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Entire Solutions for Certain Class of Non-Linear General Difference Equations

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Abstract In this paper, we investigate the entire solutions for a certain class of non-linear difference equations of the form: $f^n + q(z)e^{Q(z)}\mathcal{L}_1(z,f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}$, where $\mathcal{L}_1(z,f)$ is the generalized linear difference operator, $\alpha_1(z)$ and $\alpha_2(z)$ are non-zero small functions of f, $q(z)$ and $Q(z)$(non-constant), $\beta_1(z)$ and $\beta_2(z)$ are non-zero polynomials. Our results improve upon and generalize some previously established findings.

Keywords Non-linear difference equations, meromorphic function, entire solution, Nevanlinna theory

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1. Background information

Assuming the reader's familiarity with conventional notations and core outcomes of Nevanlinna's theory on meromorphic functions (see [9]), in this paper, we consistently refer to meromorphic functions as those meromorphic in the entire complex plane $\mathbb{C}$. For a meromorphic function f and $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, any z such that $f(z) = a$ is termed an a-point of f. In 1926, the Finnish mathematician Rolf Nevanlinna made a noteworthy breakthrough in complex analysis by investigating meromorphic functions over the complex plane. He demonstrated that a non-constant function can be uniquely determined by five distinct pre-images, including infinity, without considering multiplicities. This finding is particularly interesting because it has no counterpart in the real function theory. Later, Nevanlinna went on to prove that when multiplicities are taken into account, four points are adequate for determining the uniqueness of a pair of meromorphic functions. In such cases, either the functions coincide, or one is a bilinear transformation of the other. These seminal discoveries marked the beginning of research into the uniqueness of pairs of meromorphic functions, especially when one function is related to the other. Two meromorphic functions $f(z)$ and $g(z)$ share a CM(Counting multiplicity) or IM(Ignoring multiplicity) if $f - a$ and $g - a$ have the same set of zeros counting multiplicities or ignoring multiplicities, respectively. Further recall that the order of f is defined by

$$\rho(f) = \lim_{r\to\infty} \sup \frac{\log T(r,f)}{\log r}.$$

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Establishing the existence of solutions for complex differential equations represents a significant and challenging problem. Nevanlinna theory has found extensive application in analyzing the properties of such equations. In recent times, an increasing number of researchers have employed Nevanlinna theory to study the solutions of complex differential equations. Moreover, certain topics related to complex difference equations or complex nonlinear differential-difference equations have also been explored using the difference analogs of Nevanlinna theory (see [16], [10], [17]). Notably, in 1964, Hayman [9] examined the behavior of nonlinear differential equations of the following form:

$$f^n + P_d(z, f) = g(z), \tag{1.1}$$

where $P_d(z, f)$ is a differential polynomials in f of degree d with meromorphic coefficients of growth $S(r, f)$ and $n \geq 2$ is an integer.

In 2004, C.C. Yang and P. Li [20] demonstrated that the differential equation $4f^3 + 3f^{''} = -\sin 3z$ possesses precisely three non-constant entire solutions, namely $f_1(z) = \sin(z)$, $f_2(z) = \frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z)$, and $f_3(z) = -\frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z)$. Since $\sin(3z)$ can be expressed as a linear combination of e^{3iz} and e^{-3iz}, this result has stimulated the interest of numerous scholars to investigate the more general differential equation

$$f^n + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $P_d(z, f)$ is a polynomial in f and its derivatives with meromorphic coefficients. Subsequently, it was demonstrated in [19] that the equation:

$$f^2 + q(z)f(z+1) = p(z),$$

where $p(z)$ and $q(z)$ are polynomials, does not admit any transcendental entire solutions of finite order. As a result of the interest generated by the initial findings, numerous investigations have been undertaken by examining various forms of the function $g(z)$ in the non-linear differential equation (1.1). For a more comprehensive overview and additional details regarding non-linear differential equations, one may consult [1, 11, 15, 23, 24].

Several authors have been intrested in investigating the solution of the following type of equation

$$f^n + P_d(z, f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}, \tag{1.2}$$

where $P_d(z, f)$ is a differential polynomials in f of degree d and $\alpha_1(z)$, $\alpha_2(z)$, $\beta_1(z)$ and $\beta_2(z)$ are polynomials. Several works pertinent to the topics discussed can be found in [3, 4, 7, 12, 22]. For instance, Liu et al. [14], in their study referenced therein, investigated the existence of meromorphic solutions for the equation (1.2) and derived the following result.

Theorem 1.1. *[14] Let $n \geq 3$ be an integer and $d \leq n - 2$ be the degree of differential polynomial $P_d(z, f)$. Consider the polynomials $\beta_1(z)$, $\beta_2(z)$ of degree $k(\geq 1)$ and $\alpha_1(z)$, $\alpha_2(z)$ be two small non-zero meromorphic functions of e^{z^k}. If $\frac{\beta_1^{(k)}}{\beta_2^{(k)}} \notin \left\{\frac{n}{n-1},\ \frac{n-1}{n},\ -1,\ 1\right\}$, and any one of the following occurs*

(i) $P_d(z, f) \not\equiv 0$,

(ii) $P_d(z, f) \equiv 0$, $\frac{\beta_1^{(k)}}{\beta_2^{(k)}} \notin \left\{\frac{n}{d},\ \frac{d}{n}\right\}$,

then (1.2) *does not have the meromorphic transcendental solution* f *with* $N(r,f) = S(r,f)$.

Subsequently, in 2013, L.W. Liao et al. [13], in the referenced work, investigated differential equations of the form

$$f^n f' + P_d(z,f) = q(z)e^{Q(z)}, \tag{1.3}$$

and derived results by considering $q(z)$ as a non-zero rational function and $Q(z)$ as a non-constant polynomial.

Theorem 1.2. *[13] Let* f *be a meromorphic solution of* (1.3) *with finite number of poles. Then* $P_d \equiv 0$, $f(z) = s(z)e^{\frac{Q(z)}{n+1}}$, *for* $d \leq n-1$ *and the rational function* $s(z)$ *satisifies* $s^n[(n+1)s' + Q's] = (n+1)q$.

In 2017, M.F. Chen et al. [5], investigated the existence of finite-order entire solutions for the following non-linear difference equations

$$f^n + q(z)\Delta_c f(z) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}, \quad n \geq 2$$

and

$$f^n + q(z)e^{Q(z)}f(z+c) = \alpha_1(z)e^{\lambda z} + \alpha_2(z)e^{-\lambda z}, \quad n \geq 3,$$

where q, Q are non-zero polynomials, and c, λ, α_i, β_i $(i = 1,2)$ are non-zero constants.

Over the past two decades, researchers have primarily focused their studies on three distinct aspects concerning the solutions of shift, delay-differential, or differential equations:

1. existence and non-existence conditions,
2. order of growth,
3. different types of forms of solutions.

2. Preliminaries

The first lemma is the difference analogues of Logarithmic derivative Lemma [9] which plays an important role in the study of complex difference equations.

Lemma 2.1. *[6] Let* $f(z)$ *be a non constant meromorphic function with finite order* σ *and* c_1, c_2 *be two complex numbers such that* $c_1 \neq c_2$ *then for each* $\epsilon > 0$

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O\left(r^{\sigma-1+\epsilon}\right).$$

Lemma 2.2. *[8, **Clunie's Lemma**] Let* $f(z)$ *be a non constant finite order meromorphic solution of* $f^n(z)P(z,f) = Q(z,f)$ *where* $P(z,f)$ *and* $Q(z,f)$ *are difference polynomials in* f *with small meromorphic function as coefficient, and let* $c \in \mathbb{C}$, $\delta < 1$. *If the total degree of* $Q(z,f)$ *is a polynomial in* f *and its shifts are at most* n, *then*

$$m(r, P(z,f)) = o\left(\frac{T(r+\mid c \mid, f)}{r^\delta}\right) + o(T(r,f))$$

for all r *outside of a possible exceptional set with finite logarithmic measure.*

Lemma 2.3. *[21] If $f_k(z)$, $1 \leq k \leq m$, and $g_k(z)$, $1 \leq k \leq m$, $m \geq 2$ are entire functions that meet conditions listed below*

1. $\sum_{i=0}^{m} f_k(z)e^{g_k(z)} \equiv 0$,
2. *The orders of $f_k(z)$ are less than that of $e^{g_l(z)-g_n(z)}$,*
   for $1 \leq k \leq m$, $1 \leq k \leq l < n \leq m$, then $f_k \equiv 0$ for $1 \leq k \leq m$.

Lemma 2.4. *[21] Let f be a non-zero meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r) \quad as \ \ r \to \infty$$

if f is of finite order, and

$$m\left(r, \frac{f'}{f}\right) = O(\log T(r, f)) \quad as \ \ r \to \infty$$

possibly outside a set E of r with finite linear measure if f is of infinite order.

3. Main result

We now introduce the generalized linear difference operator of $f(z)$ as,

$$\mathcal{L}_1(z, f) = \sum_{i=1}^{k} b_i f(z + c_i)(\not\equiv 0), \tag{3.1}$$

respectively, where b_i, c_i are non negative integers, $c_0 = 0$. In view of the above discussion it is quite natural to characterize the nature of exponential polynomial as a solution of certain non linear difference equation.

Consider the following non linear difference equation of the form

$$f^n + q(z)e^{Q(z)}\mathcal{L}_1(z, f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}, \tag{3.2}$$

where n is an integer, $\alpha_1(z)$, $\alpha_2(z)$ are non zero small functions of f and $q(z)$, $Q(z)$ (non constant), $\beta_1(z)$, $\beta_2(z)$ are non zero polynomials.

Theorem 3.1. *If $f(z)$ is a finite order transcendental entire solution of (3.2) with $n \geq 3$ and $\deg \beta_1 \neq \deg \beta_2$, then the following holds:*

(i) Suppose $\deg \beta_1 < \deg \beta_2$ and $\rho(f) = \deg \beta_1$. Then every solution of f satisifies $\rho(f) < \max\{\deg \beta_1, \ \deg \beta_2\} = \deg Q(z)$ and $f = \gamma e^{\frac{\beta_1}{n}}$, where $\gamma^n = \alpha_1$.

(ii) Suppose $\deg \beta_1 < \deg \beta_2$ and $\rho(f) \geq \deg \beta_2$. Then every solution of f satisifies $\rho(f) = \deg Q(z) \geq \max\{\deg \beta_1, \ \deg \beta_2\}$.
Similarly we can get for $\deg \beta_1 > \deg \beta_2$, $\rho(f) \geq \deg \beta_1$.

Proof. Let us assume that $f(z)$ is a transcendental entire solution of finite order for the equation (3.2). To establish the proof of the theorem, we shall consider the following cases:

Case 1. If $\rho(f) < \max\{\deg \beta_1, \ \deg \beta_2\}$, then from (3.2) and Lemma 2.1 it follows that

$$T(r, e^{Q(z)}) = T\left(r, \frac{\alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)} - f^n}{q(z)\mathcal{L}_1(z, f)}\right)$$

$$
\begin{aligned}
&\leq T\left(r, \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}\right) + nT(r,f) + T\left(r,\ q(z)\mathcal{L}_1(z,f)\right)\\
&\quad + S(r,f)\\
&\leq T\left(r, \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}\right) + m\left(r, \frac{\mathcal{L}_1(z,f)}{f}\right) + m(r,\ f)\\
&\quad + nT(r,f) + S(r,f)\\
&\leq T\left(r, \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}\right) + (n+1)T(r,f) + S(r,f).
\end{aligned}
$$

That is $T(r, e^{Q(z)}) \leq T\left(r, \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}\right)$, which implies

$$\deg(Q(z)) \leq \max\left\{ \deg\beta_1,\ \deg\beta_2 \right\}. \tag{3.3}$$

Concurrently, from equation (3.2) and Lemma 2.1, we deduce the following:

$$
\begin{aligned}
T\left(r, \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}\right) &= m\left(r, f^n + q(z)\mathcal{L}_1(z,f)e^{Q(z)}\right) + S(r,f)\\
&\leq (n+1)T(r,f) + T(r, e^{Q(z)}) + S(r,f)\\
&\leq T(r, e^{Q(z)}) + S(r,f).
\end{aligned}
$$

This implies that

$$max\left\{ \deg\beta_1,\ \deg\beta_2 \right\} \leq \deg(Q(z)). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\deg(Q(z)) = max\left\{ \deg\beta_1,\ \deg\beta_2 \right\} \ \text{ and } \ \rho(f) < \deg(Q).$$

By differentiating (3.2), we get

$$nf^{n-1}f' + q\mathcal{L}_1(z,f)e^{Q}\mathcal{M} = \alpha_1(z)e^{\beta_1(z)}\mathcal{M}_1 + \alpha_2(z)e^{\beta_2(z)}\mathcal{M}_2, \tag{3.5}$$

where $\mathcal{M} = \frac{\mathcal{L}_1(z,f')}{\mathcal{L}_1(z,f)} + \frac{q'}{q} + Q'$, $\mathcal{M}_1 = \beta_1' + \frac{\alpha_1'}{\alpha_1}$, $\mathcal{M}_2 = \beta_2' + \frac{\alpha_2'}{\alpha_2}$ are small functions of f.

Eliminating the terms e^{α_1} and e^{α_2} from (3.2) and (3.5), we obtain the following expression

$$\mathcal{M}_1 f^n - nf^{n-1}f' + (\mathcal{M}_1 - \mathcal{M})q\mathcal{L}_1(z,f)e^{Q(z)} = (\mathcal{M}_1 - \mathcal{M}_2)e^{\beta_2}\alpha_2. \tag{3.6}$$

Similarly,

$$\mathcal{M}_2 f^n - nf^{n-1}f' + (\mathcal{M}_2 - \mathcal{M})q\mathcal{L}_1(z,f)e^{Q(z)} = (\mathcal{M}_2 - \mathcal{M}_1)e^{\beta_1}\alpha_1. \tag{3.7}$$

Given that $\deg(\beta_1) \neq \deg(\beta_2)$, it is evident that $\mathcal{M}_1 - \mathcal{M}_2 \neq 0$. We have $\deg(\beta_1) < \deg(\beta_2)$ and $\deg(\beta_1) = \rho(f)$. Upon differentiating equation (3.6) and eliminating the term e^{β_2}, we arrive at the following expression:

$$\mathcal{A}_1 e^{Q(z)} + \mathcal{A}_2 = 0, \tag{3.8}$$

where

$$
\begin{aligned}
\mathcal{A}_1 &= \left[\mathcal{M}_3 - \left(\tfrac{(\mathcal{M}_1 - \mathcal{M})'}{\mathcal{M}_1 - \mathcal{M}} + \mathcal{M}\right)\right](\mathcal{M}_1 - \mathcal{M})q\mathcal{L}_1(z,f),\\
\mathcal{A}_2 &= f^{n-2}\left[(\mathcal{M}_1\mathcal{M}_3 - \mathcal{M}_1')f^2 - n(\mathcal{M}_1 + \mathcal{M}_3)ff' + n(n-1)(f')^2 + nff''\right],
\end{aligned}
$$

$$\mathcal{M}_4 = \tfrac{(\mathcal{M}_1-\mathcal{M}_2)'}{\mathcal{M}_1-\mathcal{M}_2} + \beta_2' + \tfrac{\alpha_2'}{\alpha_2}.$$

Since $\rho(f) < \deg(Q)$, by equation (3.8) and Lemma 2.3, we get $\mathcal{A}_1 \equiv \mathcal{A}_2 \equiv 0$.

From $\mathcal{A}_1 \equiv 0$, we must have either $\mathcal{M}_1 - \mathcal{M} \equiv 0$ or $\mathcal{M}_4 - \left(\tfrac{(\mathcal{M}_1-\mathcal{M})'}{\mathcal{M}_1-\mathcal{M}} + \mathcal{M}\right) \equiv 0$.

Subcase 1.1. Supposing $\mathcal{M}_1 - \mathcal{M} \equiv 0$, we have

$$qe^{Q(z)}\mathcal{L}_1(z,f) = y_1\alpha_1 e^{\beta_1}, \quad y_1 \neq 0. \tag{3.9}$$

If $y_1 = 1$, then by substituting (3.9) into (3.2), we get $f^n = \alpha_2 e^{\beta_2}$. Since $\rho(f) < \deg(\beta_2)$, which is absurd.

If $y_1 \neq 1$, then by substituting (3.9) into (3.2), we get

$$f^n + qe^{Q(z)}\mathcal{L}_1(z,f)\left(1 - \frac{1}{y_1}\right) = \alpha_2 e^{\beta_2}. \tag{3.10}$$

On differentiating (3.10), and eliminating e^{β_2}, we get

$$\mathcal{A}_3 e^{Q(z)} + \mathcal{A}_4 = 0, \tag{3.11}$$

where

$$\mathcal{A}_3 = \left(1 - \frac{1}{y_1}\right)(\mathcal{M}_2 - \mathcal{M})q\mathcal{L}_1(z,f),$$
$$\mathcal{A}_4 = f^{n-1}\left[f\mathcal{M}_2 - nf'\right].$$

Analogous to Case 1, we obtain $\mathcal{A}_3 \equiv \mathcal{A}_4 \equiv 0$. From the condition $\mathcal{A}_3 = 0$, we must have $\mathcal{M}_2 - \mathcal{M} = 0$. However, since $y_1 \neq 1$ and $q\mathcal{L}_1(z,f) \neq 0$, we arrive at $\mathcal{M}_2 = \mathcal{M}$, which is a contradiction.

This implies that $\mathcal{M}_1 - \mathcal{M}_2 \equiv 0$.

Subcase 1.2. Suppose $\mathcal{M}_4 - \left(\tfrac{(\mathcal{M}_1-\mathcal{M})'}{\mathcal{M}_1-\mathcal{M}} + \mathcal{M}\right) \equiv 0$. By integrating we get

$$(\mathcal{M}_1 - \mathcal{M})qe^{Q(z)}\mathcal{L}_1(z,f) = y_2(\mathcal{M}_1 - \mathcal{M}_2)\alpha_2 e^{\beta_2}, \quad y_2 \neq 0. \tag{3.12}$$

Substituting (3.12) into (3.6), we have

$$\mathcal{M}_1 f^n - nf^{n-1}f' = (1 - y_2)(\mathcal{M}_1 - \mathcal{M}_2)\alpha_2 e^{\beta_2}. \tag{3.13}$$

Given that f is a transcendental entire function, by invoking the Hadamard factorization theorem, f can be expressed in the following form:

$$f(z) = \mathcal{H}(z)e^{v(z)}. \tag{3.14}$$

In the representation above, $\mathcal{H}(z)$ denotes the canonical product formed by the zeros of f, and $v(z)$ is a non-constant polynomial satisfying $\deg(v) = \rho(f)$.

Substituting the expression (3.14) into (3.13), we obtain:

$$(\mathcal{H}(z)e^{v(z)})^n\left[\mathcal{M}_1 - n\mathcal{H}(v'(z)+1)\right] = (1 - y_2)(\mathcal{M}_1 - \mathcal{M}_2)\alpha_2 e^{\beta_2}. \tag{3.15}$$

Combining (3.15) with $\deg(v) = \rho(f)$, we see that $\deg(v) = \deg(\beta_2)$, which is a contradiction.

Therefore $y_2 = 1$. From (3.13) we have

$$\mathcal{M}_1 f - nf' = 0. \tag{3.16}$$

On integrating, we get $f^n = y_3\alpha_1 e^{\beta_1}$, $y_3 \neq 0$, and we claim that $y_3 = 1$; otherwise by substituting (3.16) into (3.2), we get

$$(y_3 - 1)\alpha_1 e^{\beta_1} = \alpha_2 e^{\beta_2} - qe^{Q(z)}\mathcal{L}_1(z, f). \tag{3.17}$$

Since $\deg(\beta_2) = \deg(Q(z)) > \deg(\beta_1)$ and by Lemma 2.3, we get $(y_3 - 1)\alpha_1 = 0$, and since $\alpha_1 \neq 0$, we must have $y_3 = 1$. Similarly we can prove another case as well.

Case 2. If $\rho(f) > \max\{\deg(\beta_1),\ \deg(\beta_2)\}$, it follows from Lemma 2.1 and (3.2)

$$\begin{aligned} T(r, e^{Q(z)}) &= m\left(r, \frac{\alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)} - f^n}{q(z)\mathcal{L}_1(z, f)}\right) + S(r, f) \\ &\leq m(r, e^{\beta_1}) + m(r, e^{\beta_2}) + (n + 1)m(r, f) + S(r, f), \end{aligned}$$

i.e.,

$$T(r, e^{Q(z)}) \leq (n + 1)T(r, f) + S(r, f).$$

This implies that $\deg(Q(z)) \leq \rho(f)$.

We now prove that $\deg(Q) = \rho(f)$. Otherwise, if $\deg(Q) < \rho(f)$, denoting $\mathcal{D} = qe^Q$ and $\mathcal{P} = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}$. $T(r, \mathcal{P}) = S(r, f)$ and $T(r, \mathcal{D}) = S(r, f)$, substituting $\mathcal{D}$ and $\mathcal{P}$ into (3.2), we have $f^n = \mathcal{P} - \mathcal{D}\mathcal{L}_1(z, f)$ and using Lemma 2.2, we get $m(r, f) = S(r, f)$ and $N(r, f) = S(r, f)$, therefore $T(r, f) = S(r, f)$, which is absurd.

Therefore

$$\deg(Q(z)) = \rho(f) > \max\{\deg(\beta_1),\ \deg(\beta_2)\}.$$

Case 3. If $\rho(f) = \max\{\deg(\beta_1),\ \deg(\beta_2)\}$, it follows from Lemma 2.1 and (3.2) that

$$\begin{aligned} T(r, e^{Q(z)}) &= m(r, e^{Q(z)}) + S(r, f) \\ &\leq T(r, e^{\beta_1}) + T(r, e^{\beta_2}) + (n + 1)m(r, f) + S(r, f) \\ &\leq 2\rho(f) + S(r, f), \end{aligned}$$

which implies that $\deg(Q(z)) \leq \rho(f)$.

We now prove that $\deg(Q(z)) = \rho(f)$, if $\deg(Q(z)) < \rho(f)$ and denoting $\mathcal{E} = qe^Q$, then $T(r, \mathcal{E}) = S(r, f)$ and (3.2) becomes

$$f^n + \mathcal{E}\mathcal{L}_1(z, f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}. \tag{3.18}$$

Differentiating the above equation and eliminating e^{β_1}, we get

$$\mathcal{M}_1 f^n - nf^{n-1}f' + \mathcal{G}_1(z, f) = \alpha_2\mathcal{M}_3 e^{\beta_2} \tag{3.19}$$

$$\mathcal{M}_2 f^n - nf^{n-1}f' + \mathcal{G}_2(z, f) = -\alpha_1\mathcal{M}_3 e^{\beta_1}, \tag{3.20}$$

where

$$\mathcal{G}_1(z, f) = \mathcal{M}_1\mathcal{E}\mathcal{L}_1(z, f) - (\mathcal{E}\mathcal{L}_1(z, f))',$$

$$\mathcal{G}_2(z, f) = \mathcal{M}_2\mathcal{E}\mathcal{L}_1(z, f) - (\mathcal{E}\mathcal{L}_1(z, f))' \text{ and } \mathcal{M}_3 = \mathcal{M}_1 - \mathcal{M}_2.$$

On differentiating (3.19), we get

$$f^{n-2}\left[(\mathcal{M}_4\mathcal{M}_1 - \mathcal{M}_1')f^2 - n(\mathcal{M}_4 + \mathcal{M}_1)ff' + n(n-1)(f')^2 + nf''f\right]$$

$$= \mathcal{G}_1'(z,f) - \mathcal{M}_4\mathcal{G}_1(z,f). \tag{3.21}$$

For the sake of simplicity, we denote

$$\Phi(z) = (\mathcal{M}_3\mathcal{M}_1 - \mathcal{M}_1')f^2 - n(\mathcal{M}_3 + \mathcal{M}_1)ff' + n(n-1)(f')^2 + nf''f,$$

$$\Psi(z) = \mathcal{G}_1' - \mathcal{M}_3\mathcal{G}_1,$$

$$\mathcal{M}_3 = \frac{\mathcal{M}_4'}{\mathcal{M}_4} + \frac{\alpha_2'}{\alpha_2} + \beta_2'.$$

Therefore,

$$f^{n-1}\Phi(z) = \Psi(z). \tag{3.22}$$

Suppose $\Psi(z) = 0$, we have $\mathcal{G}_1' - \mathcal{M}_3\mathcal{G}_1 = 0$.

If $\mathcal{G}_1 = 0$, on integration we get $\mathcal{E}\mathcal{L}_1(z,f) = y_4\alpha_1 e^{\beta_1}$ where $y_4 \neq 0$, from this we have $f(z) = \mathcal{H}_1(z)e^{v_1(z)}$, where $\mathcal{H}_1(z)$ is the canonical product formed by zeros of f and $v_1(z)$ is a non constant polynomial which satisifies $\deg(v_1) = \rho(f)$.

This implies that $\deg(\beta_1) = \deg(v_1) = \rho(f) > \deg(\beta_2)$, which is a contradiction. Therefore $\mathcal{G}_1 \neq 0$, then we have $\mathcal{G}_1'(z,f) = \mathcal{M}_3\mathcal{G}_1(z,f)$, which implies that $\mathcal{G}_1(z,f) = y_5\mathcal{M}_3\alpha_2 e^{\beta_2}$, $y_5 \neq 0$. Substituting this into (3.19), we get

$$f^{n-1}\left[f\mathcal{M}_1 - nf'\right] = \left(\frac{1}{y_5} - 1\right)\mathcal{G}_1(z,f).$$

Since $n \geq 3$, whether or not $y_5 = 1$, we get from Lemma 2.2 that $f\mathcal{M}_1 - nf' = 0$. On integrating we get $f^n = y_6\alpha_1 e^{\beta_1}$, $y_6 \neq 0$ and $\rho(f) = \deg(\beta_1)$, which is a contradiction.

Therefore $\mathcal{G}_2(z,f) \neq 0$ and it follows that $\Phi(z) \neq 0$.

Consider

$$\Phi(z) = h_1 f^2 + h_2 ff' + h_3(f')^2 + h_4 ff'', \tag{3.23}$$

where $h_1 = \mathcal{M}_3\mathcal{M}_1 - \mathcal{M}_1'$, $h_2 = -n(\mathcal{M}_3 + \mathcal{M}_1)$, $h_3 = n(n-1)$, $h_4 = n$ where h_1 and h_2 are meromorphic functions that are non zero with $T(r,h_i) = S(r,f)$, $i = 1,2$. We now turn into the following cases.

Subcase 3.1 If f has a finite number of zeros, then it is possible to assume that f is of the form $f(z) = \mathcal{R}_1(z)e^{\mathcal{R}_2(z)}$ where $\mathcal{R}_1(\neq 0)$, $\mathcal{R}_2$ are polynomials, and $\deg(\mathcal{R}_2) = \deg(\beta_2)$, $\deg(\mathcal{R}_2) = \deg(Q(z))$.

Substituting $f(z)$ into (3.19), we get

$$\begin{aligned} &\left[\mathcal{M}_1\mathcal{R}_1 - n\mathcal{R}_1^{n-1}(\mathcal{R}_1' + \mathcal{R}_1\mathcal{R}_2')\right]e^{n\mathcal{R}_2} \\ &+ \sum_{i=1}^{k}\left\{\mathcal{M}_1\mathcal{E}\mathcal{R}_1(z+c_i) - \mathcal{E}'\left(\mathcal{R}_1'(z+c_i) + \mathcal{R}_1(z+c_i)\mathcal{R}_2'(z+c_i)\right)\right\}e^{\mathcal{R}_2(z+c_i)} \\ =&\alpha_2\mathcal{M}_3 e^{\beta_2}. \end{aligned} \tag{3.24}$$

If $\left[\mathcal{M}_1\mathcal{R}_1 - n\mathcal{R}_1^{n-1}(\mathcal{R}_1' + \mathcal{R}_1\mathcal{R}_2')\right]e^{n\mathcal{R}_2} = 0$, then on integrating we get

$$\left(\beta_1' + \frac{\alpha_1'}{\alpha_1}\right)\mathcal{R}_1 - n\mathcal{R}_1^{n-1}\left(\mathcal{R}_1' + \mathcal{R}_1\mathcal{R}_2'\right)e^{n\mathcal{R}_2} = 0,$$

$$y_7\alpha_1 e^{\beta_1} = \mathcal{R}_1^n e^{n\mathcal{R}_2}, \ \ y_7 \neq 0,$$

and since $\deg(\beta_1) < \deg(\mathcal{R}_2)$, it follows from Lemma 2.3 that $\alpha_1 = 0$, which is absurd.

Therefore $\left[\mathcal{M}_1\mathcal{R}_1 - n\mathcal{R}_1^{n-1}(\mathcal{R}_1' + \mathcal{R}_1\mathcal{R}_2')\right] e^{n\mathcal{R}_2} \neq 0$.

Suppose $\mathcal{R}_2(z) = d_n z^n + \cdots + a_0$ and $\beta_2(z) = a_n z^n + \cdots + b_0$ where $d_i,\ a_i (0 \leq i \leq n)$ are constants and $d_n,\ a_n \neq 0$. This implies that

$$\begin{aligned}&\left[\mathcal{M}_1\mathcal{R}_1 - n\mathcal{R}_1^{n-1}(\mathcal{R}_1' + \mathcal{R}_1\mathcal{R}_2')\right] e^{(nd_n - a_n)z^k + \cdots + (nd_0 - a_0)}\\ &+ \sum_{i=1}^{k} \left\{\mathcal{M}_1\mathcal{E}\mathcal{R}_1(z+c_i) - \mathcal{E}'\left(\mathcal{R}_1'(z+c_i) + \mathcal{R}_1(z+c_i)\mathcal{R}_2'(z+c_i)\right)\right\} e^{\mathcal{R}_2(z+c_i) - \beta_2}\\ =&\alpha_2\mathcal{M}_3.\end{aligned}$$

From (3.3), we get a contradiction.

Subcase 3.2. Suppose f has infinitely many zeros. Then by (3.23) and Lemma 2.3 we can get $m\left(r, \frac{\Phi}{f^2}\right) = S(r,f)$, which implies that $m\left(r, \frac{1}{f}\right) = S(r,f)$. From (3.23), we get

$$\begin{aligned}N_{(2}\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{1}{\Phi}\right) + S(r,f)\\ &\leq T(r,\Phi) + S(r,f) = S(r,f),\end{aligned} \tag{3.25}$$

where $N_{(2}\left(r, \frac{1}{f}\right)$ is the counting function of f for zeros with multiplicity minimun 2. Thus, we have

$$\begin{aligned}T(r,f) = T\left(r, \frac{1}{f}\right) + S(r,f) &= m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r,f)\\ = N_{1)}\left(r, \frac{1}{f}\right) + S(r,f).\end{aligned}$$

Therefore we deduce that f has infinitely many simple zeros. By differentiating (3.23) we get

$$\Phi'(z) = f^2 h_1' + (2h_1 + h_2')ff' + h_2(f')^2 + h_2 f f^{(2)} + (2h_3 + h_4)f'f^{(2)} + h_4 f f^{(3)}. \tag{3.26}$$

From (3.23) and (3.26), we get

$$\begin{aligned}&f'\left((h_2\Phi - h_3\Phi')f' + (2h_3 + h_4)\Phi f^{(2)}\right)\\ =&f[(h_1\Phi' - h_1'\Phi)f + (h_2\Phi' - (2h_1 + h_2'))f'\\ &+ (h_4\Phi' - h_2\Phi)f^{(2)} - h_4\Phi f^{(3)}].\end{aligned} \tag{3.27}$$

If z_0 is a simple zero of f and not the zero and pole of the coefficient of (3.27), substituting z_0 into (3.27), we observe that z_0 is a zero of $(h_2\Phi - h_3\Phi')f' + (2h_3 + h_4)\Phi f^{(2)}$.

Let

$$\mathcal{J}(z) = \frac{(h_2\Phi - h_3\Phi')f' + (2h_3 + h_4)\Phi f^{(2)}}{f}. \tag{3.28}$$

Then we can deduce that $T(r,\mathcal{J}) = O(\log r)$ by Lemma 2.4, so $\mathcal{J}(z)$ is a rational function. It follows from (3.28)

$$f^{(2)} = \left(\frac{-h_2}{n(2n-1)} + \frac{n(n-1)\Phi'}{n(2n-1)\Phi}\right) f' + \frac{\mathcal{J} f}{n(2n-1)\Phi}. \tag{3.29}$$

Substituting (3.29) into (3.23), we obtain

$$\Phi(z) = w_1 f^2 + w_2 f f' + w_3 (f')^2, \tag{3.30}$$

where $w_1 = h_1 + \frac{\mathcal{J}}{(2n-1)\Phi}$, $w_2 = (n-1)\left[\frac{2h_2}{2n-1} + \frac{n\Phi'}{(2n-1)\Phi}\right]$, $w_3 = n(n-1)$ are rational functions and

$$T(r, w_i) = S(r, f). \tag{3.31}$$

By the similar argument of [2, (from the equation (3.19) to (3.20))] we get

$$w_3(w_2^2-4w_1w_3)\frac{\Phi'}{\Phi}+w_2w_2^2-4w_1w_2w_3+w_3'w_2^2-4w_1w_3w_3' = w_3(w_2^2-4w_1w_3)', \tag{3.32}$$

representing $w_2^2 - 4w_1w_3 = \psi$. Now we will discuss the following cases.

Subcase 3.2.1. If $\psi(z) \neq 0$ then we get $\frac{w_2}{w_3} = \frac{\psi'}{\psi} - \frac{\Phi'}{\Phi} - \frac{w_3'}{w_3}$. On substituting all the parameters and integrating, we get

$$e^{\beta_1+\beta_2} = \frac{\mathcal{K}}{\alpha_1\alpha_2\mathcal{M}_3}\psi^{\frac{-(2n-1)}{2}}\Phi^{n-1} \in S(r,f).$$

This is possible when $\beta_1 = -\beta_2$, which is a contradiction since $\deg(\beta_1) < \deg(\beta_2)$.

Subcase 3.2.2 If $\psi \equiv 0$, then (3.30) becomes

$$\begin{aligned}\Phi(z) &= \left(\frac{w_2^2}{4w_3}\right) f^2 + w_2 f f' + w_3 (f')^2 \\ &= w_3\left[f' + \frac{w_2 f}{2w_3}\right]^2. \end{aligned} \tag{3.33}$$

Let $\Gamma = f' + \frac{w_2 f}{2w_3}$, we know that Γ is a non zero rational function from (3.33).

Substituting $f' = \Gamma - \frac{w_2 f}{2w_3}$ into (3.19), we get

$$\left(\mathcal{M}_1 + n\frac{w_2}{2w_3}\right) f^n - n\Gamma f^{n-1} + \mathcal{G}_1(z,f) = \alpha_2\mathcal{M}_3 e^{\beta_2}, \tag{3.34}$$

$$\left(\mathcal{M}_2 + \frac{w_2}{2w_3}\right) f^n - n\Gamma f^{n-1} + \mathcal{G}_2(z,f) = \alpha_1\mathcal{M}_3 e^{\beta_1}. \tag{3.35}$$

If $\mathcal{M}_1 + n\frac{w_2}{2w_3} \equiv 0$ and $\mathcal{M}_2 + \frac{w_2}{2w_3} \equiv 0$ then we get $\mathcal{M}_3 = 0$ which is absurd. Consequently we claim

$$\left(\mathcal{M}_1 + n\frac{w_2}{2w_3}\right)\left(\mathcal{M}_2 + \frac{w_2}{2w_3}\right) \equiv 0.$$

Otherwise, since $\mathcal{M}_4 \neq 0$ and $\alpha_2 \neq 0$, from (3.34), we have

$$N\left(r, \frac{1}{\mathcal{G}_i}\right) + N(r,f) = N\left(r, \frac{1}{\mathcal{M}_3}\right) + N(r,f) = S(r,f),\ i = 1, 2.$$

Therefore from Lemma 2.4, equation (3.31) and $T(r,\Gamma) = S(r,f)$, there exist two small functions $\beta_1,\ \beta_2$ of f such that

$$\mathcal{H}_1 = \left(\mathcal{M}_1 + n\frac{w_2}{2w_3}\right)(f-\beta_1)^n = \alpha_2\mathcal{M}_3 e^{\beta_2}, \tag{3.36}$$

$$\mathcal{H}_2 = \left(\mathcal{M}_2 + n\frac{w_2}{2w_3}\right)(f-\beta_2)^n = -\alpha_1\mathcal{M}_3 e^{\beta_1}. \tag{3.37}$$

Based on Nevanlinna's second fundamental theorem concerning small functions $\beta_1 \equiv \beta_2$, from (3.36), we get

$$e^{\beta_1-\beta_2} = -\left(\frac{\mathcal{M}_2 + n\frac{w_2}{2w_3}}{\mathcal{M}_1 + n\frac{w_2}{2w_3}}\right) \in\ S(r,f),$$

which is possible when $\beta_1 = \beta_2$, which is a contradiction since $\deg(\beta_1) < \deg(\beta_2)$. Therefore

$$\rho(f) = \deg(Q) = \max\{\deg\beta_1,\ \deg\beta_2\}.$$

□

Some examples have been given to illustrate the sharpness of our result.

Example 3.1. Take $\mathcal{L}_1(z,f) = f(z+2)$. Then the function $f(z) = 2ze^{-z}$ satisifies the difference equation $f^3 + ze^{z^2+z+2}f(z+2) = 8z^3e^{-3z} + 2z(z+2)e^{z^2}$, where $n = 3,\ q(z) = z,\ Q(z) = z^2+z+2,\ \alpha_1 = 8z^3,\ c_1 = 2,\ c_i = 0(i = 2 \text{ to } k),\ \beta_1 = -3z,\ \beta_2 = z^2,\ ,\alpha_2 = 2z(z+2)$. Then clearly we can see that $\deg\beta_1 = 1 < 2 = \deg\beta_2$ and $\rho(f) = \deg\beta_1 = 1 < \max\{1,2\} = 2 = \deg Q(z)$. Thus the conclusion (i) of Theorem 3.1 holds.

Example 3.2. Take $\mathcal{L}_1(z,f) = f(z+1)$. Then the function $f(z) = ze^{-z^2}$ satisifies the difference equation $f^3 + ze^{z^2+z+1}f(z+1) = z^3e^{-3z^2} + z(z+1)e^{-z}$, where $n = 3,\ q(z) = z,\ Q(z) = z^2+z+1,\ \alpha_1 = z^3,\ c_1 = 2,\ c_i = 0(i = 2 \text{ to } k),\ \beta_1 = -3z^2,\ \beta_2 = -z,\ ,\alpha_2 = z(z+1)$. Then clearly we can see that $\deg\beta_2 = 1 < 2 = \deg\beta_1$ and $\rho(f) = \deg Q(z) = 2 \geq \deg\beta_1$. Thus the conclusion (ii) of Theorem 3.1 holds.

The following example demonstrates that a function fulfilling the conditions of Theorem 3.1 and satisfying equation 3.2 can be constructed even in the case where $\deg\beta_1 = \deg\beta_2$.

Example 3.3. Take $\mathcal{L}_1(z,f) = f(z+1)$. Then the function $f(z) = e^z$ satisifies the difference equation $f^3(z)+e^z[f(z+1)] = e^{3z}+e^{2z+1}$, where $n = 3,\ q(z) = 1,\ Q(z) = z,\ k = 1,\ b_1 = 1,\ c_1 = 1,\ \alpha_1(z) = 1, \beta_1(z) = 3z,\ \alpha_2(z) = 1,\ \beta_2(z) = 2z+1$. This example satisfies the conditions of the Theorem 3.1 with $deg\beta_1 = 1 = deg\beta_2$, and $\rho(f) = 1 = deg\beta_1$.

4. Conclusion

In this paper, we explored the existence of transcendent entire solutions for a particular class of nonlinear difference equation (3.2). Our main result, characterizes the possible entire solution of the finite order in terms of their order of growth in relation to the degrees of polynomials $\beta_1(z)$ and $\beta_2(z)$. Specifically, if the order $\rho(f)$ is smaller than the maximum of the degrees β_1 and β_2, either f has a smaller order than the degree $Q(z)$, or f takes the form $f = \gamma e^{\beta_1/n}$, where $\gamma^n = \alpha_1$. On the other hand, if $\rho(f)$ is equal to the maximum degree, then $\rho(f)$ must be the degree of $Q(z)$. Our findings are demonstrated by providing examples. These results improve and generalize some previously established results on the existence of meromorphic solutions for related nonlinear difference or differential equations.

Open problem

1. For the class of non-linear difference equation 3.2, do meromorphic solutions exist, and if yes, how can they be characterized?
2. What are the conditions under which the non-linear difference equation 3.2 has solutions of infinite order, and how can its growth properties be explained?
3. What can be said about the distribution and nature of zeros and poles of transcendental solutions to these types of non-linear difference equations?

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Authors Contributions

All authors have contributed equally.

Conflict of interest

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References

[1] Alotaibi, Abdullah, and J. K. Langley, *The separation of zeros of solutions of higher order linear differential equations with entire coefficients*, Results in Mathematics, 2013, 63, 1365–1373.

[2] Chen W, Hu P, Wang Q, *Entire solutions of two certain types of non-linear differential-difference equations*, Computational Methods and Function Theory, 2021, 21(2), 199–218.

[3] Chen JF, Feng YY, *On meromorphic solutions of nonlinear complex differential equations*, Analysis Mathematica, 2023, 49(3), 699–719.

[4] Chen, Min Feng, Zhi Bo Huang, and Zong Sheng Gao, *Meromorphic solutions of three certain types of non-linear difference equations*, AIMS Mathematics, 2021, 6(11), 11708–11722.

[5] Chen, Min-Feng, Zong-Sheng Gao, and Ji-Long Zhang, *Entire solutions of certain type of non-linear difference equations*, Computational Methods and Function Theory, 2019, 19, 17–36.

[6] Chiang, Yik-Man, and Shao-Ji Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, The Ramanujan Journal, 2008, 16(1), 105–129.

[7] Feng, Yan-Yan; Chen, Jun-Fan, *Meromorphic Solutions of a Certain Type of Nonlinear Differential Equations*, Acta Mathematica Vietnamica, 2024, 49(2), 173–186.

[8] Halburd, R. G., and RJ Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, Journal of Mathematical Analysis and Applications, 2006, 314(2), 477–487.

[9] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[10] Hongxun, Yi. *On a theorem of Tumura and Clunie for a differential polynomial*, Bulletin of the London Mathematical Society, 1988, 20(6), 593–596.

[11] Li, Wen-Wen, and Jun-Fan Chen, *Transcendental meromorphic solutions of certain nonlinear difference equations*, Journal of Mathematical Analysis and Applications, 2024, 539(1), 128497.

[12] Li, Xiao-Min, Chen-Shuang Hao, and Hong-Xun Yi, *On the existence of meromorphic solutions of certain nonlinear difference equations*, Rocky Mountain Journal of Mathematics, 2021, 51(5), 1723–1748.

[13] Liao, Liang-Wen, Chung-Chun Yang, and Jian-Jun Zhang, *On meromorphic solutions of certain type of non-linear differential equations*, Annales Fennici Mathematici, 2013, 38(2), 581–593.

[14] Liu, Huifang, and Zhiqiang Mao, *On the non-existence of meromorphic solutions of certain types of non-linear differential equations*, Computational Methods and Function Theory, 2019, 19, 383–399.

[15] Liu, Huifang, and Zhiqiang Mao, *Meromorphic solutions of certain types of non-linear differential equations*, Computational Methods and Function Theory, 2020, 20(2), 319–332.

[16] Reyzl, Ingrid, *On the theorem of Tumura-Clunie*, Complex Variables and Elliptic Equations, 1995, 28(2), 175–188.

[17] Waghamore, H.P. and Manjunath, B. E.,*Unicity relation to entire functions and their differential diference polynomials*, Journal of Classical Analysis, 2024, 23(2), 79–94.

[18] C.C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 2003.

[19] Yang, C.C. and Laine, I., *On analogies between nonlinear difference and differential equations*, Proc. Jpn. Acad., Ser. A, 2010, 86, 10–14.

[20] Yang, C-C., and Ping Li, *On the transcendental solutions of a certain type of nonlinear differential equations*, Archiv der Mathematik, 2004, 82(5), 442–448.

[21] Yi, Hong-Xun, *Uniqueness theory of meromorphic functions*, Pure Appl. Math. Monographs, 1995.

[22] Zhao, MingXin, and Zhigang Huang, *On Meromorphic Solutions of Non-linear Differential-Difference Equations*, Journal of Nonlinear Mathematical Physics, 2023, 30(4), 1444–1466.

[23] Zhang, Jianjun., Xinping. Xu., and Liangwen. Liao, *Meromorphic solutions of nonlinear complex differential equations*, Scientia Sinica Mathematica, 2017, 49(3), 699–719.

[24] Zhang, Jie, and Liang-Wen Liao, *On entire solutions of a certain type of nonlinear differential and difference equations*, Taiwanese Journal of Mathematics, 2011, 15(5), 2145–2157.