+4}{(s+1)(s+2)}+\frac{6}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+1}\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\frac{2}{(s+1)(s+2)}\left|\xi'\left(g\right)\right|\right)$$

and

$$\left|\frac{1}{4}\left(\xi\left(f\right)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$

$$\leq \frac{g-f}{9(p+1)^{\frac{1}{p}}}\left(\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Fractional calculus emerges as a prominent field in mathematical analysis, stemming from the traditional definitions of integral and derivative operators but extended to non-integer orders. This discipline offers a valuable framework to characterize memory and hereditary traits within diverse materials and processes. During the past decade, numerous fractional counterparts of integral inequalities have been effectively introduced, as evidenced by notable works such as [9, 10, 13, 14, 17, 23, 24, 28–30, 32, 34, 35].

Definition 1.1 ([15]). Let $\xi \in L^1[f,g]$. The Riemann-Liouville fractional integrals $I^{\gamma}_{f^+}\xi$ and $I^{\gamma}_{g^-}\xi$ of order $\gamma > 0$ with $f \geq 0$ are defined by

$$I^{\gamma}_{f^+}\xi(x) = \frac{1}{\Gamma(\gamma)}\int_f^x (x-t)^{\gamma-1}\xi(t)dt, \qquad x > f,$$

$$I^{\gamma}_{g^-}\xi(x) = \frac{1}{\Gamma(\gamma)}\int_x^g (t-x)^{\gamma-1}\xi(t)dt, \qquad g > x,$$

respectively, where $\Gamma(\gamma) = \int_0^\infty e^{-t}t^{\gamma-1}dt$ is the gamma function and $I^0_{f^+}\xi(x) = I^0_{g^-}\xi(x) = \xi(x)$.

Recently, Sitthiwirattham et al. [31], established the following fractional Simpson's second formula inequalities for functions with convex derivatives

$$\begin{aligned}&\Bigg|\frac{1}{8}\left(\xi(f)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi(g)\right)\\&-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\left(I^{\gamma}_{f^+}\xi\left(\frac{2f+g}{3}\right)+I^{\gamma}_{\frac{2f+g}{3}^+}f\left(\frac{f+2g}{3}\right)+I^{\gamma}_{\frac{f+2g}{3}^+}f(g)\right)\Bigg|\\ \leq &\frac{g-f}{27}\left(\left|\xi'(g)\right|\left(3\mathcal{A}_2(\gamma)-\mathcal{A}_1(\gamma)+2\mathcal{A}_4(\gamma)-\mathcal{A}_3(\gamma)+\mathcal{A}_6(\gamma)-\mathcal{A}_5(\gamma)\right)\right.\\&\left.\left|\xi'(f)\right|\left(\mathcal{A}_1(\gamma)+\mathcal{A}_4(\gamma)+\mathcal{A}_3(\gamma)+2\mathcal{A}_6(\gamma)+\mathcal{A}_5(\gamma)\right)\right),\end{aligned}$$

where

$$\mathcal{A}_1(\gamma) = \frac{\gamma}{\gamma+2}\left(\frac{3}{8}\right)^{\frac{\gamma+2}{\gamma}} + \frac{1}{\gamma+2} - \frac{3}{16},$$

$$\mathcal{A}_2(\gamma) = \frac{2\gamma}{\gamma+1}\left(\frac{3}{8}\right)^{\frac{\gamma+1}{\gamma}} + \frac{1}{\gamma+1} - \frac{3}{8},$$

$$\mathcal{A}_3(\gamma) = \frac{\gamma}{\gamma+2}\left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma}} + \frac{1}{\gamma+2} - \frac{1}{4},$$

$$\mathcal{A}_4(\gamma) = \frac{2\gamma}{\gamma+1}\left(\frac{1}{2}\right)^{\frac{\gamma+1}{\gamma}} + \frac{1}{\gamma+1} - \frac{1}{2},$$

$$\mathcal{A}_5(\gamma) = \frac{\gamma}{\gamma+2}\left(\frac{5}{8}\right)^{\frac{\gamma+2}{\gamma}} + \frac{1}{\gamma+2} - \frac{5}{16}$$

and

$$\mathcal{A}_6(\gamma) = \frac{2\gamma}{\gamma+1}\left(\frac{5}{8}\right)^{\frac{\gamma+1}{\gamma}} + \frac{1}{\gamma+1} - \frac{5}{8}.$$

They also established the following inequalities for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\Bigg| \frac{1}{8}\left(\xi(f) + 3\xi\left(\frac{2f+g}{3}\right) + 3\xi\left(\frac{f+2g}{3}\right) + \xi(g)\right) \\
&- \frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\left(I^{\gamma}_{f^+}\xi\left(\frac{2f+g}{3}\right) + I^{\gamma}_{\frac{2f+g}{3}^+} f\left(\frac{f+2g}{3}\right) + I^{\gamma}_{\frac{f+2g}{3}^+} f(g)\right)\Bigg| \\
\leq & \frac{g-f}{27}\left(\left(\int_0^1 \left|u^{\gamma} - \tfrac{3}{8}\right|^p du\right)^{\frac{1}{p}} \left(\frac{5|\xi'(g)|^q + |\xi'(f)|^q}{6}\right)^{\frac{1}{q}} + \left(\int_0^1 \left|u^{\gamma} - \tfrac{1}{2}\right|^p du\right)^{\frac{1}{p}}\right. \\
&\left. \times \left(\frac{|\xi'(g)|^q + |\xi'(f)|^q}{2}\right)^{\frac{1}{q}} + \left(\int_0^1 \left|u^{\gamma} - \tfrac{5}{8}\right|^p du\right)^{\frac{1}{p}} \left(\frac{|\xi'(g)|^q + 5|\xi'(f)|^q}{6}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Inspired by the aforementioned findings, our study embarks on introducing a novel fractional identity characterized by two parameters. This identity reclaims Simpson's second formula and also encapsulates left- and right-Radau 2-point rules. Leveraging this identity, we formulate a series of bi-parametrized integral inequalities for functions whose first derivatives are extended s-convex in the second sense. Furthermore, we derive specific instances of these inequalities for the various formulas mentioned earlier and for different classes of functions.

2. Auxiliary results

In this section, we will review certain special functions detailed in [15] and present results that will be employed later in our study.

Definition 2.1. For any complex numbers x, y such that $Re(x), Re(y) > 0$. The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. For any complex numbers x, y such that $Re(x), Re(y) > 0$. The incomplete beta function is given by

$$B_a(x,y) = \int_0^a t^{x-1}(1-t)^{y-1} dt, \quad 0 \leq a < 1.$$

Definition 2.3. The hypergeometric function is defined for $Re(c) > Re(b) > 0$ and $|z| < 1$, as follows

$${}_2F_1(a,b,c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

where $B(.,.)$ is the beta function.

Remark 2.1. If $a=0$, then ${}_2F_1(0,b,c;z)=1$ for all b,c,z as in Definition 2.3.

Remark 2.2. If $c=2b=2$, the ${}_2F_1(a,1,2;z)=\frac{1-(1-z)^{1-a}}{z(1-a)}$ for all a,z as in Definition 2.2 with $a\neq 1$ and $z\neq 0$.

The following lemma plays a key role in establishing the results of this work.

Lemma 2.1. *Let $\xi: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I, $f,g\in I$ with $f<g$, and $\xi'\in L^1[f,g]$. Then for all real numbers $\gamma>0$ and $\lambda,\vartheta\geq 0$ with $\lambda+\vartheta\neq 0$, the following equality holds*

$$\begin{aligned}
&\tfrac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\tfrac{2f+g}{3}\right)+3\lambda\xi\left(\tfrac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\tfrac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_\gamma(f,g,\xi)\\
&=\tfrac{g-f}{9}\left(\int_0^1\left(t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)f+t\tfrac{2f+g}{3}\right)dt\right.\\
&\quad-\int_0^1\left((1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)\tfrac{2f+g}{3}+t\tfrac{f+2g}{3}\right)dt\\
&\quad\left.+\int_0^1\left(t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)\tfrac{f+2g}{3}+tg\right)dt\right),
\end{aligned}$$

where

$$\mathcal{K}_\gamma(f,g,\xi)=I^\gamma_{\frac{2f+g}{3}^-}\xi(f)+I^\gamma_{\frac{2f+g}{3}^+}\xi\left(\tfrac{f+2g}{3}\right)+I^\gamma_{g^-}\xi\left(\tfrac{f+2g}{3}\right).$$

Proof. Let

$$\mathcal{I}=\mathcal{I}_1-\mathcal{I}_2+\mathcal{I}_3, \tag{2.1}$$

where

$$\mathcal{I}_1=\int_0^1\left(t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)f+t\tfrac{2f+g}{3}\right)dt,$$

$$\mathcal{I}_2=\int_0^1\left((1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)\tfrac{2f+g}{3}+t\tfrac{f+2g}{3}\right)dt$$

and

$$\mathcal{I}_3=\int_0^1\left(t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\xi'\left((1-t)\tfrac{f+2g}{3}+tg\right)dt.$$

Integrating by parts $\mathcal{I}_1$, we get

$$\begin{aligned}
\mathcal{I}_1&=\tfrac{3}{g-f}\left(t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)\xi\left((1-t)f+t\tfrac{2f+g}{3}\right)\Big|_0^1\\
&\quad-\tfrac{3\gamma}{g-f}\int_0^1 t^{\gamma-1}\xi\left((1-t)f+t\tfrac{2f+g}{3}\right)dt\\
&=\tfrac{3(\lambda+4\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\tfrac{2f+g}{3}\right)+\tfrac{9\lambda}{4(\lambda+\vartheta)(g-f)}\xi(f)-\tfrac{3^{\gamma+1}\gamma}{(g-f)^{\gamma+1}}\int_f^{\frac{2f+g}{3}}(u-f)^{\gamma-1}\xi(u)\,du
\end{aligned}$$

$$=\frac{3(\lambda+4\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{2f+g}{3}\right)+\frac{9\lambda}{4(\lambda+\vartheta)(g-f)}\xi\left(f\right)-\frac{3^{\gamma+1}\Gamma(\gamma+1)}{(g-f)^{\gamma+1}}I^{\gamma}_{\frac{2f+g}{3}-}\xi(f). \tag{2.2}$$

Similarly, we obtain

$$\begin{aligned}\mathcal{I}_2 &= \frac{3}{g-f}\left((1-t)^{\gamma}-\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\xi\left((1-t)\frac{2f+g}{3}+t\frac{f+2g}{3}\right)\Big|_0^1\\ &\quad+\frac{3\gamma}{g-f}\int_0^1(1-t)^{\gamma-1}\xi\left((1-t)\frac{2f+g}{3}+t\frac{f+2g}{3}\right)dt\\ &=-\frac{3(5\lambda-\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{f+2g}{3}\right)-\frac{3(5\vartheta-\lambda)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{2f+g}{3}\right)\\ &\quad+\frac{3^{\gamma+1}\gamma}{(g-f)^{\gamma+1}}\int_{\frac{2f+g}{3}}^{\frac{f+2g}{3}}\left(\frac{f+2g}{3}-u\right)^{\gamma-1}\xi\left(u\right)du\\ &=-\frac{3(5\lambda-\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{f+2g}{3}\right)-\frac{3(5\vartheta-\lambda)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{2f+g}{3}\right)+\frac{3^{\gamma+1}\Gamma(\gamma+1)}{(g-f)^{\gamma+1}}I^{\gamma}_{\frac{2f+g}{3}+}f\left(\frac{f+2g}{3}\right)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathcal{I}_3 &= \frac{3}{g-f}\left(t^{\gamma}-\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\xi\left((1-t)\frac{f+2g}{3}+tg\right)\Big|_0^1\\ &\quad-\frac{3\gamma}{g-f}\int_0^1 t^{\gamma-1}\xi\left((1-t)\frac{f+2g}{3}+tg\right)dt\\ &=\frac{9\vartheta}{4(\lambda+\vartheta)(g-f)}\xi\left(g\right)+\frac{3(4\lambda+\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{f+2g}{3}\right)\\ &\quad-\frac{3^{\gamma+1}\gamma}{(g-f)^{\gamma+1}}\int_{\frac{f+2g}{3}}^{g}\left(u-\frac{f+2g}{3}\right)^{\gamma-1}\xi\left(u\right)du\\ &=\frac{9\vartheta}{4(\lambda+\vartheta)(g-f)}\xi\left(g\right)+\frac{3(4\lambda+\vartheta)}{4(\lambda+\vartheta)(g-f)}\xi\left(\frac{f+2g}{3}\right)-\frac{3^{\gamma+1}\Gamma(\gamma+1)}{(g-f)^{\gamma+1}}I^{\gamma}_{g-}f\left(\frac{f+2g}{3}\right).\end{aligned} \tag{2.4}$$

Using (2.2)-(2.4) in (2.1). Multiplying the resulting equality by $\frac{g-f}{9}$, we get the desired result. □

Now, we present additional results that will be utilized later in the proof of the results.

Lemma 2.2. *Let $l\in[0,1]$. For $\gamma>0$ and $s\in(-1,1]$, we have*

$$\Upsilon_1\left(\gamma,s,l\right)=\int_0^1\left|t^{\gamma}-l\right|(1-t)^{s}\,dt \tag{2.5}$$

$$=\frac{l}{s+1}\left(1-2\left(1-l^{\frac{1}{\gamma}}\right)^{s+1}\right)-B_{l^{\frac{1}{\gamma}}}\left(\gamma+1,s+1\right)+B_{1-l^{\frac{1}{\gamma}}}\left(s+1,\gamma+1\right).$$

Proof. We have

$$\Upsilon_1\left(\gamma,s,l\right)=\int_0^1\left|t^{\gamma}-l\right|(1-t)^{s}\,dt$$

$$
\begin{aligned}
&=\int_0^{l^{\frac{1}{\gamma}}} (l-t^{\gamma})(1-t)^s\,dt+\int_{l^{\frac{1}{\gamma}}}^1 (t^{\gamma}-l)(1-t)^s\,dt\\
&=l\int_0^{l^{\frac{1}{\gamma}}} (1-t)^s\,dt-\int_0^{l^{\frac{1}{\gamma}}} t^{\gamma}(1-t)^s\,dt+\int_{l^{\frac{1}{\gamma}}}^1 t^{\gamma}(1-t)^s\,dt-l\int_{l^{\frac{1}{\gamma}}}^1 (1-t)^s\,dt\\
&=l\int_0^{l^{\frac{1}{\gamma}}} (1-t)^s\,dt-\int_0^{l^{\frac{1}{\gamma}}} t^{\gamma}(1-t)^s\,dt+\int_0^{1-l^{\frac{1}{\gamma}}} (1-t)^{\gamma}t^s dt-l\int_{l^{\frac{1}{\gamma}}}^1 (1-t)^s\,dt\\
&=\tfrac{l}{s+1}\left(1-2\left(1-l^{\frac{1}{\gamma}}\right)^{s+1}\right)-B_{l^{\frac{1}{\gamma}}}(\gamma+1,s+1)+B_{1-l^{\frac{1}{\gamma}}}(s+1,\gamma+1).
\end{aligned}
$$

The proof is complete. □

Remark 2.3. From Lemma 2.2, we have

$$\Upsilon_1(\gamma,0,l)=\tfrac{2\gamma}{\gamma+1}l^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-l$$

and

$$\Upsilon_1(\gamma,1,l)=\tfrac{1}{(\gamma+1)(\gamma+2)}-\tfrac{l}{2}+\tfrac{2\gamma}{\gamma+1}l^{\frac{\gamma+1}{\gamma}}-\tfrac{\gamma}{\gamma+2}l^{\frac{\gamma+2}{\gamma}}.$$

Lemma 2.3. *Let $l\in[0,1]$. For $\gamma>0$ and $s\in(-1,1]$, we have*

$$\Upsilon_2(\gamma,s,l)=\int_0^1 |t^{\gamma}-l|\,t^s dt=\tfrac{2\gamma l^{\frac{\gamma+s+1}{\gamma}}}{(s+1)(\gamma+s+1)}+\tfrac{1}{\gamma+s+1}-\tfrac{l}{s+1}. \tag{2.6}$$

Proof. We have

$$
\begin{aligned}
\Upsilon_2(\gamma,s,l)&=\int_0^1 |t^{\gamma}-l|\,t^s dt=\int_0^{l^{\frac{1}{\gamma}}} (l-t^{\gamma})\,t^s dt+\int_{l^{\frac{1}{\gamma}}}^1 (t^{\gamma}-l)\,t^s dt\\
&=\int_0^{l^{\frac{1}{\gamma}}} \left(lt^s-t^{\gamma+s}\right)dt+\int_{l^{\frac{1}{\gamma}}}^1 \left(t^{\gamma+s}-lt^s\right)dt\\
&=\left(\tfrac{l}{s+1}t^{s+1}-\tfrac{1}{\gamma+s+1}t^{\gamma+s+1}\right)\Big|_0^{l^{\frac{1}{\gamma}}}+\left(\tfrac{1}{\gamma+s+1}t^{\gamma+s+1}-\tfrac{l}{s+1}t^{s+1}\right)\Big|_{l^{\frac{1}{\gamma}}}^1\\
&=\tfrac{2\gamma l^{\frac{\gamma+s+1}{\gamma}}}{(s+1)(\gamma+s+1)}+\tfrac{1}{\gamma+s+1}-\tfrac{l}{s+1}.
\end{aligned}
$$

The proof is complete. □

Remark 2.4. From Lemma 2.3, we have

$$\Upsilon_2(\gamma,0,l)=\tfrac{2\gamma}{\gamma+1}l^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-l$$

and

$$\Upsilon_2(\gamma,1,l)=\tfrac{\gamma}{\gamma+2}l^{\frac{\gamma+2}{\gamma}}+\tfrac{1}{\gamma+2}-\tfrac{l}{2}.$$

Lemma 2.4. *Let $l \in \mathbb{R}$. For $\gamma > 0$ and $s \in (-1, 1]$, we have*

$$\Psi_1(\gamma, s, l) = \int_0^1 |(1-t)^\gamma - l| (1-t)^s \, dt = \int_0^1 |t^\gamma - l| \, t^s dt$$

$$= \begin{cases} \frac{1}{s+1} l - \frac{1}{\gamma+s+1} & \text{for} \quad l > 1, \\ \Upsilon_2(\gamma, s, l) & \text{for} \quad 0 \le l \le 1, \\ \frac{1}{\gamma+s+1} - \frac{1}{s+1} l & \text{for} \quad l < 0. \end{cases} \tag{2.7}$$

Proof. For $l > 1$

$$\Psi_1(\gamma, s, l) = \int_0^1 |(1-t)^\gamma - l| (1-t)^s \, dt = \int_0^1 |t^\gamma - l| \, t^s dt$$
$$= \int_0^1 (l - t^\gamma) t^s dt = \int_0^1 \left(l t^s - t^{\gamma+s}\right) dt = \frac{1}{s+1} l - \frac{1}{\gamma+s+1}.$$

For $0 < l < 1$

$$\Psi_1(\gamma, s, l) = \Upsilon_2(\gamma, s, l).$$

For $l \le 0$

$$\Psi_1(\gamma, s, l) = \int_0^1 |(1-t)^\gamma - l| (1-t)^s \, dt = \int_0^1 |t^\gamma - l| \, t^s dt$$
$$= \int_0^1 (t^\gamma - l) t^s dt = \int_0^1 \left(t^{\gamma+s} - l t^s\right) dt$$
$$= \left(\frac{1}{\gamma+s+1} t^{\gamma+s+1} - \frac{1}{s+1} l t^{s+1} \right) \Big|_0^1$$
$$= \frac{1}{\gamma+s+1} - \frac{1}{s+1} l.$$

The proof is completed. □

Remark 2.5. From Lemma 2.4, we have

$$\Psi_1(\gamma, 0, l) = \begin{cases} l - \frac{1}{\gamma+1} & \text{for} \quad l > 1, \\ \Upsilon_2(\gamma, 0, l) & \text{for} \quad 0 \le l \le 1, \\ \frac{1}{\gamma+1} - l & \text{for} \quad l < 0 \end{cases}$$

and

$$\Psi_1(\gamma, 1, l) = \begin{cases} \frac{1}{2} l - \frac{1}{\gamma+2} & \text{for} \quad l > 1, \\ \Upsilon_2(\gamma, 1, l) & \text{for} \quad 0 \le l \le 1, \\ \frac{1}{\gamma+2} - \frac{1}{2} l & \text{for} \quad l < 0. \end{cases}$$

Lemma 2.5. *Let $l \in \mathbb{R}$. For $\gamma > 0$ and $s \in (-1, 1]$, we have*

$$\Psi_2(\gamma, s, l) = \int_0^1 |(1-t)^\gamma - l|\, t^s dt = \begin{cases} \frac{1}{s+1} l - B(s+1, \gamma+1) & \text{for} \quad l > 1, \\ \Upsilon_1(\gamma, s, l) & \text{for} \quad 0 \le l \le 1, \\ B(s+1, \gamma+1) - \frac{1}{s+1} l & \text{for} \quad l < 0. \end{cases} \tag{2.8}$$

Proof. For $l \geq 1$

$$\begin{aligned} \Psi_2(\gamma, s, l) &= \int_0^1 |(1-t)^\gamma - l|\, t^s dt = \int_0^1 (l - (1-t)^\gamma)\, t^s dt \\ &= \int_0^1 (l t^s - (1-t)^\gamma t^s)\, dt = \tfrac{1}{s+1} l - B(s+1, \gamma+1). \end{aligned}$$

For $0 < l < 1$

$$\Psi_2(\gamma, s, l) = \int_0^1 |(1-t)^\gamma - l|\, t^s dt = \int_0^1 |t^\gamma - l|\, (1-t)^s\, dt = \Upsilon_1(\gamma, s, l).$$

For $l \leq 0$

$$\begin{aligned} \Psi_2(\gamma, s, l) &= \int_0^1 |(1-t)^\gamma - l|\, t^s dt \\ &= \int_0^1 ((1-t)^\gamma - l)\, t^s dt = \int_0^1 (t^s (1-t)^\gamma - l t^s)\, dt \\ &= B(s+1, \gamma+1) - \tfrac{1}{s+1} l. \end{aligned}$$

The proof is completed. □

Remark 2.6. From Lemma 2.5, we have

$$\Psi_2(\gamma, 0, l) = \begin{cases} l - \frac{1}{\gamma+1} & \text{for} \quad l > 1, \\ \Upsilon_1(\gamma, 0, l) & \text{for} \quad 0 \le l \le 1, \\ \frac{1}{\gamma+1} - l & \text{for} \quad l < 0 \end{cases}$$

and

$$\Psi_2(\gamma, 1, l) = \begin{cases} \frac{1}{2} l - \frac{1}{(\gamma+1)(\gamma+2)} & \text{for} \quad l > 1, \\ \Upsilon_1(\gamma, 1, l) & \text{for} \quad 0 \le l \le 1, \\ \frac{1}{(\gamma+1)(\gamma+2)} - \frac{1}{2} l & \text{for} \quad l < 0. \end{cases}$$

Lemma 2.6. *Let $l \in \mathbb{R}$. For $\gamma > 0$ and $p > 1$, we have*

$$\Delta(\gamma, p, l) = \int_0^1 |t^\gamma - l|^p \, dt \tag{2.9}$$

$$= \begin{cases} l^p \, {}_2F_1\left(-p, \frac{1}{\gamma}, \frac{1}{\gamma}+1; \frac{1}{l}\right) & \text{for} \quad l \geq 1, \\ \frac{l^{p+\frac{1}{\gamma}} B\left(\frac{1}{\gamma}, p+1\right)}{\gamma} + \frac{(1-l)^{p+1} \, {}_2F_1\left(1-\frac{1}{\gamma}, 1, p+2; (1-l)\right)}{\gamma(p+1)} & \text{for } 0 < l < 1, \\ (1-l)^p \, {}_2F_1\left(-p, 1, \frac{1}{\gamma}+1; \frac{1}{1-l}\right) & \text{for} \quad l \leq 0. \end{cases}$$

Proof. For $l \geq 1$

$$\begin{aligned} \int_0^1 |t^\gamma - l|^p \, dt &= \int_0^1 (l - t^\gamma)^p \, dt = \frac{1}{\gamma} \int_0^1 (l-u)^p \, u^{\frac{1}{\gamma}-1} du \\ &= \frac{l^p}{\gamma} \int_0^1 u^{\frac{1}{\gamma}-1} \left(1 - \frac{1}{l} u\right)^p du \\ &= \frac{l^p}{\gamma} \, {}_2F_1\left(-p, \frac{1}{\gamma}, \frac{1}{\gamma}+1; \frac{1}{l}\right) B\left(\frac{1}{\gamma}, 1\right) \\ &= l^p \, {}_2F_1\left(-p, \frac{1}{\gamma}, \frac{1}{\gamma}+1; \frac{1}{l}\right). \end{aligned}$$

For $0 < l < 1$

$$\begin{aligned} \int_0^1 |t^\gamma - l|^p \, dt &= \int_0^{l^{\frac{1}{\gamma}}} (l - t^\gamma)^p \, dt + \int_{l^{\frac{1}{\gamma}}}^1 (t^\gamma - l)^p \, dt \\ &= \frac{1}{\gamma} \int_0^l (l-u)^p \, u^{\frac{1}{\gamma}-1} du + \frac{1}{\gamma} \int_l^1 (u-l)^p \, u^{\frac{1}{\gamma}-1} du \\ &= \frac{1}{\gamma} l^{p+\frac{1}{\gamma}} \int_0^1 (1-y)^p \, y^{\frac{1}{\gamma}-1} dy + \frac{(1-l)^{p+1}}{\gamma} \int_0^1 (1-y)^p \left(1 - (1-l)\, y\right)^{\frac{1}{\gamma}-1} dy \\ &= \frac{l^{p+\frac{1}{\gamma}} B\left(\frac{1}{\gamma}, p+1\right)}{\gamma} + \frac{(1-l)^{p+1} \, {}_2F_1\left(1-\frac{1}{\gamma}, 1, p+2; 1-l\right)}{\gamma(p+1)}. \end{aligned}$$

For $l \leq 0$

$$\begin{aligned} \int_0^1 |t^\gamma - l|^p \, dt &= \int_0^1 (t^\gamma - l)^p \, dt = \frac{1}{\gamma} \int_0^1 (u-l)^p \, u^{\frac{1}{\gamma}-1} du \\ &= \frac{1}{\gamma} \int_0^1 \left((1-l) - u\right)^p (1-u)^{\frac{1}{\gamma}-1} du \\ &= \frac{(1-l)^p}{\gamma} \int_0^1 (1-u)^{\frac{1}{\gamma}-1} \left(1 - \frac{1}{1-l} u\right)^p du \end{aligned}$$

$$=\frac{(1-l)^p}{\gamma}\ {}_2F_1\left(-p,1,\tfrac{1}{\gamma}+1;\tfrac{1}{1-l}\right)B\left(\tfrac{1}{\gamma},1\right)$$
$$=(1-l)^p\ {}_2F_1\left(-p,1,\tfrac{1}{\gamma}+1;\tfrac{1}{1-l}\right).$$

The proof is completed. □

3. Main results

Theorem 3.1. *Let ξ be in accordance with the assumptions of Lemma 2.1. If $|\xi'|$ is extended s-convex in the second sense with $s\in(-1,1]$, then we have*

$$\begin{aligned}
&\left|\tfrac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\tfrac{2f+g}{3}\right)+3\lambda\xi\left(\tfrac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\tfrac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^\gamma}\mathcal{K}_\gamma(f,g,\xi)\right|\\
&\leq\tfrac{g-f}{9}\left(\Upsilon_1\left(\gamma,s,\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)|\xi'(f)|+\Upsilon_2\left(\gamma,s,\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)|\xi'(g)|\right.\\
&\quad+\left(\Upsilon_2\left(\gamma,s,\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)+\Psi_1\left(\gamma,s,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|\\
&\quad\left.+\left(\Psi_2\left(\gamma,s,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)+\Upsilon_1\left(\gamma,s,\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|\right),
\end{aligned}$$

where $\lambda,\vartheta\geq0$ with $\lambda+\vartheta\neq0$ and $\Upsilon_1,\Upsilon_2,\Psi_1$ and Ψ_2 are as defined in (2.5)-(2.8), *respectively.*

Proof. From Lemma 2.1, modulus and extended s-convexity of $|\xi'|$, we have

$$\begin{aligned}
&\left|\tfrac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\tfrac{2f+g}{3}\right)+3\lambda\xi\left(\tfrac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\tfrac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^\gamma}\mathcal{K}_\gamma(f,g,\xi)\right|\\
&\leq\tfrac{g-f}{9}\left(\int_0^1\left|t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right|\left|\xi'\left((1-t)f+t\tfrac{2f+g}{3}\right)\right|dt\right.\\
&\quad+\int_0^1\left|\left((1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right|\left|\xi'\left((1-t)\tfrac{2f+g}{3}+t\tfrac{f+2g}{3}\right)\right|dt\\
&\quad\left.+\int_0^1\left|t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|\left|\xi'\left((1-t)\tfrac{f+2g}{3}+tg\right)\right|dt\right)\\
&\leq\tfrac{g-f}{9}\left(\int_0^1\left|t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right|\left((1-t)^s|\xi'(f)|+t^s\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|\right)dt\right.\\
&\quad+\int_0^1\left|(1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right|\left((1-t)^s\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|+t^s\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|\right)dt\\
&\quad\left.+\int_0^1\left|t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|\left((1-t)^s\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|+t^s|\xi'(g)|\right)dt\right)\\
&=\tfrac{g-f}{9}\left(\left(\int_0^1\left|t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right|(1-t)^s\,dt\right)|\xi'(f)|+\left(\int_0^1\left|t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|t^s dt\right)|\xi'(g)|\right.
\end{aligned}$$

$$+\left(\int_0^1\left|t^\gamma-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right|t^s dt+\int_0^1\left|(1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right|(1-t)^s\,dt\right)\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|$$
$$+\left(\int_0^1\left|(1-t)^\gamma-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right|t^s dt+\int_0^1\left|t^\gamma-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|(1-t)^s\,dt\right)\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|\Bigg)$$
$$=\tfrac{g-f}{9}\Bigg(\Upsilon_1\left(\gamma,s,\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)|\xi'(f)|+\Upsilon_2\left(\gamma,s,\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)|\xi'(g)|$$
$$+\left(\Upsilon_2\left(\gamma,s,\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)+\Psi_1\left(\gamma,s,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|$$
$$+\left(\Psi_2\left(\gamma,s,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)+\Upsilon_1\left(\gamma,s,\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|\Bigg),$$

where $\Upsilon_1,\Upsilon_2,\Psi_1$ and Ψ_2 are as defined in (2.5)-(2.8), respectively. The proof is completed. □

Corollary 3.1. *Assume that all the assumptions of Theorem 3.1 hold. If $|\xi'|$ is a P-function, then we have*

$$\left|\tfrac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\tfrac{2f+g}{3}\right)+3\lambda\xi\left(\tfrac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\tfrac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^\gamma}\mathcal{K}_\gamma(f,g,\xi)\right|$$
$$\leq\frac{g-f}{9}\Bigg(\left(\tfrac{2\gamma}{\gamma+1}\left(\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)|\xi'(f)|$$
$$+\left(\left(\tfrac{2\gamma}{\gamma+1}\left(\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{3\lambda}{4(\lambda+\vartheta)}\right)+\Psi_1\left(\gamma,0,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{2f+g}{3}\right)\right|$$
$$+\left(\Psi_2\left(\gamma,0,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)+\left(\tfrac{2\gamma}{\gamma+1}\left(\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\tfrac{f+2g}{3}\right)\right|$$
$$+\left(\tfrac{2\gamma}{\gamma+1}\left(\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)|\xi'(g)|\Bigg),$$

where

$$\Psi_1\left(\gamma,0,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)=\begin{cases}\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}-\tfrac{1}{\gamma+1} & \text{for}\quad \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}>1,\\ \tfrac{2\gamma}{\gamma+1}\left(\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)} & \text{for}\quad 0\leq\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\leq 1,\\ \tfrac{1}{\gamma+1}-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)} & \text{for}\quad \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}<0\end{cases}$$

and

$$\Psi_2\left(\gamma,0,\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)=\begin{cases}\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}-\tfrac{1}{\gamma+1} & \text{for}\quad \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}>1,\\ \tfrac{2\gamma}{\gamma+1}\left(\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}+\tfrac{1}{\gamma+1}-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)} & \text{for}\quad 0\leq\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\leq 1,\\ \tfrac{1}{\gamma+1}-\tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)} & \text{for}\quad \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}<0.\end{cases}$$

Corollary 3.2. *Assume that all the assumptions of Theorem 3.1 are valid. If $|\xi'|$ is convex, then we have*

$$\left|\tfrac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\tfrac{2f+g}{3}\right)+3\lambda\xi\left(\tfrac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\tfrac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^\gamma}\mathcal{K}_\gamma(f,g,\xi)\right|$$

$$\le \frac{g-f}{9}\left(\left(\frac{1}{(\gamma+1)(\gamma+2)}-\frac{3\lambda}{8(\lambda+\vartheta)}+\frac{2\gamma}{\gamma+1}\left(\frac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}-\frac{\gamma}{\gamma+2}\left(\frac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}\right)\left|\xi'(f)\right|\right.$$
$$+\left(\frac{\gamma}{\gamma+2}\left(\frac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{3\lambda}{8(\lambda+\vartheta)}+\Psi_1\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|$$
$$+\left(\Psi_2\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)+\frac{1}{(\gamma+1)(\gamma+2)}-\frac{4\lambda+\vartheta}{8(\lambda+\vartheta)}+\frac{2\gamma}{\gamma+1}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}\right.$$
$$\left.\left.-\frac{\gamma}{\gamma+2}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\left(\frac{\gamma}{\gamma+2}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{4\lambda+\vartheta}{8(\lambda+\vartheta)}\right)\left|\xi'(g)\right|\right),$$

where

$$\Psi_1\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)=\begin{cases}\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)}-\frac{1}{\gamma+2} & \text{for } \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}>1,\\ \frac{\gamma}{\gamma+2}\left(\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)} & \text{for } 0\le\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\le 1,\\ \frac{1}{\gamma+2}-\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)} & \text{for } \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}<0\end{cases} \tag{3.1}$$

and

$$\Psi_2\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)$$
$$=\begin{cases}\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)}-\frac{1}{(\gamma+1)(\gamma+2)} & \text{for } \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}>1,\\ \frac{1}{(\gamma+1)(\gamma+2)}-\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)}+\frac{2\gamma}{\gamma+1}\left(\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}} & \\ -\frac{\gamma}{\gamma+2}\left(\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}} & \text{for } 0\le\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\le 1,\\ \frac{1}{(\gamma+1)(\gamma+2)}-\frac{5\lambda-\vartheta}{8(\lambda+\vartheta)} & \text{for } \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}<0.\end{cases} \tag{3.2}$$

Corollary 3.3. *In Theorem 3.1, if we take $\lambda=0$, we obtain the following right-Radau type inequality*

$$\left|\frac{1}{4}\left(3\xi\left(\frac{2f+g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|$$
$$\le\frac{g-f}{9}\left[B(s+1,\gamma+1)\left|\xi'(f)\right|+\left(\frac{2}{\gamma+s+1}+\frac{1}{4(s+1)}\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|+\left(B(s+1,\gamma+1)\right.\right.$$
$$\left.+\frac{\left(1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\gamma}}\right)^{s+1}\right)}{2(s+1)}-B_{\left(\frac{1}{4}\right)^{\frac{1}{\gamma}}}(\gamma+1,s+1)+B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\gamma}}}(s+1,\gamma+1)\right)$$
$$\left.\times\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\left(\frac{\gamma}{2^{\frac{\gamma+2s+2}{\gamma}}(s+1)(\gamma+s+1)}+\frac{1}{\gamma+s+1}-\frac{1}{4(s+1)}\right)\left|\xi'(g)\right|\right].$$

Corollary 3.4. *In Corollary 3.3, taking $s=0$ we get*

$$\left|\frac{1}{4}\left(3\xi\left(\frac{2f+g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|$$
$$\le\frac{g-f}{9}\left(\frac{1}{\gamma+1}\left|\xi'(f)\right|+\frac{9+\gamma}{4(\gamma+1)}\left|\xi'\left(\frac{2f+g}{3}\right)\right|\right.$$
$$\left.+\frac{\gamma+2^{\frac{2\gamma+2}{\gamma}}}{2^{\frac{\gamma+2}{\gamma}}(\gamma+1)}\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\frac{3\times2^{\frac{\gamma+2}{\gamma}}-\left(2^{\frac{\gamma+2}{\gamma}}-4\right)\gamma}{2^{\frac{3\gamma+2}{\gamma}}(\gamma+1)}\left|\xi'(g)\right|\right).$$

Corollary 3.5. *In Corollary 3.3, taking $s=1$ we get*

$$\begin{aligned}&\left|\frac{1}{4}\left(3\xi\left(\frac{2f+g}{3}\right)+\xi\left(g\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}\left(f,g,\xi\right)\right|\\ \leq&\frac{g-f}{9}\left(B\left(2,\gamma+1\right)\left|\xi'\left(f\right)\right|+\frac{18+\gamma}{8(\gamma+2)}\left|\xi'\left(\frac{2f+g}{3}\right)\right|\right.\\ &+\frac{2^{3+\frac{4}{\gamma}}+2^{\frac{\gamma+2}{\gamma}}\gamma(\gamma+2)-\gamma(\gamma+1)}{2^{2+\frac{4}{\gamma}}(\gamma+1)(\gamma+2)}\left|\xi'\left(\frac{f+2g}{3}\right)\right|\\ &\left.+\left(\frac{\gamma}{2^{\frac{2\gamma+4}{\gamma}}(\gamma+2)}-\frac{6-\gamma}{8(\gamma+2)}\right)\left|\xi'\left(g\right)\right|\right).\end{aligned}$$

Corollary 3.6. *In Theorem 3.1, if we take $\vartheta=0$, we obtain the following left-Radau type inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left(\xi\left(f\right)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}\left(f,g,\xi\right)\right|\\ \leq&\frac{g-f}{9}\left(\Upsilon_1\left(\gamma,s,\tfrac{3}{4}\right)\left|\xi'\left(f\right)\right|+\left(\frac{2\gamma}{(s+1)(\gamma+s+1)}\left(\frac{3}{4}\right)^{\frac{\gamma+s+1}{\gamma}}+\frac{1}{2(s+1)}\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|\right.\\ &\left.+\left(\frac{9}{4(s+1)}-2B\left(\gamma+1,s+1\right)\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\frac{\gamma}{(s+1)(\gamma+s+1)}\left|\xi'\left(g\right)\right|\right),\end{aligned}$$

where

$$\begin{aligned}\Upsilon_1\left(\gamma,s,\tfrac{3}{4}\right)=&\frac{3}{4(s+1)}\left(1-2\left(1-\left(\frac{3}{4}\right)^{\frac{1}{\gamma}}\right)^{s+1}\right)-B_{\left(\frac{3}{4}\right)^{\frac{1}{\gamma}}}\left(\gamma+1,s+1\right)\\ &+B_{1-\left(\frac{3}{4}\right)^{\frac{1}{\gamma}}}\left(s+1,\gamma+1\right).\end{aligned}$$

Remark 3.1. For $\gamma=1$, Corollary 3.6 represents a refinement of the result obtained in Theorem 1 from [19], given that the latter can be deduced using the s-convexity of $|\xi'|$, i.e. $\left|\xi'\left(\frac{2f+g}{3}\right)\right|\leq\left(\frac{2}{3}\right)^{s}\left|\xi'\left(f\right)\right|+\left(\frac{1}{3}\right)^{s}\left|\xi'\left(g\right)\right|$, and this holds for $s\in(0,1]$.

Corollary 3.7. *In Corollary 3.6, taking $s=0$ we get*

$$\begin{aligned}&\left|\frac{1}{4}\left(\xi\left(f\right)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}\left(f,g,\xi\right)\right|\\ \leq&\frac{g-f}{9}\left(\left(\frac{1-3\gamma}{4(\gamma+1)}+\frac{2\gamma}{\gamma+1}\left(\frac{3}{4}\right)^{\frac{\gamma+1}{\gamma}}\right)\left|\xi'\left(f\right)\right|+\frac{\gamma}{\gamma+1}\left|\xi'\left(g\right)\right|\right.\\ &\left.+\left(\frac{1}{2}+\frac{2\gamma}{\gamma+1}\left(\frac{3}{4}\right)^{\frac{\gamma+1}{\gamma}}\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|+\frac{9\gamma+1}{4(\gamma+1)}\left|\xi'\left(\frac{f+2g}{3}\right)\right|\right).\end{aligned}$$

Corollary 3.8. *In Corollary 3.6, taking $s=1$ we get*

$$\begin{aligned}&\left|\frac{1}{4}\left(\xi\left(f\right)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}\left(f,g,\xi\right)\right|\\ \leq&\frac{g-f}{9}\left(\left(\frac{8-3(\gamma+1)(\gamma+2)}{8(\gamma+1)(\gamma+2)}+\frac{2\gamma}{\gamma+1}\left(\frac{3}{4}\right)^{\frac{\gamma+1}{\gamma}}-\frac{\gamma}{\gamma+2}\left(\frac{3}{4}\right)^{\frac{\gamma+2}{\gamma}}\right)\left|\xi'\left(f\right)\right|+\frac{\gamma}{2(\gamma+2)}\left|\xi'\left(g\right)\right|\right.\\ &\left.+\left(\frac{1}{4}+\frac{\gamma}{\gamma+2}\left(\frac{3}{4}\right)^{\frac{\gamma+2}{\gamma}}\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|+\left(\frac{9}{8}-\frac{2}{(\gamma+1)(\gamma+2)}\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|\right).\end{aligned}$$

Remark 3.2. In Corollary 3.8, taking $\gamma=1$ and using the fact that $\left|\xi'\left(\frac{2f+g}{3}\right)\right|\leq\frac{2}{3}\left|\xi'\left(f\right)\right|+\frac{1}{3}\left|\xi'\left(g\right)\right|$ (convexity of $|\xi'|$), we obtain Corollary 1 from [19].

Corollary 3.9. *In Theorem 3.1, if we take $\lambda = \vartheta$, we obtain the following Simpson second formula*

$$\begin{aligned}&\left|\frac{1}{8}\left(\xi(f)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|\\ \leq&\frac{g-f}{9}\left(\Upsilon_1\left(\gamma,s,\frac{3}{8}\right)|\xi'(f)|+\left(\Upsilon_2\left(\gamma,s,\frac{3}{8}\right)+\Upsilon_2\left(\gamma,s,\frac{1}{2}\right)\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|\right.\\ &\left.+\left(\Upsilon_1\left(\gamma,s,\frac{1}{2}\right)+\Upsilon_1\left(\gamma,s,\frac{5}{8}\right)\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\Upsilon_2\left(\gamma,s,\frac{5}{8}\right)|\xi'(g)|\right),\end{aligned}$$

with Υ_1 and Υ_2 being defined as (2.5) *and* (2.6)*, respectively.*

Remark 3.3. For $s \in (0,1]$, Corollary 3.9 will be reduced to Theorem 2.1 from [16], if we take $\gamma = 1$.

Corollary 3.10. *In Corollary 3.9, taking $s = 0$ we get*

$$\begin{aligned}&\left|\frac{1}{8}\left(\xi(f)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|\\ \leq&\frac{g-f}{9(\gamma+1)}\left(\left(\frac{5-3\gamma}{8}+2\gamma\left(\frac{3}{8}\right)^{\frac{\gamma+1}{\gamma}}\right)|\xi'(f)|+\left(\frac{3-5\gamma}{8}+2\gamma\left(\frac{5}{8}\right)^{\frac{\gamma+1}{\gamma}}\right)|\xi'(g)|\right.\\ &+\left(\frac{9-7\gamma}{8}+2\gamma\left(\left(\frac{3}{8}\right)^{\frac{\gamma+1}{\gamma}}+\left(\frac{1}{2}\right)^{\frac{\gamma+1}{\gamma}}\right)\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|\\ &\left.+\left(\frac{7-9\gamma}{8}+2\gamma\left(\left(\frac{1}{2}\right)^{\frac{\gamma+1}{\gamma}}+\left(\frac{5}{8}\right)^{\frac{\gamma+1}{\gamma}}\right)\right)\left|\xi'\left(\frac{f+2g}{3}\right)\right|\right).\end{aligned}$$

Corollary 3.11. *In Corollary 3.9, taking $s = 1$ we get*

$$\begin{aligned}&\left|\frac{1}{8}\left(\xi(f)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|\\ \leq&\frac{g-f}{9}\left(\left(\frac{16-3(\gamma+1)(\gamma+2)}{16(\gamma+1)(\gamma+2)}+\frac{2\gamma}{\gamma+1}\left(\frac{3}{8}\right)^{\frac{\gamma+1}{\gamma}}-\frac{\gamma}{\gamma+2}\left(\frac{3}{8}\right)^{\frac{\gamma+2}{\gamma}}\right)|\xi'(f)|\right.\\ &+\left(\frac{18-7\gamma}{16(\gamma+2)}+\frac{\gamma}{\gamma+2}\left(\frac{3}{8}\right)^{\frac{\gamma+2}{\gamma}}+\frac{\gamma}{\gamma+2}\left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma}}\right)\left|\xi'\left(\frac{2f+g}{3}\right)\right|\\ &+\left(\frac{32-9(\gamma+1)(\gamma+2)}{16(\gamma+1)(\gamma+2)}+\frac{2\gamma}{\gamma+1}\left(\left(\frac{1}{2}\right)^{\frac{\gamma+1}{\gamma}}+\left(\frac{5}{8}\right)^{\frac{\gamma+1}{\gamma}}\right)-\frac{\gamma}{\gamma+2}\left(\left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma}}+\left(\frac{5}{8}\right)^{\frac{\gamma+2}{\gamma}}\right)\right)\\ &\left.\times\left|\xi'\left(\frac{f+2g}{3}\right)\right|+\left(\frac{6-5\gamma}{16(\gamma+2)}+\frac{\gamma}{\gamma+2}\left(\frac{5}{8}\right)^{\frac{\gamma+2}{\gamma}}\right)|\xi'(g)|\right).\end{aligned}$$

Remark 3.4. Corollary 3.11 will be reduced to Corollary 2.1 from [16], if we take $\gamma = 1$.

Remark 3.5. Corollary 3.11, will be reduced to Theorem 4 from [31], if we use the convexity of $|\xi'|$ i.e. $\left|\xi'\left(\frac{af+bg}{a+b}\right)\right| \leq \frac{a}{a+b}|\xi'(f)| + \frac{b}{a+b}|\xi'(g)|$. Moreover, if we take $\gamma = 1$, we recapture the result given in Remark 3 from [31].

Now, let us present some examples that demonstrate the precision of our findings through graphical representations, employing the Matlab software.

Example 3.1. Let's consider the function $\xi : [0,1] \to \mathbb{R}$, defined by $\xi(u) = u^2$. The derivative $\xi'(u) = 2u$ is convex on the interval $[0,1]$. From Corollary 3.2, we have

$$\left|\frac{1}{3}-\frac{11\gamma^2+27\gamma+16}{27(\gamma+1)(\gamma+2)}\right|$$

$$\leq \frac{2}{9}\left(\frac{1}{3}\left(\frac{\gamma}{\gamma+2}\left(\frac{3\lambda}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{3\lambda}{8(\lambda+\vartheta)}+\Psi_1\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\right.$$
$$+\frac{2}{3}\left(\Psi_2\left(\gamma,1,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)+\frac{1}{(\gamma+1)(\gamma+2)}-\frac{4\lambda+\vartheta}{8(\lambda+\vartheta)}+\frac{2\gamma}{\gamma+1}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+1}{\gamma}}\right.$$
$$\left.\left.-\frac{\gamma}{\gamma+2}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}\right)+\left(\frac{\gamma}{\gamma+2}\left(\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{4\lambda+\vartheta}{8(\lambda+\vartheta)}\right)\right),$$

where Ψ_1 and Ψ_2 are defined as (3.1) and (3.2), repectively.

We can graphically represent the result, which depends on three parameters, by fixing one parameter at a time. Let us look at the following situations:

Case 1: We can begin by equating λ to ϑ, which leads to the result of Simpson's second formula for convex derivatives. This is shown in detail in Table 1 and illustrated in Figure 1.

$$\left|\frac{1}{3}-\frac{11\gamma^2+27\gamma+16}{27(\gamma+1)(\gamma+2)}\right|$$
$$\leq \frac{2}{9}\left(\frac{1}{3}\left(\frac{18-7\gamma}{16(\gamma+2)}+\frac{\gamma}{\gamma+2}\left(\frac{3}{8}\right)^{\frac{\gamma+2}{\gamma}}+\frac{\gamma}{\gamma+2}\left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma}}\right)+\left(\frac{6-5\gamma}{16(\gamma+2)}+\frac{\gamma}{\gamma+2}\left(\frac{5}{8}\right)^{\frac{\gamma+2}{\gamma}}\right)\right.$$
$$\left.+\frac{2}{3}\left(\frac{32-9(\gamma+1)(\gamma+2)}{16(\gamma+1)(\gamma+2)}+\frac{2\gamma}{\gamma+1}\left(\frac{1}{2}\right)^{\frac{\gamma+1}{\gamma}}-\frac{\gamma}{\gamma+2}\left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma}}+\frac{2\gamma}{\gamma+1}\left(\frac{5}{8}\right)^{\frac{\gamma+1}{\gamma}}-\frac{\gamma}{\gamma+2}\left(\frac{5}{8}\right)^{\frac{\gamma+2}{\gamma}}\right)\right).$$

Case 2: If we set ϑ to zero, the left-Radau formula for convex derivatives is revealed. This is shown in Table 2 and Figure 2.

$$\left|\frac{1}{3}-\frac{11\gamma^2+27\gamma+16}{27(\gamma+1)(\gamma+2)}\right|$$
$$\leq \frac{2}{9}\left(\frac{1}{3}\left(\frac{\gamma}{\gamma+2}\left(\frac{3}{4}\right)^{\frac{\gamma+2}{\gamma}}+\frac{1}{\gamma+2}-\frac{3}{8}+\frac{5}{8}-\frac{1}{\gamma+2}\right)+\frac{2}{3}\left(\frac{1}{8}+\frac{\gamma(\gamma+3)}{(\gamma+1)(\gamma+2)}\right)+\frac{\gamma}{2(\gamma+2)}\right).$$

γ	Left-hand side	Right-hand side
0.10	0.0317	0.1196
0.20	0.0269	0.0991
0.30	0.0225	0.0875
0.40	0.0185	0.0817
0.50	0.0148	0.0795
0.60	0.0114	0.0793
0.70	0.0082	0.0804
0.80	0.0053	0.0822
0.90	0.0026	0.0844
1.00	0.0000	0.0868

Table 1. Numerical validation of Case 1

γ	Left hand-side	Right-hand side
0.10	0.0317	0.0622
0.20	0.0269	0.0833
0.30	0.0225	0.1016
0.40	0.0185	0.1177
0.50	0.0148	0.1319
0.60	0.0114	0.1445
0.70	0.0082	0.1558
0.80	0.0053	0.1659
0.90	0.0026	0.1750
1.00	0.0000	0.1833

Table 2. Numerical validation of Case 2

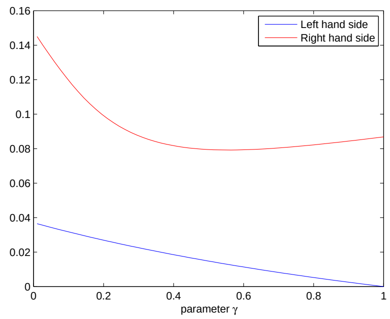

Figure 1. $\gamma \in (0,1]$

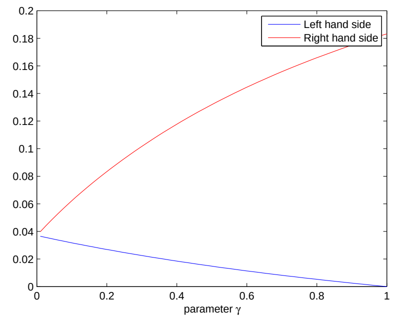

Figure 2. $\gamma \in (0,1]$

Example 3.2. Let us consider the function $\xi : [0,1] \to \mathbb{R}$, defined for some fixed $s \in (-1,1]$ by $\xi(u) = \frac{u^{s+1}}{s+1}$. The derivative $\xi'(u) = u^s$ is extended s-convex in the second sense on the interval $[0,1]$.

If we attempt to fix $\gamma = 1$, we obtain from Theorem 3.1:

$$\begin{aligned}&\left|\tfrac{1}{4(\lambda+\vartheta)(s+1)}\left(\vartheta\left(\tfrac{1}{3}\right)^s + 3\lambda\left(\tfrac{2}{3}\right)^{s+1} + \vartheta\right) - \tfrac{1}{(s+1)(s+2)}\right| \\ \le&\tfrac{1}{9}\left(\left(\Upsilon_2\left(1, s, \tfrac{3\lambda}{4(\lambda+\vartheta)}\right) + \Psi_1\left(1, s, \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)\left(\tfrac{1}{3}\right)^s\right. \\ &\left.+\left(\Psi_2\left(1, s, \tfrac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right) + \Upsilon_1\left(1, s, \tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\right)\left(\tfrac{2}{3}\right)^s + \Upsilon_2\left(1, s, \tfrac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right)\right),\end{aligned}$$

where $\lambda, \vartheta \ge 0$ with $\lambda + \vartheta \ne 0$ and $\Upsilon_1, \Upsilon_2, \Psi_1$ and Ψ_2 are as defined in (2.5)-(2.8), respectively.

Case 1: By setting $\lambda = \vartheta$, we obtain the following result regarding the Simpson's second formula for s-convex derivatives with $s \in (-1,1]$. This outcome is detailed in Table 3 and illustrated in Figure 3.

$$\begin{aligned}&\left|\tfrac{1}{8(s+1)}\left(\left(\tfrac{1}{3}\right)^s + 3\left(\tfrac{2}{3}\right)^{s+1} + 1\right) - \tfrac{1}{(s+1)(s+2)}\right| \\ \le&\tfrac{1}{9}\left(\left(\tfrac{9s+2}{8(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{3}{8}\right)^{s+1} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{1}{2}\right)^{s+1}\right)\left(\tfrac{1}{3}\right)^s\right. \\ &+\left(\tfrac{9s+2}{8(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{1}{2}\right)^{s+2} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{3}{8}\right)^{s+2}\right)\left(\tfrac{2}{3}\right)^s \\ &\left.+\tfrac{3s-2}{8(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{5}{8}\right)^{s+1}\right).\end{aligned}$$

Case 2: Now, if we set $\lambda = 0$, we obtain the result concerning the right-Radau formula for $s \in (-1,1]$ which is detailed in Table 4 and depicted in Figure 4.

$$\begin{aligned}&\left|\tfrac{1}{4(s+1)}\left(\left(\tfrac{1}{3}\right)^s + 1\right) - \tfrac{1}{(s+1)(s+2)}\right| \\ \le&\tfrac{1}{9(s+1)(s+2)}\left(\tfrac{9s+10}{4}\left(\tfrac{1}{3}\right)^s + \tfrac{9}{8}\left(\tfrac{1}{2}\right)^s + \tfrac{3s+2}{4} + \left(\tfrac{1}{2}\right)^{2s+1}\right).\end{aligned}$$

From the provided tables and figures, it is evident that the right-hand side is consistently greater than the left-hand side, thereby substantiating the accuracy of our results.

s	Left-hand side	Right-hand side
-0.80	0.3079	4.1733
-0.60	0.0720	1.3836
-0.40	0.0200	0.6481
-0.20	0.0047	0.3602
0.00	0.0000	0.2240
0.20	0.0011	0.1511
0.40	0.0010	0.1085
0.60	0.0007	0.0817
0.80	0.0003	0.0639
1.00	0.0000	0.0515

Table 3. Numerical validation of Case 1

s	Left hand-side	Right-hand side
-0.80	0.0936	2.3427
-0.60	0.0475	1.0173
-0.40	0.0216	0.5831
-0.20	0.0073	0.3747
0.00	0.0000	0.2569
0.20	0.0032	0.1842
0.40	0.0040	0.1366
0.60	0.0033	0.1042
0.80	0.0019	0.0815
1.00	0.0000	0.0652

Table 4. Numerical validation of Case 2

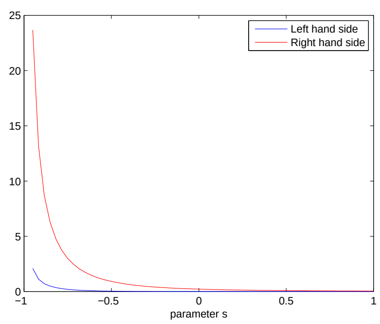

Figure 3. $s \in (-1, 1]$

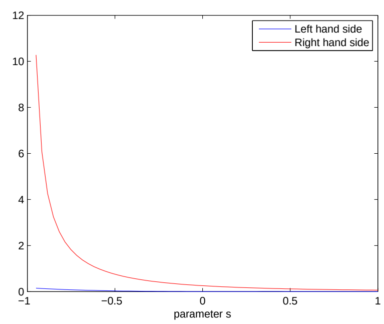

Figure 4. $s \in (-1, 1]$

Theorem 3.2. *Let ξ be as in Lemma 2.1. If $|\xi'|^q$ is extended s-convex in the second sense for some fixed $s \in (-1, 1]$ and $q > 1$ with $\frac{1}{q} + \frac{1}{p} = 1$, then we have*

$$\left|\frac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f) + 3\vartheta\xi\left(\frac{2f+g}{3}\right) + 3\lambda\xi\left(\frac{f+2g}{3}\right) + \vartheta\xi(g)\right) - \frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|$$
$$\leq \frac{g-f}{9}\left(\left(\frac{(3\lambda)^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma(4(\lambda+\vartheta))^{p+\frac{1}{\gamma}}} + \frac{(\lambda+4\vartheta)^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{\lambda+4\vartheta}{4(\lambda+\vartheta)}\right)}{\gamma(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{|\xi'(f)|^q + \left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\Delta\left(\gamma,p,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q + \left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$+\left(\frac{(4\lambda+\vartheta)^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma(4(\lambda+\vartheta))^{p+\frac{1}{\gamma}}} + \frac{(3\vartheta)^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{3\vartheta}{4(\lambda+\vartheta)}\right)}{\gamma(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg),$$

where Δ *is defined as* (2.9).

Proof. From Lemma 2.1, modulus, Hölder's inequality and extended s-convexity of $|\xi'|^q$, we have

$$\left|\frac{1}{4(\lambda+\vartheta)}\left(\lambda\xi(f)+3\vartheta\xi\left(\frac{2f+g}{3}\right)+3\lambda\xi\left(\frac{f+2g}{3}\right)+\vartheta\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^\gamma}\mathcal{K}_\gamma(f,g,\xi)\right|$$

$$\leq\frac{g-f}{9}\left(\left(\int_0^1\left|t^\gamma-\frac{3\lambda}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\xi'\left((1-t)f+t\frac{2f+g}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\left((1-t)^\gamma-\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\xi'\left((1-t)\frac{2f+g}{3}+t\frac{f+2g}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|t^\gamma-\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\xi'\left((1-t)\frac{f+2g}{3}+tg\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq\frac{g-f}{9}\left(\left(\int_0^1\left|t^\gamma-\frac{3\lambda}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|\xi'(f)\right|^q+t^s\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\left(t^\gamma-\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+t^s\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|t^\gamma-\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+t^s\left|\xi'(g)\right|^q\right)dt\right)\right)$$

$$=\frac{g-f}{9}\left(\left(\int_0^1\left|t^\gamma-\frac{3\lambda}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\left(t^\gamma-\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|t^\gamma-\frac{4\lambda+\vartheta}{4(\lambda+\vartheta)}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$=\frac{g-f}{9}\left(\left(\frac{(3\lambda)^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma(4(\lambda+\vartheta))^{p+\frac{1}{\gamma}}}+\frac{(\lambda+4\vartheta)^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{\lambda+4\vartheta}{4(\lambda+\vartheta)}\right)}{\gamma(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\Delta\left(\gamma,p,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{(4\lambda+\vartheta)^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma(4(\lambda+\vartheta))^{p+\frac{1}{\gamma}}}+\frac{(3\vartheta)^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{3\vartheta}{4(\lambda+\vartheta)}\right)}{\gamma(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg),$$

where we have used (2.9). The proof is completed. □

Corollary 3.12. *In Theorem 3.2, if we take $\gamma=1$, we obtain*

$$\left|\frac{1}{4(\lambda+\vartheta)}\left(\lambda\xi\left(f\right)+3\vartheta\xi\left(\frac{2f+g}{3}\right)+3\lambda\xi\left(\frac{f+2g}{3}\right)+\vartheta\xi\left(g\right)\right)-\frac{1}{g-f}\int_f^g\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{(3\lambda)^{p+1}+(\lambda+4\vartheta)^{p+1}}{(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$+\left(\Delta\left(1,p,\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\right)\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{(4\lambda+\vartheta)^{p+1}+(3\vartheta)^{p+1}.}{(p+1)(4(\lambda+\vartheta))^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

where

$$\Delta\left(1,p,l\right)=\begin{cases}\frac{(5\lambda-\vartheta)^{p+1}-(\lambda-5\vartheta)^{p+1}}{(p+1)(4(\lambda+\vartheta))^{p+1}} & if \quad \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\geq 1,\\ \frac{(5\lambda-\vartheta)^{p+1}+(5\vartheta-\lambda)^{p+1}}{(p+1)(4(\lambda+\vartheta))^{p+1}} & if\ 0<\frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}<1,\\ \frac{(5\vartheta-\lambda)^{p+1}-(\vartheta-5\lambda)^{p+1}}{(p+1)(4(\lambda+\vartheta))^{p+1}} & if \quad \frac{5\lambda-\vartheta}{4(\lambda+\vartheta)}\leq 0.\end{cases}$$

Corollary 3.13. *In Theorem 3.2, if we take $\lambda=0$, we obtain the following right-Radau type inequality*

$$\left|\frac{1}{4}\left(3\xi\left(\frac{2f+g}{3}\right)+\xi\left(g\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}\left(f,g,\xi\right)\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{1}{\gamma(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{5^p}{4^p}\,{}_2F_1\left(-p,1,\frac{1}{\gamma}+1;\frac{4}{5}\right)\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{B\left(\frac{1}{\gamma},p+1\right)}{4^{p+\frac{1}{\gamma}}\gamma}+\frac{3^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{3}{4}\right)}{4^{p+1}\gamma(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Corollary 3.14. *In Corollary 3.13, if we take $\gamma=1$, we obtain*

$$\left|\frac{1}{4}\left(3\xi\left(\frac{2f+g}{3}\right)+\xi\left(g\right)\right)-\frac{1}{g-f}\int_f^g\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}-1}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{1+3^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\Bigg).$$

Corollary 3.15. *In Theorem 3.2, if we take $\vartheta = 0$, we obtain the following left-Radau type inequality*

$$\left|\frac{1}{4}\left(\xi(f)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{3^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{4^{p+\frac{1}{\gamma}}\gamma}+\frac{{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{1}{4}\right)}{4^{p+1}\gamma(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{5^p}{4^p}\,{}_2F_1\left(-p,\frac{1}{\gamma},\frac{1}{\gamma}+1;\frac{4}{5}\right)\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{B\left(\frac{1}{\gamma},p+1\right)}{\gamma}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Corollary 3.16. *In Corollary 3.15, if we take $\gamma = 1$, we obtain*

$$\left|\frac{1}{4}\left(\xi(f)+3\xi\left(\frac{f+2g}{3}\right)\right)-\frac{1}{g-f}\int_f^g\xi(u)\,du\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{1+3^{p+1}}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}-1}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.$$
$$\left.\times\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Remark 3.6. In Corollary 3.16, using the fact that $\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q\leq\left(\frac{2}{3}\right)^s\left|\xi'(f)\right|^q+\left(\frac{1}{3}\right)^s\left|\xi'(g)\right|^q$ (the s-convexity of $|\xi'|^q$), we obtain Theorem 2 from [19]. Moreover, if we choose $s = 1$, we get Corollary 2 from [19].

Corollary 3.17. *In Theorem 3.2, if we take $\lambda = \vartheta$, we obtain the following Simpson second formula*

$$\left|\frac{1}{8}\left(\xi(f)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi(g)\right)-\frac{3^{\gamma-1}\Gamma(\gamma+1)}{(g-f)^{\gamma}}\mathcal{K}_{\gamma}(f,g,\xi)\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{3^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma 8^{p+\frac{1}{\gamma}}}+\frac{5^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{5}{8}\right)}{\gamma(p+1)8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'(f)\right|^q+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{B\left(\frac{1}{\gamma},p+1\right)}{2^{p+\frac{1}{\gamma}}\gamma}+\frac{{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{1}{2}\right)}{2^{p+1}\gamma(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^q+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{5^{p+\frac{1}{\gamma}}B\left(\frac{1}{\gamma},p+1\right)}{\gamma 8^{p+\frac{1}{\gamma}}}+\frac{3^{p+1}\,{}_2F_1\left(\frac{\gamma-1}{\gamma},1,p+2;\frac{3}{8}\right)}{\gamma(p+1)8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^q+\left|\xi'(g)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

Remark 3.7. In Corollary 3.17, taking $s = 1$ and using the convexity of $|\xi'|^q$ i.e. $\left|\xi'\left(\frac{af+bg}{a+b}\right)\right|^q\leq\frac{a}{a+b}\left|\xi'(f)\right|^q+\frac{b}{a+b}\left|\xi'(g)\right|^q$, we get Theorem 6 from [31].

Corollary 3.18. *In Corollary 3.17, if we take $\gamma=1$, we obtain*

$$\left|\frac{1}{8}\left(\xi\left(f\right)+3\xi\left(\frac{2f+g}{3}\right)+3\xi\left(\frac{f+2g}{3}\right)+\xi\left(g\right)\right)-\frac{1}{g-f}\int_{f}^{g}\xi\left(u\right)du\right|$$
$$\leq\frac{g-f}{9}\left(\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\xi'(f)\right|^{q}+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}+\left|\xi'(g)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\frac{1}{2^{p}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right),$$

which is the same result obtained in Theorem 2.2 from [16].

Remark 3.8. Corollary 3.18 will be reduced to Corollary 3.5. from [21], if we take $s=1$. Moreover, if we use the convexity of $|\xi'|^{q}$, we recapture Corollary 3.7 from [21].

4. Applications

For arbitrary real numbers f,g,h we have:

the arithmetic mean: $A\left(f,g\right)=\frac{f+g}{2}$ and $A\left(f,g,h\right)=\frac{f+g+h}{3}$;

the logarithmic means: $L\left(f,g\right)=\frac{g-f}{\ln g-\ln a}$, $f,g>0$ and $f\neq g$;

the p-logarithmic mean: $L_{p}\left(f,g\right)=\left(\frac{g^{p+1}-f^{p+1}}{(p+1)(g-f)}\right)^{\frac{1}{p}}$, $f,g>0,f\neq g$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 4.1. *Let $f,g\in\mathbb{R}$ with $1<f<g$. Then we have*

$$\left|3A^{-1}\left(f,f,g\right)+\frac{1}{g}-4L^{-1}\left(f,g\right)\right|$$
$$\leq\frac{g-f}{9}\left(\frac{2}{f^{2}}+\frac{45}{\left(2f+g\right)^{2}}+\frac{153}{4(f+2g)^{2}}+\frac{5}{4g^{2}}\right).$$

Proof. Applying Corollary 3.4 with $\gamma=1$ to the function $\xi\left(u\right)=\frac{1}{u}$ whose modulud of the first derivative $\left|\xi'\left(u\right)\right|=\frac{1}{u^{2}}$ is P-function. □

Proposition 4.2. *Let $f,g\in\mathbb{R}$ with $0<f<g$. Then we have*

$$\left|f^{s+1}+3A^{s+1}\left(f,g,g\right)-4L_{s+1}^{s+1}\right|$$
$$\leq\frac{g-f}{9(s+2)}\left(\frac{1+2^{2s+1}(2+3s)}{2^{2s+1}}f^{s}+\frac{3^{s+2}+2^{2s+2}(s+2)}{2^{2s+1}}\left(\frac{2f+g}{3}\right)^{s}+(9s+10)\left(\frac{f+2g}{3}\right)^{s}+4g^{s}\right).$$

Proof. Applying Corollary 3.6 with $\gamma=1$ to the function $\xi\left(u\right)=u^{s+1}$ whose modulus of the first derivative $\left|\xi'\left(u\right)\right|=\left(s+1\right)u^{s}$ is s-convex. □

5. Conclusion

In conclusion, our study has introduced a novel identity which has proven to be a powerful tool, allowing us to establish a series of new bi-parametrized fractional integral inequalities tailored for functions whose first derivatives exhibiting extended

s-convexity in the second sense, among which we derived Simpson's second formula, as well as the left- and right-Radau-type inequalities. These findings open up new avenues for fractional inegral inequalities and provide a versatile framework for addressing various challenges. To further underscore the practical utility of our results, we have presented an illustrative examples complete with graphical representations. Through some practical applications, we have demonstrated the effectiveness of our approach.

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