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Existence and Simulation of Solutions for a Class of Fractional Differential Systems with *p*-Laplacian Operators on Star Graphs*

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Abstract This paper is mainly concerned with the existence and simulation of solutions for a class of Caputo fractional differential systems with p-Laplacian operators on star graphs. The Hyers-Ulam stability of the systems on star graphs is also proved. Furthermore, an example on a formaldehyde graph is presented to demonstrate the practicality of the main results. The innovation of this paper lies in combining a fractional differential system with a formaldehyde graph, interpreting the chemical bonds as the edges of the graph, and exploring the existence and numerical simulation of solutions to the fractional differential system on this unique graph structure.

Keywords Fractional differential systems, p-Laplacian operator, star graphs, existence, Hyers-Ulam stability

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1. Introduction

Fractional differential equation is a generalization of integer-order differential equation, which can describe some complex phenomena in nature and engineering more accurately. For example, fractional differential equations provide a more appropriate model for describing diffusion processes, wave phenomena and memory effects [2,4,5,11,19,21]. In addition, fractional differential equations also show advantages in dealing with singular systems and nonlinear problems. Therefore, fractional differential equations are widely used in many fields, including physics, biomedicine and engineering [1, 6–8, 12, 16, 28]. For example, Dang [7] proposed a new fractional order model to describe the mechanical behavior of viscoelastic materials with memory effects. Abdullaeva [1] introduced a new fractional model on lithiumion batteries and discussed the application of fractional differential equations in engineering.

Graph theory is a branch of mathematics that mainly studies networks formed by the interconnection of nodes through edges. It originated in 1736 when Euler

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published a paper on "the Seven Bridges of Konigsberg Problem". Since then, graph theory has become widely applied in sociology, traffic management, telecommunications and other fields [3,23,24].

A star graph is a special graphical structure in which one central node is connected to multiple peripheral nodes without any connections between them. In [17], Mehandiratta et al. has established the sketch of the directed star graphs. The star graph consists of the vertex set $\{\nu_0, \nu_1, ..., \nu_k\}$ and the edge set $\{\overrightarrow{\nu_i \nu_0}, i = 1, 2, ..., k\}$, where $l_i = |\overrightarrow{\nu_i \nu_0}|$ and ν_0 is the joint point.

As is well known, differential equations on star graphs have profound application backgrounds, which can be applied to different fields, such as chemistry, bioengineering and so on. For instance, in the chemical molecular structure, each solution function μ_i on any edge means bond strength, bond energy and bond polarity. The origin of fractional differential equations on star graphs dates back to the research of Graef et al. [10]. They first studied fractional differential equations on a star graph with two edges and proved the existence result by the fixed point theorem. Mehandiratta et al. [17] extended the 2-sided star graph studied by Graef et al. [10] to a k-sided star graph, and converted the equation into an equivalent fractional differential system defined on $[0, 1]$ by transformation $t = \frac{x}{l_i} \in [0, 1]$. The uniqueness result was proved through the use of Banach's contraction principle. Zhang et al. [27] added a function $\lambda_i(x)$ on the basis of the reference [17], and proved the existence result by the fixed point theorem. In addition, Su et al. [18] discussed the existence of a coupled fractional differential system on a glucose graph and proved the Hyers-Ulam stability of solutions to the system. It can be seen that in literatures [10,17,27], attention was mainly focused on the existence of solutions to the fractional differential systems, while Su et al. [18,22,25,26] conducted numerical simulations without combining traditional star graphs.

Inspired by the above references [10,15,17,18,22,27], we study the existence and Hyers-Ulam stability of the solution to the boundary value problem with p-Laplacian operator on star graphs as follows

$$\begin{cases} \phi_p\left({}^cD^{\eta}_{0+}\mu_i(x)\right) = -\lambda_i(x)\hbar_i\left(x, \mu_i(x), {}^cD^{\theta}_{0+}\mu_i(x)\right), \\ \mu_i(x)|_{x=0} = 0, \ i = 1, 2, ..., k, \\ \mu_i(x)|_{x=l_i} = \mu_j(t)|_{x=l_j}, \ i, j = 1, 2, ..., k, \ i \neq j, \\ \sum_{i=1}^{k} \mu'_i(x)|_{x=l_i} = 0, \end{cases} \tag{1.1}$$

where ${}^cD^{\eta}_{0+}, {}^cD^{\theta}_{0+}$ are both Caputo fractional derivative operators, $\eta \in (1, 2]$, $\theta \in (0, 1]$, $p \in (1, 2)$, $\lambda_i(x) \neq 0$, $\lambda_i \in C[0, 1]$, $\hbar_i \in C([0, l_i] \times \mathbb{R} \times \mathbb{R})$ and $\phi_p(s) = sgn(s) \cdot |s|^{p-1}$. The existence and Hyers-Ulam stability of the solutions to system (1.1) are discussed. Moreover, the approximate graphs of the solution are obtained. The innovation of this paper lies in combining a fractional differential system with a formaldehyde graph, interpreting the chemical bonds as the edges of the formaldehyde graph, and exploring the existence and numerical simulation of solutions to the fractional differential system on this unique graph structure.

The outline of the paper is organized as follows. In Section 2, some basic definitions and lemmas are presented. In Section 3, the existence of the solution to the Caputo fractional derivative system is obtained by the Banach and the Krasnoselakii fixed point theorems. In Section 4, the Hyers-Ulam stability of the system is discussed. In Section 5, an example on a formaldehyde graph and numerical simulation on the example are given.

2. Preliminaries

In this section, some basic definitions and lemmas of the fractional integrals and fractional derivatives are recalled, which will be used in this paper.

Definition 2.1. [13] Let $\eta > 0$. Then the fractional integral of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\eta} h(t) = \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} h(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. [13] Let $\eta > 0$. Then the Caputo fractional derivative of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$${}^{c}D_{0+}^{\eta} h(t) = \int_0^t \frac{(t-s)^{n-\eta-1}}{\Gamma(n-\eta)} h^{(n)}(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\eta] + 1$.

Lemma 2.1. [13] *If $\eta > 0$, then*

$$I_{0+}^{\eta}\, {}^{c}D_{0+}^{\eta} \nu(t) = \nu(t) + \omega_1 + \omega_2 t + \omega_3 t^2 + \cdots + \omega_n t^{n-1},$$

where $\omega_1, ..., \omega_n \in \mathbb{R}$, $n = [\eta] + 1$.

Lemma 2.2. [13] *If $\eta > 0$, $\theta > 0$, then*

$${}^{c}D_{0+}^{\eta} t^{\theta} = \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\eta)} t^{\theta-\eta}.$$

Lemma 2.3. [14] *If $p > 2$, $| a |, \ | b | \leq N$, then*

$$| \phi_p(a) - \phi_p(b) | \leq (p-1) N^{p-2} | a - b | .$$

Theorem 2.1. [20] Let B_{P_i} be a closed convex and nonempty subset of a Banach space E and $\Upsilon_1, \Upsilon_2 : B_{P_i} \to E$ be the operators satisfying the following conditions:

(i) $\Upsilon_1 w + \Upsilon_2 z \in B_{P_i}$ for all $w, z \in B_{P_i}$;

(ii) Υ_2 is contraction mapping on B_{P_i};

(iii) Υ_1 is compact and continuous on B_{P_i}.

Then there exists a solution $\nu \in E$ such that $\nu = \Upsilon_1 \nu + \Upsilon_2 \nu$.

Lemma 2.4. [27] *If $\eta > 0$, μ is a function defined on $[0, l]$. Suppose that ${}^{c}D_{0+}^{\eta}\mu$ exists on $(0, l]$. Let $x \in [0, l]$, $t = \frac{x}{l}$. So $\nu(t) = \mu(lt)$, which indicates*

$${}^{c}D_{0+}^{\eta} \mu(x) = l^{-\eta} ({}^{c}D_{0+}^{\eta} \nu(t)).$$

By using Lemma 2.4, it can be seen that system (1.1) is equivalent to

$$\begin{cases} {}^cD_{0+}^{\eta}\nu_i(t) = -g_i(t)\phi_q(h_i(t,\nu_i(t), l_i^{-\theta}({}^cD_{0+}^{\theta}\nu_i(t)))),\ t\in[0,1], \\ \nu_i(t)|_{t=0} = 0,\ i=1,2,...,k, \\ \nu_i(t)|_{t=1} = \nu_j(t)|_{t=1},\ i,j=1,2,...,k,\ i\neq j, \\ \sum_{i=1}^{k} l_i^{-1}\nu_i'(t)|_{t=1} = 0, \end{cases} \tag{2.1}$$

where $g_i(t) = l_i^{\eta}\lambda_i(l_i t),\ \phi_q(s) = \phi_p^{-1}(s),\ \frac{1}{p}+\frac{1}{q}=1,\ \hbar_i(l_i t,a,b) = h_i(t,a,b)$ and $\mu_i(l_i t) = \nu_i(t)$.

Lemma 2.5. *If $\chi_i \in C[0,1],\ i=1,...,k$, then the solution of the system*

$$\begin{cases} {}^cD_{0+}^{\eta}\nu_i(t) = -\chi_i(t), \\ \nu_i(t)|_{t=0} = 0,\ i=1,2,...,k, \\ \nu_i(t)|_{t=1} = \nu_j(t)|_{t=1},\ i,j=1,2,...,k,\ i\neq j, \\ \sum_{i=1}^{k} l_i^{-1}\nu_i'(t)|_{t=1} = 0, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} \nu_i = &-\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\chi_i(s)ds + \sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}(\chi_i(s) \\ &- \chi_j(s))ds + \sum_{j=1}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_j(s)ds, \end{aligned} \tag{2.3}$$

where

$$\beta_j = \frac{l_j^{-1}}{\sum_{j=1}^{k} l_j^{-1}}, j=1,2,...,k.$$

Proof. By using Lemma 2.1, we get

$$I_{0+}^{\eta}\,{}^cD_{0+}^{\eta}\nu_i = I_{0+}^{\eta}(-\chi_i) = \nu_i - \omega_1^{(i)} - \omega_2^{(i)}t,\ i=1,2,...,k,$$

where $\omega_1^{(i)},\ \omega_2^{(i)} \in \mathbb{R}$. Noting that $\nu_i(t)|_{t=0} = 0,\ i=1,2,...k$, which indicates that $\omega_1^{(i)} = 0$, then

$$\nu_i(t) = -I_{0+}^{\eta}y(t) + \omega_2^{(i)}t = -\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\chi_i(s)ds + \omega_2^{(i)}t, \tag{2.4}$$

and

$$\nu_i'(t) = -(\eta-1)\int_0^t \frac{(t-s)^{\eta-2}}{\Gamma(\eta)}\chi_i(s)ds + \omega_2^{(i)} = -\int_0^t \frac{(t-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_i(s)ds + \omega_2^{(i)}.$$

Since $\nu_i(t)|_{t=1} = \nu_j(t)|_{t=1},\ i,j = 1,2,...,k,\ i \neq j$, we obtain

$$-\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}\chi_i(s)ds + \omega_2^{(i)} = -\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}\chi_j(s)ds + \omega_2^{(j)}. \tag{2.5}$$

It is obvious from $\sum_{i=1}^{k} l_i^{-1}\nu_i'(1) = 0$ that

$$\sum_{i=1}^{k} l_i^{-1}\left(-\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_i(s)ds - \omega_2^{(i)}\right) = 0. \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\begin{aligned}\sum_{j=1}^{k} l_j^{-1}\omega_2^{(i)} =& \sum_{\substack{j=1\\ j\neq i}}^{k} l_j^{-1}\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}(\chi_i(s)-\chi_j(s))ds \\ &+ \sum_{j=1}^{k} l_j^{-1}\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_j(s)ds,\end{aligned}$$

which implies

$$\begin{aligned}\omega_2^{(i)} =& \sum_{\substack{j=1\\ j\neq i}}^{k} \beta_j\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}(\chi_i(s)-\chi_j(s))ds \\ &+ \sum_{j=1}^{k} \beta_j\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_j(s)ds.\end{aligned}$$

By substituting the value of $\omega_2^{(i)}, i = 1,2,...,k$ into (2.4), we get the solution (2.3) to system (2.2) and the proof is completed. □

3. Main results

In this section, we will present and prove the existence and uniqueness of the solution to the Caputo fractional derivative system (1.1) by using the Krasnoselakii and the Banach fixed point theorems. Let $E = \{\nu : \nu \in C[0,1],\ {}^cD_{0^+}^{\theta}\nu \in C[0,1]\}$. Then $(E, \|\cdot\|_E)$ is a Banach space equipped with the norm

$$\|\nu\|_E = \|\nu\| + \|{}^cD_{0^+}^{\theta}\nu\|,$$

where

$$\|\nu\| = \sup_{t\in[0,1]}|\nu(t)|,\ \|{}^cD_{0^+}^{\theta}\nu\| = \sup_{t\in[0,1]}|{}^cD_{0^+}^{\theta}\nu|.$$

Thus $(E^k = E\times E\times ...\times E, \|\cdot\|_{E^k})$ is a Banach space endowed with the norm

$$\|(\nu_1,\nu_2,...,\nu_k)\|_{E^k} = \sum_{i=1}^{n}\|\nu_i\|_E.$$

In consideration of Lemma 2.5, the operator $T : E^k \to E^k$ is defined as

$$T(\nu_1, \nu_2, ..., \nu_k)(t) = (T_1(\nu_1, \nu_2, ..., \nu_k)(t), ..., T_k(\nu_1, \nu_2, ..., \nu_k)(t)),$$

and

$$\begin{aligned} &T_i(\nu_1, \nu_2, ..., \nu_k)(t) \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_j t \int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_i(s)\phi_q h_i ds - \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_j t \int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_j(s)\phi_q h_j ds \\ &\quad + \sum_{j=1}^{k} \beta_j t \int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)} g_j(s)\phi_q h_j ds - \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i(s)\phi_q h_i ds, \end{aligned}$$

where $h_i = h_i(s, \nu_i(s), l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))$.

Assume that the following conditions hold:

(H1) Let $\gamma_i(t) \in C[0,1]$. Then for any $t \in [0,1]$, (μ,ν), $(\mu',\nu') \in \mathbb{R}^2$, there is

$$| h_i(t, \mu(t), \nu(t)) - h_i(t, \mu'(t), \nu'(t)) | \le \gamma_i (|\mu - \mu'| + |\nu - \nu'|),$$

where $\gamma_i = sup|\gamma_i(t)|$, $i = 1, 2, ..., k$;

(H2) There are positive constants L_i, which satisfy the following inequality

$$| h_i(t, \mu, \nu) | \le L_i, \ (\mu,\nu) \in \mathbb{R}, \ i = 1, 2, ..., k;$$

(H3) $\sup_{t\in[0,1]} | h_i(t, 0, 0) | = \widetilde{\omega} < \infty, \ i = 1, 2, ..., k.$

For the convenience of calculation, we also use the notation as follows

$$\Lambda_1 = \frac{\Gamma(\eta) + \Gamma(\eta+1)}{\Gamma(\eta)\Gamma(\eta+1)}, \ \Lambda_2 = \frac{1}{\Gamma(\eta+1)},$$

$$\Lambda_3 = \frac{\Gamma(\eta) + \Gamma(\eta+1)}{\Gamma(\eta)\Gamma(\eta+1)\Gamma(2-\theta)}, \ \Lambda_4 = \frac{1}{\Gamma(\eta-\theta+1)}.$$

In the following, the main results of the existence of solutions for the fractional differential system (2.1) are presented.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. Then system* (2.1) *has a unique solution on each edge of the star graph if*

$$\left(\sum_{i=1}^{k} Q_i\right)\left(\sum_{i=1}^{k} \gamma_i\right) < 1,$$

where

$$Q_i = (q-1)\left[\sum_{\epsilon=1}^{4} \Lambda_\epsilon \ (1 + l_i^{-\theta}) g_i L_i^{q-2} + \sum_{\substack{j=1 \\ j\neq i}}^{k} (\Lambda_1 + \Lambda_3) \ (1 + l_j^{-\theta}) g_j L_j^{q-2}\right].$$

Proof. For any $\nu=(\nu_1,\nu_2,...,\nu_3)(t), \mu=(\mu_1,\mu_2,...,\mu_3)(t)\in E^k$, we have

$$\begin{aligned}
&\Big|T_i\nu(t)-T_i\mu(t)\Big|\\
=&\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1(1-s)^{\eta-1}g_i(s)\Big|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)ds\\
&-\phi_q\left(h_i(s,\mu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\mu_i(s)))\right)\Big|ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1(1-s)^{\eta-1}g_j(s)\Big|\phi_q\left(h_j(s,\nu_j(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j(s)))\right)ds\\
&-\phi_q\left(h_j(s,\mu_j(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\mu_j(s)))\right)\Big|ds\\
&+\sum_{j=1}^{k}\frac{\beta_j t}{\Gamma(\eta-1)}\int_0^1(1-s)^{\eta-2}g_j(s)\Big|\phi_q\left(h_j(s,\nu_j(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j(s)))\right)ds\\
&-\phi_q\left(h_j(s,\mu_j(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\mu_j(s)))\right)\Big|ds\\
&+\frac{1}{\Gamma(\eta)}\int_0^t(t-s)^{\eta-1}g_i(s)\Big|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)ds\\
&-\phi_q\left(h_i(s,\mu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\mu_i(s)))\right)\Big|ds.
\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}
&\Big|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)-\phi_q\left(h_i(s,\mu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\mu_i(s)))\right)\Big|\\
\leq&(q-1)L_i^{q-2}\Big|h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))-h_i(s,\mu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\mu_i(s)))\Big|\\
\leq&(q-1)L_i^{q-2}\gamma_i(t)\left(|\nu_i(s)-\mu_i(s)|+\Big|l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s))-l_i^{-\theta}({}^cD_{0^+}^{\theta}\mu_i(s))\Big|\right)\\
\leq&(q-1)L_i^{q-2}\gamma_i(t)\left(\|\nu_i-\mu_i\|+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right).
\end{aligned}$$

Since $0\leq t\leq 1, 0<\beta_j\leq 1, j=1,2,...,k$, we have

$$\begin{aligned}
&\|T_i\nu(t)-T_i\mu(t)\|\\
\leq&\frac{2g_i}{\Gamma(\eta+1)}(1+l_i^{-\theta})\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{g_j(1+l_j^{-\theta})}{\Gamma(\eta+1)}\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\\
&+\sum_{j=1}^{k}\frac{g_j(1+l_j^{-\theta})}{\Gamma(\eta)}\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\\
=&\left(\Lambda_1+\Lambda_2\right)g_i\gamma_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)
\end{aligned}$$

$$+\Lambda_1 \sum_{\substack{j=1\\ j\neq i}}^{k} g_j\gamma_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right).$$

By using Lemma 2.2, we can write

$$\begin{aligned}
&\left|{}^cD_{0^+}^{\theta}T_i\nu(t)-{}^cD_{0^+}^{\theta}T_i\mu(t)\right|\\
\leq&\int_0^1 \frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_i\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|\right.\\
&\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\int_0^1 \frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.\\
&\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&+\sum_{j=1}^{k}\int_0^1 \frac{t^{1-\theta}(1-s)^{\eta-2}}{\Gamma(\eta-1)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.\\
&\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&+\int_0^t \frac{(t-s)^{\eta-\theta-1}}{\Gamma(\eta-\theta)}g_i\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|\right.\\
&\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds\\
\leq&\int_0^1 \frac{1}{\Gamma(\eta+1)\Gamma(2-\theta)}g_i\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|\right.\\
&\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{1}{\Gamma(\eta+1)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.\\
&\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&+\sum_{j=1}^{k}\frac{1}{\Gamma(\eta)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.\\
&\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&+\frac{1}{\Gamma(\eta-\theta+1)}g_i\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|\right.\\
&\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds,
\end{aligned}$$

which reduces to

$$\begin{aligned}
&\left\|{}^cD_{0^+}^{\theta}T_i\nu-{}^cD_{0^+}^{\theta}T_i\mu\right\|\\
\leq&\left(\Lambda_3+\Lambda_4\right)g_i\gamma_i(1+l_i^{-\theta})(q-1)L_i^{q-2}(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|)\\
&+\Lambda_3\sum_{\substack{j=1\\ j\neq i}}^{k} g_j\gamma_j(1+l_j^{-\theta})(q-1)L_j^{q-2}(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|).
\end{aligned}$$

According to $\|\nu\|_E = \|\nu\| + \|^c D^{\theta}_{0^+}\nu\|$, we have

$$\begin{aligned}
&\| T_i\nu - T_i\mu \|_E \\
&\leq \left[\sum_{\epsilon=1}^{4} \Lambda_\epsilon \; g_i(1+l_i^{-\theta})L_i^{q-2} + \sum_{\substack{j=1\\ j\neq i}}^{k} (\Lambda_1+\Lambda_3) \; g_j(1+l_j^{-\theta})L_j^{q-2} \right] \\
&\quad \times \left(\sum_{i=1}^{k} \gamma_i \right) (q-1) \sum_{j=1}^{k} \left(\|\nu_j - \mu_j\| + \left\| ^c D^{\theta}_{0^+}\nu_j - {}^c D^{\theta}_{0^+}\mu_j \right\| \right) \\
&= Q_i \left(\sum_{i=1}^{k} \gamma_i \right) \| \nu - \mu \|_{E^k} \; .
\end{aligned}$$

Hence,

$$\| T\nu - T\mu \|_{E^k} = \sum_{j=1}^{k} \| T_i\nu - T_i\mu \|_E \leq \left(\sum_{j=1}^{k} Q_i \right) \sum_{i=1}^{k} \gamma_i \| \nu - \mu \|_{E^k} \; .$$

It follows from $\left(\sum_{i=1}^{k} Q_i\right)\left(\sum_{i=1}^{k} \gamma_i\right) < 1$ that T is a contraction. By using Banach contraction mapping principle, we can claim that system (2.1) has a unique solution on each edge of the star graph. □

Theorem 3.2. *Suppose that* (H1)-(H3) *hold. Then the fractional differential system* (2.1) *on the star graph has a solution on each edge if*

$$\left(\sum_{i=1}^{k} W_i \right) \left(\sum_{i=1}^{k} \gamma_i \right) < 1,$$

where

$$W_i = (q-1)\,(\Lambda_1+\Lambda_3) \left((1+l_i^{-\theta}) g_i L_i^{q-2} + \sum_{\substack{j=1\\ j\neq i}}^{k} (1+l_j^{-\theta}) g_j L_j^{q-2} \right).$$

Proof. Put $g_i = sup_{t\in[0,1]|} g_i(t)$ and choose a suitable constant P_i such that

$$\begin{aligned}
P_i \geq & \sum_{\epsilon=1}^{4} \Lambda_\epsilon \sum_{i=1}^{k} g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i E^* \\
& + (\Lambda_1+\Lambda_3) \sum_{\substack{j=1\\ j\neq i}}^{k} g_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\gamma_j E^* \\
& + \left(\sum_{\epsilon=1}^{4} \Lambda_\epsilon \sum_{i=1}^{k} g_i + (\Lambda_1+\Lambda_3) \sum_{\substack{j=1\\ j\neq i}}^{k} g_j \right) |\phi_q(\widetilde{\omega})|,
\end{aligned}$$

where $E^* = sup\|\nu\|_E$. Consider the set $B_{P_i} = \{\nu = (\nu_1, \nu_2, ..., \nu_k) \in E : \|\nu\| \leq P_i\}$, which is a closed convex and nonempty subset of E^k. The operators Υ_1 and Υ_2 on B_{P_i} are defined by

$$\Upsilon_1\nu = \Upsilon_1(\nu_1, \nu_2, ..., \nu_k)(t) = \left(\Upsilon_1^{(1)}(\nu_1, \nu_2, ..., \nu_k)(t), ..., \Upsilon_1^{(k)}(\nu_1, \nu_2, ..., \nu_k)(t) \right),$$

$$\Upsilon_2\nu = \Upsilon_2(\nu_1,\nu_2,...,\nu_k)(t) = \left(\Upsilon_2^{(1)}(\nu_1,\nu_2,...,\nu_k)(t),...,\Upsilon_2^{(k)}(\nu_1,\nu_2,...,\nu_k)(t)\right),$$

where

$$\Upsilon_1\nu = -\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i(s)\phi_q\left(h_i(s,\nu_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i))\right)ds,$$

$$\begin{aligned}\Upsilon_2\nu =& \sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_i(s)\phi_q\left(h_i(s,\nu_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i))\right)ds\\ &-\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_j(s)\phi_q\left(h_j(s,\nu_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j))\right)ds\\ &+\sum_{j=1}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)} g_j(s)\phi_q\left(h_j(s,\nu_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j))\right)ds.\end{aligned}$$

For $z=(z_1,z_2,...,z_k), w=(w_1,w_2,...,w_3)(t)\in B_{P_i},\ t\in[0,1],\ \beta_j\in(0,1],\ j=1,2,...,k$, we can get

$$\begin{aligned}&|\Upsilon_1 w_i(t)+\Upsilon_2 z_i(t)|\\ \le&\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,w_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}w_i))\right)\right|ds\\ &+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,z_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}z_i))\right)\right|ds\\ &+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_j\left|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\right|ds\\ &+\sum_{j=1}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)} g_j\left|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\right|ds\\ \le&\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,w_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}w_i))\right)-\phi_q\left(h_i(s,0,0)\right)\right|ds\\ &+\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,0,0)\right)\right|ds\\ &+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,z_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}z_i))\right)\right.\\ &\left.-\phi_q\left(h_i(s,0,0)\right)\right|ds+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_i\left|\phi_q\left(h_i(s,0,0)\right)\right|ds\\ &+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)} g_j\left|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\right.\end{aligned}$$

$$
\begin{aligned}
&-\phi_q\left(h_j(s,0,0)\right)\Big|ds+\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1\frac{(1-s)^{\eta-1}}{\Gamma(\eta)}g_j\Big|\phi_q\left(h_j(s,0,0)\right)\Big|ds\\
&+\sum_{j=1}^{k}\beta_j t\int_0^1\frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}g_j\Big|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\\
&-\phi_q\left(h_j(s,0,0)\right)\Big|ds+\sum_{j=1}^{k}\beta_j t\int_0^1\frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}g_j\Big|\phi_q\left(h_j(s,0,0)\right)\Big|ds\\
\leq\ &\frac{t^{\eta}g_i}{\Gamma(\eta+1)}|\phi_q(\widetilde{\omega})|+\frac{g_i}{\Gamma(\eta+1)}|\phi_q(\widetilde{\omega})|+\frac{g_j}{\Gamma(\eta+1)}|\phi_q(\widetilde{\omega})|+\frac{g_j}{\Gamma(\eta)}|\phi_q(\widetilde{\omega})|\\
&+\frac{t^{\eta}g_i}{\Gamma(\eta+1)}(q-1)L_i^{q-2}g_i\gamma_i(t)\left(\|\,w_i\,\|+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}w_i\right\|\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{g_i}{\Gamma(\eta+1)}(q-1)L_i^{q-2}\gamma_i(t)\left(\|\,z_i\,\|+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_i\right\|\right)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{g_j}{\Gamma(\eta+1)}(q-1)L_j^{q-2}\gamma_j(t)\left(\|\,z_j\,\|+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_j\right\|\right)\\
&+\sum_{j=1}^{k}\frac{g_j}{\Gamma(\eta)}(q-1)L_j^{q-2}\gamma_j(t)\left(\|\,z_j\,\|+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_j\right\|\right)\\
\leq\ &(\Lambda_1+\Lambda_2)\,g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_iE^*\\
&+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}g_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\gamma_jE^*\\
&+\Bigg((\Lambda_1+\Lambda_2)\,g_i+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}g_j\Bigg)|\phi_q(\widetilde{\omega})|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\Upsilon_1w(t)+\Upsilon_2z(t)\|\\
\leq\ &(\Lambda_1+\Lambda_2)\,g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_iE^*\\
&+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_iE^*\\
&+\Bigg((\Lambda_1+\Lambda_2)\,g_i+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}g_j\Bigg)|\phi_q(\widetilde{\omega})|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|{}^cD_{0^+}^{\theta}\Upsilon_1w(t)+{}^cD_{0^+}^{\theta}\Upsilon_2z(t)\|\\
\leq\ &\int_0^t\frac{(t-s)^{\eta-\theta-1}}{\Gamma(\eta-\theta)}g_i\Big|\phi_q\left(h_i(s,w_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}w_i))\right)\Big|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\j\neq i}}^{k}\beta_j\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_i\left|\phi_q\left(h_i(s,z_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}z_i))\right)\right|ds\\
&+\sum_{\substack{j=1\\j\neq i}}^{k}\beta_j\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_j\left|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\right|ds\\
&+\sum_{j=1}^{k}\beta_j\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-2}}{\Gamma(\eta-1)\Gamma(2-\theta)}g_j\left|\phi_q\left(h_j(s,z_j,l_j^{-\theta}({}^cD_{0^+}^{\theta}z_j))\right)\right|ds\\
\le&\frac{g_it^{\eta-\theta}(q-1)}{\Gamma(\eta-\theta+1)}L_i^{q-2}\gamma_i(t)\left(\|w_i\|+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}w_i\right\|\right)+\frac{g_it^{\eta-\theta}|\phi_q(\widetilde{\omega})|}{\Gamma(\eta-\theta+1)}\\
&+\frac{g_i|\phi_q(\widetilde{\omega})|}{\Gamma(\eta+1)\Gamma(2-\theta)}+\frac{g_j|\phi_q(\widetilde{\omega})|}{\Gamma(\eta)\Gamma(2-\theta)}+\frac{g_j|\phi_q(\widetilde{\omega})|}{\Gamma(\eta+1)\Gamma(2-\theta)}\\
&+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{g_i(q-1)}{\Gamma(\eta+1)\Gamma(2-\theta)}L_i^{q-2}\gamma_i(t)\left(\|z_i\|+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_i\right\|\right)\\
&+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{g_j(q-1)}{\Gamma(\eta+1)\Gamma(2-\theta)}L_j^{q-2}\gamma_j(t)\left(\|z_j\|+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_j\right\|\right)\\
&+\sum_{j=1}^{k}\frac{g_j(q-1)}{\Gamma(\eta)\Gamma(2-\theta)}L_j^{q-2}\gamma_j(t)\left(\|z_j\|+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}z_j\right\|\right)\\
\le&\left(\Lambda_3+\Lambda_4\right)g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_iE^*\\
&+\Lambda_3\sum_{\substack{j=1\\j\neq i}}^{k}g_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\gamma_jE^*\\
&+\left(\left(\Lambda_3+\Lambda_4\right)g_i+\Lambda_3\sum_{\substack{j=1\\j\neq i}}^{k}g_j\right)|\phi_q(\widetilde{\omega})|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\Upsilon_1w(t)+\Upsilon_2z(t)\|_E\\
\le&\sum_{\epsilon=1}^{4}\Lambda_\epsilon g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_iE^*\\
&+(\Lambda_1+\Lambda_3)\sum_{\substack{j=1\\j\neq i}}^{k}g_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\gamma_jE^*\\
&+\left(\sum_{\epsilon=1}^{4}\Lambda_\epsilon g_i+(\Lambda_1+\Lambda_3)\sum_{\substack{j=1\\j\neq i}}^{k}g_j\right)|\phi_q(\widetilde{\omega})|.
\end{aligned}
$$

This yields $\|\Upsilon_1w(t)+\Upsilon_2z(t)\|_{E^k}=\sum_{i=1}^{k}\|\Upsilon_1w(t)+\Upsilon_2z(t)\|_E\le P_i$, so it is clear that $\Upsilon_1w+\Upsilon_2z\in B_{P_i}$. In the following, we prove that Υ_2 is a contraction. For any $\nu=(\nu_1,\nu_2,...,\nu_3)(t),\mu=(\mu_1,\mu_2,...,\mu_3)(t)\in B_{P_i}$, we have

$$\begin{aligned}
&\left|\Upsilon_2\nu(t)-\Upsilon_2\mu(t)\right|\\
&\le\sum_{\substack{j=1\\j\neq i}}^{k}\int_0^1\frac{t(1-s)^{\eta-1}}{\Gamma(\eta)}g_i(s)(q-1)L_i^{q-2}\gamma_i(t)\left(\|\nu_i-\mu_i\|\right.\\
&\quad\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds\\
&\quad-\sum_{\substack{j=1\\j\neq i}}^{k}\int_0^1\frac{t(1-s)^{\eta-1}}{\Gamma(\eta)}g_j(s)(q-1)L_j^{q-2}\gamma_j(t)\left(\|\nu_j-\mu_j\|\right.\\
&\quad\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&\quad+\sum_{j=1}^{k}\int_0^1\frac{t(1-s)^{\eta-2}}{\Gamma(\eta-1)}g_j(s)(q-1)L_j^{q-2}\gamma_j(t)\left(\|\nu_j-\mu_j\|\right.\\
&\quad\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\Upsilon_2\nu(t)-\Upsilon_2\mu(t)\|\\
&\le\frac{g_i}{\Gamma(\eta+1)}(1+l_i^{-\theta})\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)\\
&\quad+\sum_{\substack{j=1\\j\neq i}}^{k}\frac{g_j(1+l_j^{-\theta})}{\Gamma(\eta+1)}\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\\
&\quad+\sum_{j=1}^{k}\frac{g_j(1+l_j^{-\theta})}{\Gamma(\eta)}\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\\
&=\Lambda_1(q-1)\Big[g_i\gamma_i(1+l_i^{-\theta})L_i^{q-2}\left(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)\\
&\quad+\sum_{\substack{j=1\\j\neq i}}^{k}g_j\gamma_j(1+l_j^{-\theta})L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\Big].
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
&\|\,{}^cD_{0^+}^{\theta}\Upsilon_2\nu(t)-{}^cD_{0^+}^{\theta}\Upsilon_2\mu(t)\,\|\\
&\le\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_i\gamma_i(q-1)L_i^{q-2}\left(\|\nu_i-\mu_i\|\right.\\
&\quad\left.+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)ds\\
&\quad+\sum_{\substack{j=1\\j\neq i}}^{k}\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.\\
&\quad\left.+l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)ds\\
&\quad+\sum_{j=1}^{k}\int_0^1\frac{t^{1-\theta}(1-s)^{\eta-2}}{\Gamma(\eta-1)\Gamma(2-\theta)}g_j\gamma_j(q-1)L_j^{q-2}\left(\|\nu_j-\mu_j\|\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad +l_j^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\Big)\,ds\\
&\leq \Lambda_3(q-1)\Big[g_i\gamma_i(1+l_i^{-\theta})L_i^{q-2}\left(\|\nu_i-\mu_i\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\mu_i\right\|\right)\\
&\quad +\sum_{\substack{j=1\\ j\neq i}}^{k} g_j\gamma_j(1+l_j^{-\theta})L_j^{q-2}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\Big].
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\quad \parallel \Upsilon_2\nu(t)-\Upsilon_2\mu(t)\parallel_E\\
&\leq (\Lambda_1+\Lambda_3)\,(q-1)\left(g_i(1+l_i^{-\theta})L_i^{q-2}+\sum_{\substack{j=1\\ j\neq i}}^{k} g_j(1+l_j^{-\theta})L_j^{q-2}\right)\\
&\quad \times\left(\sum_{i=1}^{k}\gamma_i\right)\sum_{j=1}^{k}\left(\|\nu_j-\mu_j\|+\left\|{}^cD_{0^+}^{\theta}\nu_j-{}^cD_{0^+}^{\theta}\mu_j\right\|\right)\\
&= W_i\left(\sum_{i=1}^{k}\gamma_i\right)\parallel \nu-\mu\parallel_{E^k}.
\end{aligned}
$$

Hence,

$$
\parallel \Upsilon_2\nu-\Upsilon_2\mu\parallel_{E^k}=\sum_{j=1}^{k}\parallel \Upsilon_2\nu-\Upsilon_2\mu\parallel_E\leq\left(\sum_{j=1}^{k}W_i\right)\sum_{i=1}^{k}\gamma_i\parallel \nu-\mu\parallel_{E^k}.
$$

It follows from $\left(\sum_{i=1}^{k}W_i\right)\left(\sum_{i=1}^{k}\gamma_i\right)<1$ that Υ_2 is a contraction on B_{P_i}. Now we prove that Υ_1 is uniformly bounded. For any $\nu=(\nu_1,\nu_2,...,\nu_3)(t)\in B_{P_i}$, we have

$$
\begin{aligned}
&\quad \|\Upsilon_1\nu(t)\|\\
&\leq \int_0^t\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}g_i\left|\phi_q\left(h_i(s,\nu_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i))\right)\right|ds\\
&\leq \int_0^t\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}g_i\left|\phi_q\left(h_i(s,\nu_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i))\right)-\phi_q\left(h_i(s,0,0)\right)\right|ds\\
&\quad +\int_0^t\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}g_i\left|\phi_q\left(h_i(s,0,0)\right)\right|ds\\
&\leq \frac{t^\eta}{\Gamma(\eta+1)}(q-1)L_i^{q-2}g_i\gamma_i(t)\left(\parallel\nu_i\parallel+l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i\right\|\right)+\frac{t^\eta}{\Gamma(\eta+1)}g_i|\phi_q(\widetilde{\omega})|\\
&\leq \frac{g_i}{\Gamma(\eta+1)}(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i\left(\parallel\nu_i\parallel+\left\|{}^cD_{0^+}^{\theta}\nu_i\right\|\right)+\frac{g_i}{\Gamma(\eta+1)}|\phi_q(\widetilde{\omega})|\\
&= \frac{g_i}{\Gamma(\eta+1)}(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i\|\nu_i\|_E+\frac{g_i}{\Gamma(\eta+1)}|\phi_q(\widetilde{\omega})|.
\end{aligned}
$$

By using Lemma 2.2, we know

$$
\|{}^cD_{0^+}^{\theta}\Upsilon_1\nu(t)\|
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\eta-\theta)}\int_0^t (t-s)^{\eta-\theta-1} g_i \left|\phi_q\left(h_i(s,\nu_i,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i))\right)\right| ds \\
&\leq \frac{g_i t^{\eta-\theta}}{\Gamma(\eta-\theta+1)}(q-1)L_i^{q-2}\gamma_i(t)\left(\| \nu_i \| + l_i^{-\theta}\left\|{}^cD_{0^+}^{\theta}\nu_i\right\|\right) \\
&\quad + \frac{g_i t^{\eta-\theta}}{\Gamma(\eta-\theta+1)}|\phi_q(\widetilde{\omega})| \\
&\leq \frac{g_i}{\Gamma(\eta-\theta+1)}(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i\left(\| \nu_i \| + \left\|{}^cD_{0^+}^{\theta}\nu_i\right\|\right) \\
&\quad + \frac{g_i}{\Gamma(\eta-\theta+1)}|\phi_q(\widetilde{\omega})| \\
&= \frac{g_i}{\Gamma(\eta-\theta+1)}(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i\|\nu_i\|_E + \frac{g_i}{\Gamma(\eta-\theta+1)}|\phi_q(\widetilde{\omega})|,
\end{aligned}
$$

which implies

$$
\|\Upsilon_1\nu\|_E \leq (\Lambda_2+\Lambda_4)\left(L_i^{q-2}(q-1)g_i\gamma_i(1+l_i^{-\theta})\|\nu_i\|_E + |\phi_q(\widetilde{\omega})|\right).
$$

Hence,

$$
\|\Upsilon_1\nu\|_{E^k} \leq (\Lambda_2+\Lambda_4)\left(\sum_{i=1}^{k} L_i^{q-2}(q-1)g_i\gamma_i(1+l_i^{-\theta})\|\nu_i\|_E + \sum_{i=1}^{k}|\phi_q(\widetilde{\omega})|\right).
$$

This shows that Υ_1 is uniformly bounded on B_{P_i}. Now we prove that Υ_1 is compact. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|\Upsilon_1\nu(t_2) - \Upsilon_1\nu(t_1)| \\
&\leq \frac{g_i}{\Gamma(\eta)}\int_0^{t_1}\left((t_2-s)^{\eta-1}-(t_1-s)^{\eta-1}\right) \\
&\quad \times \left|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)\right| ds \\
&\quad + \frac{g_i}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-s)^{\eta-1}\left|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)\right| ds \\
&\leq g_i\phi_q(L_i)\frac{t_2^{\eta}-t_1^{\eta}}{\Gamma(\eta+1)} + \frac{g_i}{\Gamma(\eta)}\phi_q(L_i)\int_{t_1}^{t_2}(t_2-s)^{\eta-1}ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^cD_{0^+}^{\theta}\Upsilon_1\nu(t_2) - {}^cD_{0^+}^{\theta}\Upsilon_1\nu(t_1)\right| \\
&\leq \frac{g_i}{\Gamma(\eta-\theta)}\int_0^{t_1}\left((t_2-s)^{\eta-\theta-1}-(t_1-s)^{\eta-\theta-1}\right) \\
&\quad \times \left|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)\right| ds \\
&\quad + \frac{g_i}{\Gamma(\eta-\theta)}\int_{t_1}^{t_2}(t_2-s)^{\eta-\theta-1}\left|\phi_q\left(h_i(s,\nu_i(s),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(s)))\right)\right| ds \\
&\leq g_i\phi_q(L_i)\frac{t_2^{\eta}-t_1^{\eta}}{\Gamma(\eta-\theta+1)} + \frac{g_i}{\Gamma(\eta-\theta)}\phi_q(L_i)\int_{t_1}^{t_2}(t_2-s)^{\eta-\theta-1}ds.
\end{aligned}
$$

Therefore,

$$\begin{aligned}&\|\Upsilon_1\nu(t_2)-\Upsilon_1\nu(t_1)\|_E\\ &\leq (\Lambda_2+\Lambda_4)\,(1+\eta)\,g_i\phi_q(L_i)\left(t_2^{\eta-1}-t_1^{\eta-1}+\int_{t_1}^{t_2}(t_2-s)^{\eta-1}ds\right).\end{aligned}$$

It is clear that $\|\Upsilon_1 v(t_2)-\Upsilon_1 v(t_1)\|_{E^k}=\sum_{j=1}^{k}\|\Upsilon_1 v(t_2)-\Upsilon_1 v(t_1)\|_E\to 0$ as $t_2\to t_1$. Therefore, Υ_1 is equi-continuous, and so Υ_1 is relatively compact on B_{P_i}. It follows from the Arzela-Ascoli theorem that Υ_1 is compact on B_{P_i}. By using Theorem 2.1, we infer that T has a fixed point, which is the solution for system (2.1) on each edge of the star graph. □

4. Hyers-Ulam stability

In this section, the Hyers-Ulam stability of system (2.1) is studied. For $\varepsilon_i>0, i=1,2,...,k$, assume that the following inequality holds:

$$\left|{}^cD_{0^+}^{\eta}\nu_i(t)+g_i(t)\phi_q(h_i(t,\nu_i(t),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(t))))\right|\leq\varepsilon_i, t\in[0,1]. \tag{4.1}$$

Definition 4.1. [27] The fractional differential system (2.1) is called Hyers-Ulam stable, if there is a constant $d_{f_1,f_2,...,f_n}>0$ such that for each $\varepsilon=(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)>0$ and for each $\nu=(\nu_1,\nu_2,...,\nu_n)>0$ of the inequality (4.1), there exists a solution $\nu^*=(\nu_1^*,\nu_2^*,...,\nu_n^*)\in E$ of (2.1) with

$$\|\nu-\nu^*\|_E\leq d_{h_1,h_2,...,h_n}\varepsilon, t\in[0,1].$$

Definition 4.2. [27] The fractional differential system (2.1) is called generalized Hyers-Ulam stable, if there exists a function $\psi_{f_1,f_2,...,f_n}\in\mathbb{C}(\mathbb{R}+,\mathbb{R}+)$ with $\psi_{f_1,f_2,...,f_n}(0)=0$ such that for each $\varepsilon=\varepsilon(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)>0$ and for each solution $v=(v_1,v_2,...,v_k)\in X$ of the inequality (4.1), there exists a solution $\nu^*=(\nu_1^*,\nu_2^*,...,\nu_n^*)\in E$ of (2.1) with

$$\|\nu-\nu^*\|_E\leq\psi_{h_1,h_2,...,h_n}\varepsilon, t\in[0,1].$$

Remark 4.1. If functions $\varphi_i:[0,1]\to\mathbb{R}^+$ are dependent on $\nu_i, i=1,2,...,k$, which is the solution of inequality (4.1), then

(i) $|\varphi_i(t)|\leq\varepsilon_i$, $t\in[0,1]$, $i=1,2,...,k$;

(ii) ${}^cD_{0^+}^{\eta}\nu_i(t)=-g_i(t)\phi_q(h_i(t,\nu_i(t),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(t))))+\varphi_i(t), t\in[0,1], i=1,2,...,k.$

Lemma 4.1. *Suppose that $\nu=(\nu_1,\nu_2,...,\nu_k)\in X^k$ is the solution of inequality (4.1). Then the following inequalities hold*

$$|\nu_i(t)-\widetilde{\nu}_i(t)|\leq(\Lambda_1+\Lambda_2)\,\varepsilon_i+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j,$$

$$\left|{}^cD_{0^+}^{\theta}\nu_i(t)-{}^cD_{0^+}^{\theta}\widetilde{\nu}_i(t)\right|\leq(\Lambda_3+\Lambda_4)\,\varepsilon_i+\Lambda_3\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j,$$

where

$$\widetilde{\nu}_i(t) = -\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\chi_i(s)ds + \sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)}(\chi_i(s) - \chi_j(s))ds + \sum_{j=1}^{k}\beta_j t\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)}\chi_j(s)ds,$$

$${}^cD_{0^+}^{\theta}\widetilde{\nu}_i(t) = -\int_0^t \frac{(t-s)^{\eta-\theta-1}}{\Gamma(\eta-\theta)}\chi_i(s)ds + \sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j t^{1-\theta}\int_0^1 \frac{(1-s)^{\eta-1}}{\Gamma(\eta)\Gamma(2-\theta)}(\chi_i(s) - \chi_j(s))ds + \sum_{j=1}^{k}\beta_j t^{1-\theta}\int_0^1 \frac{(1-s)^{\eta-2}}{\Gamma(\eta-1)\Gamma(2-\theta)}\chi_j(s)ds,$$

and here

$$\chi_i(s) = -g_i(t)\phi_q(h_i(t,\widetilde{\nu}_i(t), l_i^{-\theta}({}^cD_{0^+}^{\theta}\widetilde{\nu}_i(t)))), i = 1,2,...,k.$$

Proof. From Remark 4.1, there is

$$\begin{cases} {}^cD_{0^+}^{\eta}\nu_i(t) = -g_i(t)\phi_q(h_i(t,\nu_i(t), l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(t)))) + \varphi_i(t),\ t\in[0,1],\\ \nu_i(t)|_{t=0} = 0,\ i = 1,2,...,k,\\ \nu_i(t)|_{t=1} = \nu_j(t)|_{t=1},\ i,j = 1,2,...,k,\ i\neq j,\\ \sum_{i=1}^{n} l_i^{-1}\nu_i'(t)|_{t=1} = 0. \end{cases} \tag{4.2}$$

By using Lemma 2.5, the solution of system (4.2) can be given as

$$\begin{aligned} \nu_i =& \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1}\left(\chi_i(s)-\varphi_i(s)\right)ds\\ &-\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1}\left(\chi_j(s)-\varphi_j(s)\right)ds\\ &+\sum_{j=1}^{k}\frac{\beta_j t}{\Gamma(\eta-1)}\int_0^1 (1-s)^{\eta-2}\left(\chi_j(s)-\varphi_j(s)\right)ds\\ &-\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\left(\chi_i(s)-\varphi_i(s)\right)ds, \end{aligned}$$

and

$${}^cD_{0^+}^{\theta}\nu_i = \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t^{1-\theta}}{\Gamma(\eta)\Gamma(2-\theta)}\int_0^1 (1-s)^{\eta-1}\left(\chi_i(s)-\varphi_i(s)\right)ds$$

$$- \sum_{\substack{j=1\\j\neq i}}^{k} \frac{\beta_j t^{1-\theta}}{\Gamma(\eta)\Gamma(2-\theta)} \int_0^1 (1-s)^{\eta-1} \left(\chi_j(s)-\varphi_j(s)\right) ds$$
$$+ \sum_{j=1}^{k} \frac{\beta_j t^{1-\theta}}{\Gamma(\eta-1)\Gamma(2-\theta)} \int_0^1 (1-s)^{\eta-2} \left(\chi_j(s)-\varphi_j(s)\right) ds$$
$$- \int_0^t \frac{(t-s)^{\eta-\theta-1}}{\Gamma(\eta-\theta)} \left(\chi_i(s)-\varphi_i(s)\right) ds.$$

Thus

$$\begin{aligned}|\nu_i(t)-\widetilde{\nu}_i(t)| &\le \sum_{\substack{j=1\\j\neq i}}^{k} \frac{\beta_j\varepsilon_i}{\Gamma(\eta+1)} + \sum_{\substack{j=1\\j\neq i}}^{k} \frac{\beta_j\varepsilon_j}{\Gamma(\eta+1)} + \sum_{j=1}^{k} \frac{\beta_j\varepsilon_j}{\Gamma(\eta)} + \frac{\varepsilon_i}{\Gamma(\eta+1)} \\ &\le (\Lambda_1+\Lambda_2)\,\varepsilon_i + \Lambda_1 \sum_{\substack{j=1\\j\neq i}}^{k} \varepsilon_j,\end{aligned}$$

and

$$\begin{aligned}&\left|{}^cD_{0+}^{\theta}\nu_i(t) - {}^cD_{0+}^{\theta}\widetilde{\nu}_i(t)\right| \\ &\le \sum_{\substack{j=1\\j\neq i}}^{k} \frac{\beta_j\varepsilon_i}{\Gamma(\eta+1)\Gamma(2-\theta)} + \sum_{\substack{j=1\\j\neq i}}^{k} \frac{\beta_j\varepsilon_j}{\Gamma(\eta+1)\Gamma(2-\theta)} \\ &\quad + \sum_{j=1}^{k} \frac{\beta_j\varepsilon_j}{\Gamma(\eta)\Gamma(2-\theta)} + \frac{\varepsilon_i}{\Gamma(\eta-\theta+1)} \\ &\le (\Lambda_3+\Lambda_4)\,\varepsilon_i + \Lambda_3 \sum_{\substack{j=1\\j\neq i}}^{k} \varepsilon_j.\end{aligned}$$

□

Theorem 4.1. *Suppose that* (H1) *and* (H2) *hold. Then system* (2.1) *is Hyers-Ulam stable if the eigenvalues of G are all in the open unit circle, which implies that $|\lambda| < 1$, for $\lambda \in \mathbb{C}$ with $det(\lambda I - A) = 0$, where*

$$G = (q-1)$$
$$\times \begin{pmatrix} \varpi_1(1+l_1^{-\theta})a_1b_1L_1^{q-2} & \varpi_2(1+l_2^{-\theta})a_2b_2L_2^{q-2} & \cdots & \varpi_2(l_k^{\eta}+l_k^{\eta-\theta})a_kb_kL_k^{q-2} \\ \varpi_2(1+l_2^{-\theta})a_1b_1L_1^{q-2} & \varpi_1(1+l_2^{-\theta})a_2b_2L_2^{q-2} & \cdots & \varpi_2(1+l_2^{-\theta})a_kb_kL_k^{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \varpi_2(1+l_2^{-\theta})a_1b_1L_1^{q-2} & \varpi_2(1+l_2^{-\theta})a_2b_2L_2^{q-2} & \cdots & \varpi_1(1+l_2^{-\theta})a_kb_kL_k^{q-2} \end{pmatrix}.$$

Proof. Let $\nu = (\nu_1, \nu_2, ..., \nu_k) \in E^k$ be the solution of the following inequality

$$\left|{}^cD_{0+}^{\eta}\nu_i(t) + g_i(t)\phi_q(h_i(t,\nu_i(t), l_i^{-\theta}({}^cD_{0+}^{\theta}\nu_i(t))))\right| \le \varepsilon_i, t \in [0,1], i = 1, 2, ..., k,$$

and $\nu^*=(\nu_1^*,\nu_2^*,...,\nu_k^*)\in E^k$ be the solution of the system as follows

$$\begin{cases} {}^cD_{0^+}^{\eta}\nu_i^*(t)+g_i(t)\phi_q(h_i(t,\nu_i^*(t),l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i^*(t))))=0,\ t\in[0,1],\\ \nu_i^*(t)|_{t=0}=0,\ i=1,2,...,k,\\ \nu_i^*(t)|_{t=1}=\nu_j^*(t)|_{t=1},\ i,j=1,2,...,k,\ i\neq j,\\ \sum_{i=1}^{n} l_i^{-1}(\nu^*{}_i(t))'|_{t=1}=0. \end{cases} \tag{4.3}$$

According to Lemma 2.5, the solution of (4.3) can be given as

$$\begin{aligned} \nu_i^* =& \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\beta_j t}{\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} g_i(s)\phi_q\left(h_i(s,\nu_i^*,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i^*))\right)ds\\ &-\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\beta_j t}{\Gamma(\eta)}\int_0^1 (1-s)^{\eta-1} g_j(s)\phi_q\left(h_j(s,\nu_j^*,l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j^*))\right)ds\\ &+\sum_{j=1}^{k} \frac{\beta_j t}{\Gamma(\eta-1)}\int_0^1 (1-s)^{\eta-2} g_j(s)\phi_q\left(h_j(s,\nu_j^*,l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j^*))\right)ds\\ &-\int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} g_i(s)\phi_q\left(h_i(s,\nu_i^*,l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i^*))\right)ds. \end{aligned}$$

For convenience, here we make

$$z_i(s)=\phi_q\left(h_j(s,\nu_j(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j(s)))\right)-\phi_q\left(h_j(s,\nu_j^*(s),l_j^{-\theta}({}^cD_{0^+}^{\theta}\nu_j^*(s)))\right).$$

By using Lemma 4.1, for $t\in[0,1]$, one can get

$$\begin{aligned} &|\nu_i(t)-\nu_i^*(t)|\\ \leq& |\nu_i(t)-\widetilde{\nu}_i(t)|+|\widetilde{\nu}_i(t)-\nu_i^*(t)|\\ \leq& (\Lambda_1+\Lambda_2)\varepsilon_i+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j+\sum_{j=1}^{k}\frac{\beta_j t}{\Gamma(\eta-1)}\int_0^1(1-s)^{\eta-2}|g_j(s)z_i(s)|ds\\ &+\int_0^t\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}|g_i(s)z_i(s)|ds+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1(1-s)^{\eta-1}|g_i(s)z_i(s)|ds\\ &+\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j t}{\Gamma(\eta)}\int_0^1(1-s)^{\eta-1}|g_j(s)z_i(s)|ds\\ \leq& (\Lambda_1+\Lambda_2)\varepsilon_i+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j+(\Lambda_1+\Lambda_2)g_i\gamma_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\\ &\times\left(\|\nu_i-\nu_i^*\|+\left\|{}^cD_{0^+}^{\theta}\nu_i-{}^cD_{0^+}^{\theta}\nu_i^*\right\|\right)+\Lambda_1\sum_{\substack{j=1\\ j\neq i}}^{k}g_j\gamma_j(l+l_j^{-\theta}) \end{aligned}$$

$$\times(q-1)L_j^{q-2}\left(\|\nu_j-\nu_j^*\|+\left\|{}^cD_{0+}^{\theta}\nu_j-{}^cD_{0+}^{\theta}\nu_j^*\right\|\right),$$

and

$$\begin{aligned}
&\left|{}^cD_{0+}^{\theta}T_i\nu(t)-{}^cD_{0+}^{\theta}T_i\nu^*(t)\right|\\
&\le(\Lambda_3+\Lambda_4)\,\varepsilon_i+\Lambda_3\sum_{\substack{j=1\\ j\neq i}}^{k}\varepsilon_j+(\Lambda_3+\Lambda_4)\,g_i\gamma_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\\
&\quad\times\left(\|\nu_i-\nu_i^*\|+\left\|{}^cD_{0+}^{\theta}\nu_i-{}^cD_{0+}^{\theta}\nu_i^*\right\|\right)+\Lambda_3\sum_{\substack{j=1\\ j\neq i}}^{k}g_j\gamma_j(1+l_j^{-\theta})\\
&\quad\times(q-1)L_j^{q-2}\left(\|\nu_j-\nu_j^*\|+\left\|{}^cD_{0+}^{\theta}\nu_j-{}^cD_{0+}^{\theta}\nu_j^*\right\|\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\nu_i-\nu_i^*\|_E\\
&=\|\nu_i-\nu_i^*\|+\left\|{}^cD_{0+}^{\theta}T_i\nu(t)-{}^cD_{0+}^{\theta}T_i\nu^*(t)\right\|\\
&\le\sum_{\epsilon=1}^{4}\Lambda_\epsilon\varepsilon_i+\sum_{\substack{j=1\\ j\neq i}}^{k}(\Lambda_1+\Lambda_3)\varepsilon_j+\sum_{\epsilon=1}^{4}\Lambda_\epsilon g_i(1+l_i^{-\theta})(q-1)L_i^{q-2}\gamma_i\|\nu_i-\nu_i^*\|_E\\
&\quad+\sum_{\substack{j=1\\ j\neq i}}^{k}(\Lambda_1+\Lambda_3)g_j(1+l_j^{-\theta})(q-1)L_j^{q-2}\gamma_j\|\nu_j-\nu_j^*\|_E.
\end{aligned}$$

We obtain

$$\begin{aligned}
&(\|\nu_1-\nu_1^*\|_E,\|\nu_2-\nu_2^*\|_E,...,\|\nu_k-\nu_k^*\|_E)^T\\
&\le H(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)^T+G(\|\nu_1-\nu_1^*\|_E,\|\nu_2-\nu_2^*\|_E,...,\|\nu_k-\nu_k^*\|_E)^T,
\end{aligned}$$

where

$$H_{k\times k}=\begin{pmatrix}\varpi_1&\varpi_2&\cdots&\varpi_2\\ \varpi_2&\varpi_1&\cdots&\varpi_2\\ \vdots&\vdots&\ddots&\vdots\\ \varpi_2&\varpi_2&\cdots&\varpi_1\end{pmatrix}.$$

Thus

$$(\|\nu_1-\nu_1^*\|_E,\|\nu_2-\nu_2^*\|_E,...,\|\nu_k-\nu_k^*\|_E)^T\le(I-G)^{-1}H(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)^T.$$

Set

$$\widetilde{E}_{k\times k}=(I-G)^{-1}H=\begin{pmatrix}\widetilde{e}_{11}&\widetilde{e}_{12}&\cdots&\widetilde{e}_{1k}\\ \widetilde{e}_{21}&\widetilde{e}_{22}&\cdots&\widetilde{e}_{2k}\\ \vdots&\vdots&\ddots&\vdots\\ \widetilde{e}_{k1}&\widetilde{e}_{k2}&\cdots&\widetilde{e}_{kk}\end{pmatrix}.$$

Obviously, $\widetilde{e}_{ij} > 0$. Let $\varepsilon = max\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_k\}$. Then there is

$$\|\nu - \nu^*\|_E \le \big(\sum_{j=1}^{k}\sum_{i=1}^{k}\widetilde{e}_{ij}\big)\varepsilon. \tag{4.4}$$

Thus, it can be concluded that system (2.1) is Hyers-Ulam stable. □

Remark 4.2. By substituting the right-hand side of (4.4) with $\psi_{f_1,f_2,...,f_k}(\varepsilon)$, we obtain $\psi_{f_1,f_2,...,f_k}(0) = 0$. According to Definition 4.2, system (2.1) is generalized Ulam-Hyers stable.

5. Example

Star graphs are often applied in the discussion of formaldehyde graphs. In this section, we take a formaldehyde graph as a special case for numerical simulation and prove the applicability of the obtained conclusions.

Example 5.1. Consider the existence and stability of the solution to the fractional differential problem on a formaldehyde graph as follows

$$\begin{cases} \phi_{\frac{4}{3}}\left({}^cD_{0^+}^{\frac{7}{4}}\mu_1(x)\right) + \frac{\sqrt{\pi}}{8}\left(1 + \frac{1}{7(x+3)^5}\left(sin(\mu_1(x)) + \frac{|{}^cD_{0^+}^{\frac{2}{3}}\mu_1(x)|}{1+|{}^cD_{0^+}^{\frac{2}{3}}\mu_1(x)|}\right)\right) = 0, \\ 0 \le x \le \frac{1}{2}, \\ \phi_{\frac{4}{3}}\left({}^cD_{0^+}^{\frac{7}{4}}\mu_2(x)\right) + \frac{\sqrt{\pi}}{18}\left(1 + \frac{1}{2(x+3)^6}\left(sin|\mu_2(x)| + \frac{|{}^cD_{0^+}^{\frac{2}{3}}\mu_2(x)|}{1+|{}^cD_{0^+}^{\frac{2}{3}}\mu_2(x)|}\right)\right) = 0, \\ 0 \le x \le \frac{2}{3}, \\ \phi_{\frac{4}{3}}\left({}^cD_{0^+}^{\frac{7}{4}}\mu_3(x)\right) + \frac{\sqrt{\pi}}{9}\left(1 + \frac{x}{25}|arcsin(\mu_3(x))| + \frac{x|{}^cD_{0^+}^{\frac{2}{3}}\mu_3(x)|}{25+25|{}^cD_{0^+}^{\frac{2}{3}}\mu_3(x)|}\right) = 0, \\ 0 \le x \le \frac{3}{4}, \\ \mu_1(x)|_{x=0} = \mu_2(x)|_{x=0} = \mu_3(x)|_{x=0} = 0, \\ \mu_1(\frac{1}{2}) = \mu_2(\frac{2}{3}) = \mu_3(\frac{3}{4}), \\ \mu_1'(\frac{1}{2}) + \mu_2'(\frac{2}{3}) + \mu_3'(\frac{3}{4}) = 0. \end{cases} \tag{5.1}$$

According to system (1.1), it can be seen that

$$k = 3, \eta = \frac{7}{4}, \theta = \frac{2}{3}, p = \frac{4}{3}, l_1 = \frac{1}{2}, l_2 = \frac{2}{3}, l_3 = \frac{3}{4}.$$

The molecular structure of formaldehyde comprises one oxygen atom, two hydrogen atoms, and one carbon atom. Here, we interpret the chemical bonds between atoms as the edges of the formaldehyde graph (Figure 1). Then we establish coordinate systems with ν_1, ν_2, and ν_3 as coordinate origins respectively on the formaldehyde graph with 3 edges (Figure 2), where ν_1 is the solution of system (5.1) on $\overrightarrow{\nu_1\nu_0}$, ν_2 is the solution of system (5.1) on $\overrightarrow{\nu_2\nu_0}$ and ν_3 is the solution of system (5.1) on $\overrightarrow{\nu_3\nu_0}$. It is clear that

$$l_1 = |e_1| = |\overrightarrow{\nu_1\nu_0}| = \frac{1}{2},$$

$$l_2 = |e_2| = |\overrightarrow{\nu_2\nu_0}| = \frac{2}{3},$$

and

$$l_3 = |e_3| = |\overrightarrow{\nu_3\nu_0}| = \frac{3}{4}.$$

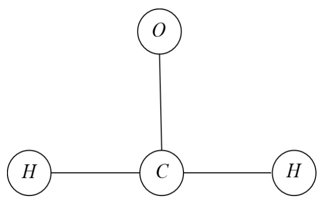

Figure 1. Molecular structure of CH_2O

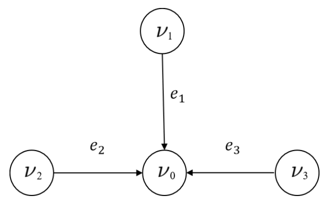

Figure 2. Sketch of a formaldehyde graph

By using Lemma 2.4, the equivalent system can be obtained as follows

$$\begin{cases} {}^cD_{0^+}^{\frac{7}{4}}\nu_1(t) + \frac{\sqrt{\pi}}{8}\cdot(\frac{1}{2})^{\frac{7}{4}}\phi_4\left(1+\frac{1}{7(t+3)^5}\left(sin(\nu_1(t))+\frac{(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}{1+(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}\right)\right)=0,\\ {}^cD_{0^+}^{\frac{7}{4}}\nu_2(t) + \frac{\sqrt{\pi}}{18}\cdot(\frac{2}{3})^{\frac{7}{4}}\phi_4\left(1+\frac{1}{2(t+3)^6}\left(sin|\nu_2(t)|+\frac{(\frac{2}{3})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_2(t)|}{1+(\frac{2}{3})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_2(t)|}\right)\right)=0,\\ {}^cD_{0^+}^{\frac{7}{4}}\nu_3(t) + \frac{\sqrt{\pi}}{9}\cdot(\frac{3}{4})^{\frac{7}{4}}\phi_4\left(1+\frac{t}{25}|arcsin(\nu_3(t))|+\frac{t\times(\frac{3}{4})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_3(t)|}{25+25\times(\frac{3}{4})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_3(t)|}\right)=0,\\ \nu_1(t)|_{t=0}=\nu_2(t)|_{t=0}=\nu_3(t)|_{t=0}=0,\\ \nu_1(t)|_{t=1}=\nu_2(t)|_{t=1}=\nu_3(t)|_{t=1},\\ (\frac{1}{2})^{-\frac{2}{3}}\nu_1'(1)+(\frac{2}{3})^{-\frac{2}{3}}\nu_2'(1)+(\frac{3}{4})^{-\frac{2}{3}}\nu_3'(1)=0, \end{cases} \tag{5.2}$$

where $t \in [0,1]$, and

$$h_1(t,\nu_1(t),l_1D_{0^+}^{\theta}\nu_1(t)) = 1+\frac{1}{7(t+3)^5}\left(sin(\nu_1(t))+\frac{(\frac{1}{3})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}{1+(\frac{1}{3})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}\right),$$

$$h_2(t,\nu_2(t),l_2D_{0^+}^{\theta}\nu_2(t)) = \frac{1}{2(t+3)^6}\left(sin|\nu_2(t)|+\frac{(\frac{1}{5})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_2(t)|}{1+(\frac{1}{5})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_2(t)|}\right)+1,$$

$$h_3(t,\nu_3(t),l_3D_{0^+}^{\theta}\nu_3(t)) = 1+\frac{t}{25}|arcsin(\nu_3(t))|+\frac{t\times(\frac{1}{6})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_3(t)|}{25+25\times(\frac{1}{6})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_3(t)|}.$$

For any μ, ν, μ_1, ν_1, it is clear that

$$h_1(t,\mu,\nu)-h_1(t,\mu_1,\nu_1) \le \frac{1}{7(t+3)^5}\left(|\mu-\mu_1|+|\nu-\nu_1|\right),$$

$$h_2(t,\mu,\nu)-h_2(t,\mu_2,\nu_2) \le \frac{1}{2(t+3)^6}\left(|\mu-\mu_2|+|\nu-\nu_2|\right),$$

and

$$h_3(t,\mu,\nu) - h_3(t,\mu_3,\nu_3) \leq \frac{t}{25}\left(|\mu-\mu_3| + |\nu-\nu_3|\right).$$

So we have

$$\gamma_1 = \sup|\gamma_1(t)| = \frac{1}{1701}, \gamma_2 = \sup|\gamma_2(t)| = \frac{1}{1458}, \gamma_3 = \sup|\gamma_3(t)| = \frac{1}{25},$$

$$L_1 = 1.0012, L_2 = 1.0014, L_3 = 1.0800,$$

$$N_1 = 7.0406, N_2 = 6.8264, N_3 = 7.5384,$$

and

$$(N_1+N_2+N_3)(b_1+b_2+b_3) = 0.8835 < 1.$$

By using Theorem 3.1, the uniqueness of the solution to system (5.1) can be deduced.

Now we prove the Hyers-Ulam stability of system (5.2). We can write

$$u_1 = 4.8426, \quad u_2 = 3.6245.$$

$$A = \begin{pmatrix} 1.4590e-03\ 8.3682e-04\ 0.1335 \\ 1.0920e-03\ 1.1180e-03\ 0.1335 \\ 1.0920e-03\ 8.3682e-04\ 0.1784 \end{pmatrix}.$$

Let

$$0 = det(\lambda I - A) = (\lambda - 0.1799)(\lambda - 0.0008)(\lambda - 0.0003).$$

So we have

$$\lambda_1 = 0.1799 < 1, \quad \lambda_2 = 0.0008 < 1, \quad \lambda_3 = 0.0003 < 1.$$

According to Theorem 4.1, it has been verified that system (5.2) is Hyers-Ulam stable.

In the following, numerical simulation is carried out. Let $\mu_i(t) = l_i^{-\theta}({}^cD_{0^+}^{\theta}\nu_i(t))$, where $\nu_{i,0} = \mu_{i,0} = 0$. The iteration sequence is

$$\begin{aligned}
&\nu_{1,n+1}(t) \\
&= \frac{(\frac{2}{3})^{-\frac{2}{3}}(\frac{1}{2})^{-\frac{7}{4}}\sqrt{\pi}t}{\left((\frac{1}{2})^{-\frac{2}{3}} + (\frac{2}{3})^{-\frac{2}{3}} + (\frac{3}{4})^{-\frac{2}{3}}\right)\times 8\Gamma(\frac{7}{4})}\int_0^1 (1-s)^{\frac{3}{4}}\phi_4\Bigg(\frac{1}{7(t+3)^5} \\
&\quad\times\left(sin(\nu_{1,n}(t)) + \frac{(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_{1,n}(t)|}{1+(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_{1,n}(t)|}\right)+1\Bigg)ds \\
&\quad+\frac{(\frac{3}{4})^{-\frac{2}{3}}(\frac{1}{2})^{-\frac{7}{4}}\sqrt{\pi}t}{\left((\frac{1}{2})^{-\frac{2}{3}} + (\frac{2}{3})^{-\frac{2}{3}} + (\frac{3}{4})^{-\frac{2}{3}}\right)\times 8\Gamma(\frac{7}{4})}\int_0^1 (1-s)^{\frac{3}{4}}\phi_4\Bigg(1+\frac{1}{7(t+3)^5} \\
&\quad\times\left(sin(\nu_1(t)) + \frac{(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}{1+(\frac{1}{2})^{-\frac{2}{3}}|{}^cD_{0^+}^{\frac{2}{3}}\nu_1(t)|}\right)\Bigg)ds
\end{aligned}$$

$$-\frac{(\frac{2}{3})^{\frac{13}{12}}\sqrt{\pi}t}{\Big((\frac{1}{2})^{-\frac{2}{3}}+(\frac{2}{3})^{-\frac{2}{3}}+(\frac{3}{4})^{-\frac{2}{3}}\Big)18\Gamma(\frac{7}{4})}\int_0^1(1-s)^{\frac{3}{4}}\phi_4\Bigg(1+\frac{1}{2(t+3)^6}$$

$$\times\Bigg(sin|\nu_{2,n}(t)|+\frac{(\frac{2}{3})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{2,n}(t)|}{1+(\frac{2}{3})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{2,n}(t)|}\Bigg)\Bigg)ds$$

$$-\frac{(\frac{3}{4})^{\frac{13}{12}}\sqrt{\pi}t}{\Big((\frac{1}{2})^{-\frac{2}{3}}+(\frac{2}{3})^{-\frac{2}{3}}+(\frac{3}{4})^{-\frac{2}{3}}\Big)\times 9\times\Gamma(\frac{7}{4})}\int_0^1(1-s)^{\frac{3}{4}}\phi_4\Bigg(1+\frac{t}{25}$$

$$\times|arcsin(\nu_3(t))|+\frac{(\frac{3}{4})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_3(t)|t}{25+25\times(\frac{3}{4})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_3(t)|}\Bigg)ds$$

$$+\frac{(\frac{1}{2})^{\frac{13}{12}}\sqrt{\pi}t}{\Big((\frac{1}{2})^{-\frac{2}{3}}+(\frac{2}{3})^{-\frac{2}{3}}+(\frac{3}{4})^{-\frac{2}{3}}\Big)8\Gamma(\frac{3}{4})}\int_0^1(1-s)^{-\frac{1}{4}}\phi_4\Bigg(1+\frac{1}{7(t+3)^5}$$

$$\times\Bigg(sin(\nu_{1,n}(t))+\frac{(\frac{1}{2})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{1,n}(t)|}{1+(\frac{1}{2})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{1,n}(t)|}\Bigg)\Bigg)ds$$

$$+\frac{(\frac{2}{3})^{\frac{13}{12}}\sqrt{\pi}t}{\Big((\frac{1}{2})^{-\frac{2}{3}}+(\frac{2}{3})^{-\frac{2}{3}}+(\frac{3}{4})^{-\frac{2}{3}}\Big)18\Gamma(\frac{3}{4})}\int_0^1(1-s)^{-\frac{1}{4}}\phi_4\Bigg(\frac{1}{2(t+3)^6}$$

$$\times\Bigg(sin|\nu_{2,n}(t)|+\frac{(\frac{2}{3})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{2,n}(t)|}{1+(\frac{2}{3})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{2,n}(t)|}\Bigg)+1\Bigg)ds$$

$$-\frac{(\frac{3}{4})^{\frac{13}{12}}\sqrt{\pi}t}{\Big((\frac{1}{2})^{-\frac{2}{3}}+(\frac{2}{3})^{-\frac{2}{3}}+(\frac{3}{4})^{-\frac{2}{3}}\Big)\times 9\times\Gamma(\frac{3}{4})}\int_0^1(1-s)^{-\frac{1}{4}}\phi_4\Bigg(1+\frac{t}{25}$$

$$\times|arcsin(\nu_{3,n}(t))|+\frac{(\frac{3}{4})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{3,n}(t)|t}{25(1+(\frac{3}{4})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{3,n}(t)|)}\Bigg)ds$$

$$-\frac{(\frac{1}{2})^{\frac{7}{4}}\sqrt{\pi}t}{8\Gamma(\frac{7}{4})}\int_0^t(t-s)^{\frac{3}{4}}\phi_4\Bigg(\frac{1}{7(t+3)^5}\Bigg(sin(\nu_{1,n}(t))$$

$$+\frac{(\frac{1}{2})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{1,n}(t)|}{1+(\frac{1}{2})^{-\frac{2}{3}}|^cD_{0+}^{\frac{2}{3}}\nu_{1,n}(t)|}\Bigg)+1\Bigg)ds.$$

The iterative sequences of $\nu_{2,n+1}$ and $\nu_{3,n+1}$ are similar to $\nu_{1,n+1}$. According to the iterative sequences, we can get the approximate graphs of solutions. Figure 3 is the approximate solution graph of system (5.1) on $\overrightarrow{\nu_1\nu_0}$ after 10 iterations. Figure 4 means the approximate solution graph of system (5.1) on $\overrightarrow{\nu_2\nu_0}$ after 10 iterations, and Figure 5 represents the approximate solution graph of system (5.1) on $\overrightarrow{\nu_3\nu_0}$ after 10 iterations, showing the approximate solutions to the fractional differential system (5.1) on different edges of the formaldehyde graph (Figure 2). From Figure 3, it can be seen that the solution exhibits a certain degree of volatility over time, which reflects the characterization of system memory by fractional derivatives. Figures 4 and 5 also show the complexity and dynamic changes of the solutions, but the specific forms and amplitudes vary depending on the edge length, reflecting the diversity of fractional order differential systems in different geometric structures.

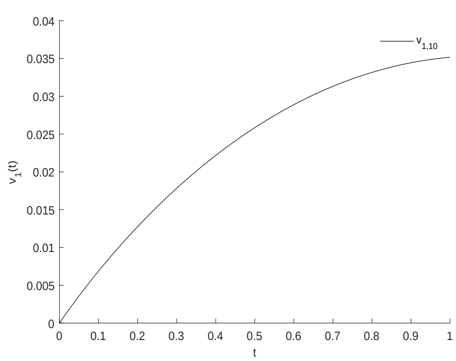

Figure 3. The approximate solution of ν_1

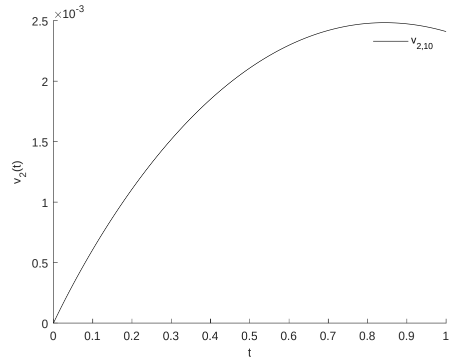

Figure 4. The approximate solution of ν_2

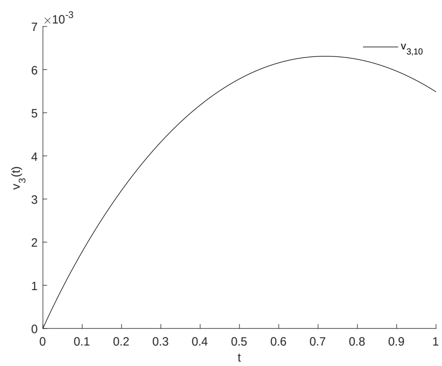

Figure 5. The approximate solution of ν_3

It is worth noting that based on the existence of solutions, Hyers-Ulam stability, and the completion of numerical simulations, we can reasonably infer that under appropriate conditions, the approximate solution obtained using the method discussed in this article will gradually converge to the exact solution as the computational accuracy improves, and we can further increase the number of iterations or optimize algorithm parameters as needed to obtain more accurate approximate solutions.

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