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Inertial Self-Adaptive Method for Solving Fixed Point Constraint Split Common Null Point Problem

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Abstract In this manuscript, we study the split null point problem in the settings of real Hilbert spaces using two different iterative methods. In our first method, we propose a self-adaptive algorithm with an inertial technique for solving split common null point problem and fixed point of a finite family of a demimetric mapping without the computation of the resolvent of a monotone operator. In our second method, we propose a self-adaptive algorithm with a multi-step inertial technique to approximate a solution of the aforementioned problems and to accelerate the rate of convergence of our iterative method. The selection of the stepsize employed in our iterative algorithms does not require prior knowledge of the operator norm. Lastly, we present a numerical example to show the performance of our iterative algorithms. The result discussed in this article extends and complements many related results in literature.

Keywords Fixed point problem, split common null point problem, demimetric mappings, iterative method

MSC(2010) 47H06, 47H09, 47J05, 47J25.

1. Introduction

Throughout this manuscript, let $\mathcal{H}$ denote a real Hilbert space with inner product $\langle ., . \rangle$ and the induced norm $\|.\|$. Let I be the identity operator on $\mathcal{H}$, $\mathbb{N}$ be the set of all natural numbers and $\mathbb{R}$ be the set of real numbers. For a self-operator Ψ on $\mathcal{H}$, we denote by $Fix(\Psi) = \{p \in \mathcal{H} : \Psi(p) = p\}$, the set of all fixed points of Ψ.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and $B_j : \mathcal{H}_1 \to \mathcal{H}_2$ $(1 \le j \le m)$ be bounded linear operator. The split common null point problem (in short, SCNPP) is to find a point

$$x^* \in \mathcal{H}_1 \text{ such that } 0 \in \bigcap_{i=1}^{r} \Psi_i(x^*), \tag{1.1}$$

and such that the point

$$y_j^* = B_j x^* \in \mathcal{H}_2 \text{ solves } 0 \in \Delta_j(y_j^*),\ j = 1, 2, \cdots, m, \tag{1.2}$$

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where $\Psi_i : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ $(1 \le i \le r)$ and $\Delta_j : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ $(1 \le j \le m)$ are set-valued mappings.

The SCNPP (1.1)-(1.2) includes several optimization problems such as variational inequalities, convex feasibility problem and many constrained optimization problems as special cases, (see [7, 9, 24, 25, 29, 32]).

If $m = r = 1$, the SCNPP (1.1)-(1.2) reduces to the following split null point problem (in short, SNPP) which is to find a point

$$x^* \in \mathcal{H}_1 \text{ such that } 0 \in \Psi_1(x^*), \tag{1.3}$$

and the point

$$y^* = B_1 x^* \in \mathcal{H}_2 \text{ solves } 0 \in \Delta_1(y^*). \tag{1.4}$$

We denote by Θ the solution set of SNPP (1.3)-(1.4).

$$x^* \text{ solves } SNPP(1.3)\text{-}(1.4) \Longleftrightarrow x^* = J_\lambda^{\Psi_1}(x^* - \gamma B_1^*(I - J_\lambda^{\Delta_1})B_1 x^*), \tag{1.5}$$

where $\lambda > 0$, $\gamma > 0$ and $J_\lambda^\Psi = (I + \lambda\Psi)^{-1}$ denotes the resolvent of a monotone operator Ψ.

In 2012, Bryne *et al.* [7] introduced the following forward-backward algorithm to solve SNPP (1.3)-(1.4): find $x_1 \in \mathcal{H}_1$

$$x_{t+1} = J_\lambda^\Psi(x_t - \gamma B^*(I - J_\lambda^\Delta)Bx_t) \tag{1.6}$$

where the stepsize $\gamma \in (0, \frac{2}{L})$ with $L = \|B^*B\|$, $J_\lambda^\Psi = (I + \lambda\Psi)^{-1}$ and $J_\lambda^\Delta = (I + \lambda\Delta)^{-1}$ are the resolvents of Ψ and Δ respectively.

Recently, Kazmi and Rizvi [22] studied the SNPP and fixed point of a nonexpansive mapping. They proposed the following iterative method to approximate the solution of the aforementioned problems as follows:

$$\begin{cases} y_t = J_\lambda^\Psi(x_t + \gamma B^*(J_\lambda^\Delta - I)Bx_t), \\ x_{t+1} = \alpha_t f(x_t) + (1 - \alpha_t)Sy_t, \end{cases}$$

where f and S are contraction and nonexpansive mappings respectively.

In 2018, Jailoka and Suantai [20] proposed the following iterative method for approximating the solution of SNPP and fixed point of a multivalued demicontractive mappings as follows:

$$\begin{cases} x_1 \in \mathcal{H}_1, \\ y_t = J_{\lambda_t}^\Psi(x_t + \gamma B^*(J_{\lambda_t}^\Delta - I)Bx_t), \\ u_t = (1 - \delta)y_t + \delta z_t, \ z_t \in Ty_t, \\ x_{t+1} = \alpha_t u + (1 - \alpha_t)u_t, \ t \in \mathbb{N}, \end{cases}$$

where γ, δ and the sequences $\{\alpha_t\}$ and $\{\lambda_t\}$ satisfy the following conditions:

(i) $\gamma \in \left(0, \frac{2}{\|B\|^2}\right)$ and $\delta \in (0, 1 - k)$,

(ii) $\alpha_t \in (0, 1)$ such that $\lim_{t\to\infty} \alpha_t = 0$ and $\sum_{t=1}^{\infty} \alpha_t = \infty$,

(iii) $\lambda_t \in (0, \infty)$ such that $\liminf_{t\to\infty} \lambda_t > 0$.

They established a strong convergence result using the above iterative algorithm.

The common drawback in the results mentioned above (see [7, 12, 20, 22]) is that at each step of the iterative processes, one has to compute the resolvent of one of the operators which is certainly not convenient. Another drawback of these iterative algorithms is the need to calculate the stepsize which solely depends on the operator norm $\|B^*B\|$. In order to overcome this difficulty, linesearch and self-adaptive step size algorithms have been proposed (see [3, 5, 6, 13, 28, 34]). Readers can consult [26, 31, 33, 37] for more details on SCNPP.

In recent years, authors have been concerned with effective iterative methods with a faster rate of convergence. In this direction, there have been several extrapolation methods employed by researchers. One of such methods is the inertial-type method which originates from the heavy ball method (an implicit discretization) in time of second-order dynamical systems (see [4]). The inertial technique finds crucial application in the construction of effective and accelerated algorithms in optimization theory (see [1, 2, 4, 10, 27, 30]). In this method, the next iterate is determined by two preceding iterates (x_{t-1} and x_t) and an inertial parameter θ_t which controls the momentum $x_t - x_{t-1}$.

In 2016, Liang [19] proposed a multi-step inertial splitting method. Let $Q = \{0, 1, \cdots, q-1\}$, $q \in \mathbb{N}^+$, and the multi-step inertial form is as follows:

$$y_t = x_t + \sum_{i\in Q} \delta_{i,t}(x_{t-i} - x_{t-i-1}).$$

In this paper, we propose two self-adaptive algorithms with inertial extrapolation method for solving the SCNPP (1.1)-(1.2) and the fixed point of a finite family of a demimetric mappings in the settings of real Hilbert spaces. Under suitable conditions, we establish that the sequence generated by our iterative method converges strongly to a solution of the aforementioned problems without the computation of the monotone operator. Also, the selection of our stepsize does not need prior knowledge of the operator norms. Secondly, we propose a self-adaptive method with a multi-step inertial technique for solving SCNPP and fixed point of a finite family of demimetric mappings in real Hilbert space, we prove a strong convergence theorem under suitable conditions. Lastly, we present a numerical example to illustrate the performance of our algorithms. Our results extend and generalize many related results in the literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$" respectively.

Let Ω be a nonempty, closed and convex subset of real Hilbert space $\mathcal{H}$. A point $p \in \Delta$ is said to be a fixed point of a mapping $\Psi : \mathcal{H} \to \mathcal{H}$ if $\Psi p = p$. We denote by $Fix(\Psi)$, the set of all fixed points of Ψ. Also, in the sequel, we use P_Ω to denote the projection from $\mathcal{H}$ onto Ω, namely:

$$P_\Omega x := arg\min\{\|x-y\| : y \in \Omega\},\ x \in \mathcal{H}.$$

The following is a characterization of the projection P_Ω: Given $x \in \mathcal{H}$ and $y \in \Omega$,

$$P_\Omega x = z \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \ \forall \ y \in C. \tag{2.1}$$

The following property of the projection P_Ω is known as firmly nonexpansive

$$\langle x - y, P_\Omega x - P_\Omega\rangle \geq \|P_\Omega x - P_\Omega y\|^2, \ \forall \ x, y \in \mathcal{H}.$$

Definition 2.1. Let $\mathcal{H}$ be a real Hilbert space. An operator $\Delta : \mathcal{H} \to \mathcal{H}$ is said to be μ- inverse strongly monotone (μ- ism), if there exists a number $\mu > 0$ such that

$$\langle \Delta(x) - \Delta(y), x - y\rangle \geq \mu\|\Delta(x) - \Delta(y)\|^2, \ \forall \ x, y \in \mathcal{H}.$$

It is easy to find that if Δ is μ- ism, then Δ is Lipschitz (see definition below) with constant $\frac{1}{\mu}$, i.e.

$$\|\Delta x - \Delta y\| \leq \tfrac{1}{\mu}\|x - y\|, \ \forall \ x, y \in \mathcal{H}.$$

Definition 2.2. Let $\mathcal{H}$ be a real Hilbert space. The mapping $\Psi : \mathcal{H} \to \mathcal{H}$ is called

(i) k-contractive, if there exists a constant $k \in [0, 1)$ such that

$$\|\Psi x - \Psi y\| \leq k\|x - y\|, \ \forall \ x, y \in \mathcal{H}.$$

(ii) Lipschitz with the constant $k > 0$, if

$$\|\Psi x - \Psi y\| \leq k\|x - y\|, \ \forall \ x, y \in \mathcal{H}.$$

(iii) nonexpansive, if

$$\|\Psi x - \Psi y\| \leq \|x - y\|, \ \forall \ x, y \in \mathcal{H}.$$

(iv) quasi-nonexpansive, if $Fix(\Psi) \neq \emptyset$ and

$$\|\Psi x - p\| \leq \|x - p\|, \ \forall \ x \in \mathcal{H} \text{ and } p \in Fix(\Psi).$$

(v) β-strict pseudo-contraction [20], if there exists a constant $\beta \in [0, 1)$ such that

$$\|\Psi x - \Psi y\|^2 \leq \|x - y\|^2 + \beta\|(x - \Psi x) - (y - \Psi y)\|^2, \ \forall \ x, y \in \mathcal{H}.$$

Definition 2.3. [35] Let $\mathcal{H}$ be a real Hilbert space and let η be a real number with $\eta \in (-\infty, 1)$. Let $\Psi : \mathcal{H} \to \mathcal{H}$ with $Fix(\Psi) \neq \emptyset$ be called η-demimetric, if for any $x \in \mathcal{H}$ and $p \in Fix(\Psi)$

$$\langle x - p, x - \Psi x\rangle \geq \frac{1 - \eta}{2}\|x - \Psi x\|^2.$$

We give the following example of η-demimetric mapping in real Hilbert space.

Example 2.1. Let $\mathcal{H}$ be the real line and $\Omega = [-2, 1]$. Define

$$\Psi x = \begin{cases} \frac{x+9}{10}, & x \in [0, 1] \\ \frac{3+x}{4}, & x \in [-2, 0). \end{cases}$$

Obviously, $Fix(\Psi) = \{1\}$. We will show that there exists $\eta \in (-\infty, 1)$ such that

$$|\Psi x - 1|^2 \leq |x - 1|^2 + \eta|x - \Psi x|^2, \ \forall \ x \in [-2, , 1].$$

Consider the following two cases:
Case 1: Let $x \in [0,1]$. Then

$$\big|x-\Psi x\big|^2 = \big|x-\frac{x+9}{10}\big|^2 = \big|\frac{9}{10}(x-1)\big|^2 = \frac{81}{100}\big|x-1\big|^2.$$

Also,

$$\begin{aligned}\big|\Psi x-1\big| &= \big|\frac{x+9}{10}-1\big|^2 = \frac{1}{100}\big|x-1\big|^2\\ &= \big|x-1\big|^2 - \frac{99}{100}\big|x-1\big|^2\\ &= |x-1|^2 - \frac{99}{81}\times\frac{81}{100}\big|x-1\big|^2\\ &\le |x-1|^2 + \eta_1\times\frac{81}{100}|x-1|^2,\end{aligned}$$

for any $\eta_1 \in [-\frac{99}{81},1)$. Hence $\big|\Psi x-1\big|^2 \le \big|x-1\big|^2 + \eta_1\big|x-\Psi x\big|^2$.
Case 2: Let $x \in [-2,0)$. Thus

$$\big|x-\Psi x\big|^2 = \big|x-\frac{3+x}{4}\big|^2 = \big|\frac{3(x-1)}{4}^2\big|^2 = \frac{9}{16}\big|x-1\big|^2.$$

Then

$$\begin{aligned}\big|\Psi x-1\big|^2 &= \big|\frac{3+x}{4}-1\big|^2 = \big|\frac{x-1}{4}\big|^2 = \frac{1}{16}\big|x-1\big|^2\\ &= \big|x-1\big|^2 - \frac{15}{16}\big|x-1\big|^2\\ &= \big|x-1\big|^2 - \frac{15}{9}\times\frac{9}{16}\big|x-1\big|^2\\ &\le \big|x-1\big|^2 + \eta_2\times\frac{9}{16}\big|x-1\big|^2,\end{aligned}$$

for any $\eta_2 \in [-\frac{15}{9},1)$. Hence $\big|\Psi x-1\big|^2 \le \big|x-1\big|^2 + \eta_2\big|x-\Psi x\big|^2$.
In particular, choose $\eta = \min\{\eta_1,\eta_2\}$. Thus, Ψ is $\frac{-15}{9}$- demimetric.

It has been established that the class of demimetric mappings is more general than the class of strict pseudo-contractive mappings and quasi-nonexpansive mappings, (see [35]).

Lemma 2.1. *[11] Let $\mathcal{H}$ be a real Hilbert space. Then $\forall\ x,y \in \mathcal{H}$ and $\alpha \in (0,1)$, we have*

(i) $2\langle x,y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2$,
(ii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$,
(iii) $||x+y||^2 \le ||x||^2 + 2\langle y, x+y\rangle$.

Lemma 2.2. *[35] Let $\mathcal{H}$ be a real Hilbert space and let η be a real number with $\eta \in (-\infty,1)$. Let $\Psi : \mathcal{H} \to \mathcal{H}$ be an η- demimetric mapping. Then $Fix(\Psi)$ is closed and convex.*

Definition 2.4. Let $\Psi : \mathcal{H} \to \mathcal{H}$ be a mapping. Then $I-\Psi$ is said to be demiclosed at 0, if for any sequence $\{x_t\}$ in $\mathcal{H}$, the condition $x_t \rightharpoonup x$ and $\lim_{t\to\infty} \|\Psi x_t - x_t\| = 0$, imply $x = \Psi x$.

Lemma 2.3. *[14] Let $\mathcal{H}$ be a real Hilbert space, and let $\Psi : \mathcal{H} \to \mathcal{H}$ be $\eta-$ strict pseudo-contractive mapping. Then $I - T$ is demiclosed at 0.*

Lemma 2.4. *[18] Assume that $\{b_t\}$ is a sequence of nonnegative real numbers such that*

$$b_{t+1} \leq (1 - \sigma_t)b_t + \sigma_t a_t,\ t \geq 0,$$

$$b_{t+1} \leq b_t - \eta_t + \varphi_t,\ t \geq 0,$$

where $\{\sigma_t\}$ is a sequence in $(0, 1)$, $\{\eta_t\}$ is a sequence of nonnegative real numbers, and $\{a_t\}$ and $\{\varphi_t\}$ are two sequences in $\mathbb{R}$ satisfying

(i) $\sum_{t=0}^{\infty} \sigma_t = \infty$,

(ii) $\lim_{t\to\infty} \varphi_t = 0$,

(iii) $\lim_{k\to\infty} \eta_{t_k} = 0$, *implies* $\limsup_{k\to\infty} a_{t_k} \leq 0$ *for any subsequence* $\{t_k\} \subset \{t\}$. *Then,* $\lim_{t\to\infty} b_t = 0$.

3. Main result

In this section, we present our iterative method and establish its convergence result for solving split null point problem and fixed point problem.

We state our assumptions as follows:

Assumption 1. (L1)$\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, and $B_j : \mathcal{H}_1 \to \mathcal{H}_2,\ j = 1, 2, \cdots, m$ are bounded linear operators with $B_j^* : \mathcal{H}_2 \to \mathcal{H}_1$ being the adjoint of B_j. Let ϕ be a $k-$ contractive mapping on $\mathcal{H}_1$ with $0 \leq \kappa < 1$.

(L2) For $i = 1, 2, \cdots, r$, $\{\varphi_i\}_{i=1}^r \subset (-\infty, 1)$ and let $\{U_i\}_{i=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of φ_i- demimetric mappings such that $U_i - I$ is demiclosed at 0, and $\varphi = \min\{\varphi_1, \varphi_2, \cdots, \varphi_r\}$.

(L3) For $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, m$, let $\Psi_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $\Delta_j : \mathcal{H}_2 \to \mathcal{H}_2$ be λ_i and θ_j- inverse strongly monotone mapping, respectively.

(L4) Let

$$\Gamma := \{x^* \in \bigcap_{i=1}^{r} \Psi_i^{-1}(0) \bigcap \bigcap_{i=1}^{r} Fix(U_i) \text{ and } 0 \in \Delta_j(B_j x^*), j = 1, 2, \cdots, m\}.$$

We assume that $\Gamma \neq \emptyset$.

Assumption 2. (M1) Define the mapping:

$$h_i(x) = \frac{1}{2}\|\Psi_i x\|^2, \qquad s_j(x) = \frac{1}{2}\|\Delta_j(B_j x)\|^2.$$

$$\sigma^2(x) = \left(\sum_{i=1}^{r} \|\Psi_i x\|\right)^2 + \left(\sum_{j=1}^{m} \|B_j^* \Delta_j(B_j x)\|\right)^2.$$

(M2) Choose sequences $\{\epsilon_t\}, \{\alpha_{t,i}\}, \{\beta_t\}$ and $\{\rho_t\}$ such that

(i) $\beta_t \in [a,b] \subset (0,1)$, $\lim_{t\to\infty} \beta_t = 0$ and $\sum_{t=1}^{\infty} \beta_t = \infty$;

(ii) $\{\rho_t\} \subset (0,\omega)$, where $\omega = \min\{4\lambda, 4\theta\}$ with $\lambda = \min\{\lambda_1, \lambda_2, \cdots \lambda_r\}$ and $\theta = \min\{\theta_1, \theta_2, \cdots \theta_m\}$;

(iii) $\lim_{t\to\infty}(1-\beta_t)\beta_t > 0$, $\inf_t \rho_t(\omega - \rho_t) > 0$ and $\sum_{i=1}^{r} \alpha_{t,i} = 1$;

(iv) $\epsilon_t = \circ(\beta_t)$, i.e. $\lim_{t\to\infty} \frac{\epsilon_t}{\beta_t} = 0$.

Algorithm 3.1. Extrapolation method for split common null point and fixed point problem.

Initialization: Given $\epsilon > 0$, $\delta > 3$, let $q_0, q_1 \in \mathcal{H}_1$ and $\{\rho_t\} \subset (0,\omega)$.

Step 1: Given q_{t-1}, q_t and compute

$$w_t = q_t + \xi_t(q_t - q_{t-1}),$$

where ξ_t satisfies $0 \leq |\xi_t| \leq \bar{\xi}_t$ with $\bar{\xi}_t$ defined by

$$\bar{\xi}_t = \begin{cases} \min\left\{\frac{t-1}{t+\delta-1}, \quad \frac{\varepsilon_t}{||q_t - q_{t-1}\|}\right\}, & q_t \neq q_{t-1} \\ \frac{t-1}{t+\delta-1}, & q_t = q_{t-1}. \end{cases}$$

Step 2: Compute

$$y_t = w_t - \gamma_t \sum_{i=1}^{r} \Psi_i(w_t),$$

where

$$\gamma_t = \begin{cases} \frac{\rho_t \sum_{i=1}^{r} h_i(w_t)}{\sigma^2(w_t)}, & \sigma^2(w_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: Compute

$$z_t = y_t + \sum_{i=1}^{r} \alpha_{t,i} \frac{1-\varphi_i}{2}(U_i - I)y_t.$$

Step 4: Compute

$$u_t = z_t - \tau_t \sum_{j=1}^{m} B_j^* \Delta_j(B_j z_t),$$

where

$$\tau_t = \begin{cases} \frac{\rho_t \sum_{j=1}^{m} s_j(z_t)}{\sigma^2(z_t)}, & \sigma^2(z_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 5: Compute

$$q_{t+1} = \beta_t \phi(w_t) + (1-\beta_t)u_t.$$

Step 6: If $\|q_{t+1} - q_t\| \leq \varepsilon$, then the iterative process stops. Otherwise, set $t := t+1$ and go to **Step 1.**

Theorem 3.1. *Suppose Assumption 1 and Assumption 2 hold. Then the sequence* $\{q_t\}$ *generated by Algorithm 3.1 converges in norm to* $p = P_\Gamma(0)$ *(i.e. the minimum norm element of the solution set* Γ*).*

Proof. Let $p \in \Gamma$. Thus $p \in \Psi_i^{-1}(0)$. Since $\Psi_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $\Delta_j : \mathcal{H}_2 \to \mathcal{H}_2$ are λ_i $(1 \leq i \leq r)$ and θ_j $(1 \leq j \leq m)$-inverse strongly monotone, respectively, we have for all $t \geq \mathbb{N}$

$$\begin{aligned} \langle \Psi_i w_t, w_t - p\rangle &= \langle \Psi_i w_t - \Psi_i p, w_t - p\rangle \\ &\geq \lambda_i \|\Psi_i w_t\|^2 \\ &= 2\lambda_i h_i(w_t) \\ &\geq 2\lambda h_i(w_t), \end{aligned}$$

and

$$\begin{aligned} \langle B_j^* \Delta_j(B_j z_t), z_t - p\rangle &= \langle \Delta_j(B_j z_t), B_j z_t - B_j p\rangle \\ &\geq \theta_j \|\Delta_j(B_j z_t)\|^2 \\ &= 2\theta_j s_j(z_t) \\ &\geq 2\theta s_j(z_t). \end{aligned}$$

Now, using Step 2 of Algorithm 3.1, we get

$$\begin{aligned} \|y_t - p\|^2 &= \|w_t - \gamma_t \sum_{i=1}^{r} \Psi_i w_t - p\|^2 \\ &\leq \|w_t - p\|^2 + \gamma_t^2 \Big\| \sum_{i=1}^{r} \Psi_i w_t \Big\|^2 - 2\gamma_t \Big\langle \sum_{i=1}^{r} \Psi_i w_t, w_t - p \Big\rangle \\ &\leq \|w_t - p\|^2 + \gamma_t^2 \big(\sum_{i=1}^{r} \|\Psi_i w_t\| \big)^2 - 4\lambda\gamma_t \sum_{i=1}^{r} h_i(w_t) \\ &\leq \|w_t - p\|^2 + \frac{\rho_t(\rho_t - 4\lambda)\big(\sum_{i=1}^{r} h_i(w_t)\big)^2}{\sigma^2(w_t)}. \end{aligned} \tag{3.1}$$

Also, using Step 4 of Algorithm 3.1, we get

$$\begin{aligned} \|u_t - p\|^2 &= \|z_t - \tau_t \sum_{j=1}^{m} B_j^* \Delta_j(B_j z_t) - p\|^2 \\ &= \|z_t - p\|^2 + \tau_t^2 \Big\| \sum_{j=1}^{m} B_j^* \Delta_j(B_j z_t) \Big\|^2 - 2\tau_t \Big\langle \sum_{j=1}^{m} B_j^* \Delta_j(B_j z_t), z_t - p \Big\rangle \\ &\leq \|z_t - p\|^2 + \tau_t^2 \left(\sum_{j=1}^{m} \|B_j^* \Delta_j(B_j z_t)\| \right)^2 - 4\theta\tau_t \sum_{j=1}^{m} s_j(z_t) \end{aligned}$$

$$\leq \|z_t - p\|^2 + \frac{\rho_t(\rho_t - 4\theta)\big(\sum\limits_{j=1}^{m} s_j(z_t)\big)^2}{\sigma^2(z_t)}. \tag{3.2}$$

By utilizing the convexity of $\|.\|^2$, we obtain from Step 3 of Algorithm 3.1 that

$$\begin{aligned}
&\|z_t - p\|^2 \\
=&\Big\|y_t + \sum_{i=1}^{r}\alpha_{t,i}\frac{1-\varphi_i}{2}(U_i - I)y_t - p\Big\|^2 \\
\leq&\sum_{i=1}^{r}\alpha_{t,i}\Big\|y_t + \frac{1-\varphi_i}{2}(U_i - I)y_t - p\Big\|^2 \\
=&\sum_{i=1}^{r}\alpha_{t,i}\bigg(\|y_t - p\|^2 + \big(\frac{1-\varphi_i}{2}\big)^2\|(U_i - I)y_t\|^2 + 2\big(\frac{1-\varphi_i}{2}\big)\langle y_t - p, (U_i - I)y_t\rangle\bigg) \\
=&\sum_{i=1}^{r}\alpha_{t,i}\bigg(\|y_t - p\|^2 + (\frac{1-\varphi_i}{2})^2\big\|(U_i - I)y_t\big\|^2 - 2\big(\frac{1-\varphi_i}{2}\big)\big(\frac{1-\varphi_i}{2}\big)\big\|(U_i - I)y_t\big\|^2\bigg) \\
\leq&\|y_t - p\|^2 - \sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\big\|(U_i - I)y_t\big\|^2.
\end{aligned} \tag{3.3}$$

By substituting (3.3) into (3.2), we get

$$\begin{aligned}
\|u_t - p\|^2 \leq&\|y_t - p\|^2 - \sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\big\|(U_i - I)y_t\big\|^2 \\
&+ \frac{\rho_t(\rho_t - 4\theta)\big(\sum\limits_{j=1}^{m} s_j(z_t)\big)^2}{\sigma^2(z_t)}.
\end{aligned} \tag{3.4}$$

On substituting (3.1) into (3.4), we have

$$\begin{aligned}
\|u_t - p\|^2 \leq&\|w_t - p\|^2 - \sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i - I)y_t\|^2 + \frac{\rho_t(\rho_t - 4\lambda)\big(\sum\limits_{i=1}^{r} h_i(w_t)\big)^2}{\sigma^2(w_t)} \\
&+ \frac{\rho_t(\rho_t - 4\theta)\big(\sum\limits_{j=1}^{m} s_j(z_t)\big)^2}{\sigma^2(z_t)} \\
=&\|w_t - p\|^2 - \sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i - I)y_t\|^2 - \frac{\rho_t(4\lambda - \rho_t)\big(\sum\limits_{i=1}^{r} h_i(w_t)\big)^2}{\sigma^2(w_t)} \\
&- \frac{\rho_t(4\theta - \rho_t)\big(\sum\limits_{j=1}^{m} s_j(z_t)\big)^2}{\sigma^2(z_t)} \qquad (3.5) \\
\leq&\|w_t - p\|^2. \qquad (3.6)
\end{aligned}$$

Using Step 1 of Algorithm 3.1, we get

$$\|u_t - p\| \leq \|w_t - p\|$$

$$
\begin{aligned}
&= \|q_t + \xi_t(q_t - q_{t-1}) - p\| \\
&\le \|q_t - p\| + |\xi_t| \cdot \|q_t - q_{t-1}\|. \qquad (3.7)
\end{aligned}
$$

Utilizing Step 5 of Algorithm 3.1 and (3.7), we have

$$
\begin{aligned}
\|q_{t+1} - p\| &= \|\beta_t \phi(w_t) + (1-\beta_t)u_t - p\| \\
&= \|\beta_t(\phi(w_t) - p) + (1-\beta_t)(u_t - p)\| \\
&\le \beta_t\|\phi(w_t) - \phi(p)\| + \beta_t\|\phi(p) - p\| + (1-\beta_t)\|u_t - p\| \\
&\le \beta_t\kappa\|w_t - p\| + \beta_t\|\phi(p) - p\| + (1-\beta_t)\big(\|q_t - p\| + |\xi_t| \cdot \|q_t - q_{t-1}\|\big) \\
&\le (1-(1-\kappa)\beta_t)\|q_t - p\| + (1-\kappa)\beta_t \frac{\|\phi(p) - p\| + |\xi_t| \cdot \frac{\|q_t - q_{t-1}\|}{\beta_t}}{1-\kappa} \\
&\le \max\left\{\|q_t - p\|, \frac{\|\phi(p) - p\| + |\xi_t| \cdot \frac{\|q_t - q_{t-1}\|}{\beta_t}}{1-\kappa}\right\}.
\end{aligned}
$$

From condition (iv) of Assumption 2 and the definition of $\bar{\xi}_t$, we get $\big\{|\xi_t| \cdot \frac{\|q_t - q_{t-1}\|}{\beta_t}\big\}$ is bounded. Thus, there exists some constant $M_1 > 0$ such that

$$
M_1 = \sup_{t\ge 1}\left\{\frac{\|\phi(p) - p\| + |\xi_t| \cdot \frac{\|q_t - q_{t-1}\|}{\beta_t}}{1-\kappa}\right\}.
$$

Then, by the mathematical induction, we conclude that

$$
\|q_t - p\| \le \max\{\|q_1 - p\|, M_1\}.
$$

Therefore, $\{q_t\}$ is bounded. Consequently, $\{w_t\}$, $\{z_t\}$, $\{u_t\}$ and $\{\phi(w_t)\}$ are bounded. It is obvious to see from Step 1 of Algorithm 3.1 that

$$
\begin{aligned}
\|w_t - p\|^2 &= \|q_t + \xi_t(q_t - q_{t-1}) - p\|^2 \\
&\le \|q_t - p\|^2 + 2\langle q_t - p + \xi_t(q_t - q_{t-1}), \xi_t(q_t - q_{t-1})\rangle \\
&\le \|q_t - p\|^2 + 2\big(\|q_t - p\| + |\xi_t| \cdot \|q_t - q_{t-1}\|\big)|\xi_t| \cdot \|q_t - q_{t-1}\| \\
&\le \|q_t - p\|^2 + 2M_2|\xi_t| \cdot \|q_t - q_{t-1}\| \\
&\le \|q_t - p\|^2 + 2M_2\epsilon_t, \qquad (3.8)
\end{aligned}
$$

where $M_2 = \sup_{t\ge 1}\{\|q_t - p\| + |\xi_t| \cdot \|q_t - q_{t-1}\|\}$.

On the other hand, using (3.5), we have

$$
\begin{aligned}
\|q_{t+1} - p\|^2 =& \|\beta_t\phi(w_t) + (1-\beta_t)u_t - p\|^2 \\
=& \|\beta_t(\phi(w_t) - p) + (1-\beta_t)(u_t - p)\|^2 \\
\le& \beta_t\|\phi(w_t) - p\|^2 + (1-\beta_t)\|u_t - p\|^2 \\
=& \beta_t\|\phi(w_t) - \phi(p) + \phi(p) - p\|^2 + (1-\beta_t)\|u_t - p\|^2 \\
\le& \beta_t\big(\kappa^2\|w_t - p\|^2 + 2\langle\phi(w_t) - \phi(p), \phi(p) - p\rangle + \|\phi(p) - p\|^2\big) \\
&+ (1-\beta_t)\|u_t - p\|^2 \\
=& \beta_t\big(\kappa^2\|w_t - p\|^2 + 2\langle\phi(w_t) - p + p - \phi(p), \phi(p) - p\rangle + \|\phi(p) - p\|^2\big) \\
&+ (1-\beta_t)\|u_t - p\|^2 \\
=& \beta_t\big(\kappa^2\|w_t - p\|^2 + 2\langle\phi(w_t) - p, \phi(p) - p\rangle - \|\phi(p) - p\|^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\beta_t)\|u_t-p\|^2\\
\leq&\beta_t\|w_t-p\|^2+(1-\beta_t)\Bigg(\|w_t-p\|^2-\sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i-I)y_t\|^2\\
&-\frac{\rho_t(4\lambda-\rho_t)\big(\sum_{i=1}^{r}h_i(w_t)\big)^2}{\sigma^2(w_t)}-\frac{\rho_t(4\theta-\rho_t)\big(\sum_{j=1}^{m}s_j(z_t)\big)^2}{\sigma^2(z_t)}\Bigg)\\
&+2\beta_t\langle\phi(w_t)-p,\phi(p)-p\rangle\\
=&\|w_t-p\|^2-(1-\beta_t)\Bigg(\sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i-I)y_t\|^2\\
&+\frac{\rho_t(4\lambda-\rho_t)\big(\sum_{i=1}^{r}h_i(w_t)\big)^2}{\sigma^2(w_t)}+\frac{\rho_t(4\theta-\rho_t)\big(\sum_{j=1}^{m}s_j(z_t)\big)^2}{\sigma^2(z_t)}\Bigg)\\
&+2\beta_t\langle\phi(w_t)-p,\phi(p)-p\rangle\\
\leq&\|q_t-p\|^2-(1-\beta_t)\Bigg(\sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i-I)y_t\|^2\\
&+\frac{\rho_t(4\lambda-\rho_t)\big(\sum_{i=1}^{r}h_i(w_t)\big)^2}{\sigma^2(w_t)}+\frac{\rho_t(4\theta-\rho_t)\big(\sum_{j=1}^{m}s_j(z_t)\big)^2}{\sigma^2(z_t)}\Bigg)\\
&+2M_2\epsilon_t+2\beta_t\langle\phi(w_t)-p,\phi(p)-p\rangle.
\end{aligned}
\tag{3.9}
$$

Set

$$
\begin{aligned}
\vartheta_t:=&(1-\beta_t)\Bigg(\sum_{i=1}^{r}\alpha_{t,i}\frac{(1-\varphi_i)^2}{4}\|(U_i-I)y_t\|^2-\frac{\rho_t(4\lambda-\rho_t)\big(\sum_{i=1}^{r}h_i(w_t)\big)^2}{\sigma^2(w_t)}\\
&-\frac{\rho_t(4\theta-\rho_t)\big(\sum_{j=1}^{m}s_j(z_t)\big)^2}{\sigma^2(z_t)}\Bigg),
\end{aligned}
$$

and

$$\chi_t:=2M_2\epsilon_t+2\beta_t\langle\phi(w_t)-p,\phi(p)-p\rangle.$$

Thus (3.9) implies that

$$r_{t+1}\leq r_t-\vartheta_t+\chi_t. \tag{3.10}$$

By the boundedness of $\{\phi(w_t)\}$ and $\beta\to 0$, we have that $\lim_{t\to\infty}\chi_t=0$. Thus, $\{\chi_t\}$ satisfies condition (ii) of Lemma 2.4. In order to complete the proof, it suffices to verify that $\vartheta_{t_k}\to 0$ $(k\to\infty)$. Noticing $\{\rho_{t_k}\}\leq\min\{4\lambda,4\theta\}$, (3.9) implies that

$$\sum_{t=1}^{\infty}\rho_{t_k}(\omega-\rho_{t_k})\Bigg(\frac{(\sum_{i=1}^{r}h_i(w_{t_k}))^2}{\sigma^2(w_{t_k})}+\frac{(\sum_{j=1}^{m}s_j(z_{t_k}))^2}{\sigma^2(z_{t_k})}\Bigg)<\infty.$$

Since $\inf_k \rho_{t_k}(\omega - \rho_{t_k}) > 0$ and the boundedness of $\sigma^2(w_{t_k})$ and $\sigma^2(z_{t_k})$, it turns out that

$$\lim_{k\to\infty} h_i(w_{t_k}) = 0 = \lim_{k\to\infty} s_j(z_{t_k}), \text{ for } i = 1, \cdots, r \text{ and } j = 1, 2, \cdots, m. \tag{3.11}$$

Also, from the condition on $\alpha_{t_k,i}$, it turns out that

$$\lim_{k\to\infty} \|y_{t_k} - U_i y_{t_k}\| = 0, \; i = 1, 2, \cdots, r. \tag{3.12}$$

From Step 1 of Algorithm 3.1, we have that

$$\|w_{t_k} - q_{t_k}\| = |\xi_{t_k}| \cdot \|q_{t_k} - q_{t_{k-1}}\| \le \epsilon_{t_k} \to 0 \text{ as } k \to \infty. \tag{3.13}$$

Also, using Algorithm 3.1 and (3.12), it can be easily seen that

$$\lim_{k\to\infty} \|z_{t_k} - y_{t_k}\| = 0. \tag{3.14}$$

Since $\{w_t\}, y_t, \{u_t\}$ and $\{z_t\}$ are bounded, there exist subsequences $\{w_{t_k}\}, \{y_{t_k}\}$, $\{u_{t_k}\}, \{z_{t_k}\}$ and a constant $M > 0$ such that $\sum_{i=1}^{r} \|\Psi_i(w_{t_k})\| \le M$ and $\sum_{j=1}^{m} \|B_j^* \Delta_j(B_j z_{t_k})\| \le M$. This together with Step 2 and Step 4 of Algorithm 3.1 implies that

$$\|y_{t_k} - w_{t_k}\| \le M\gamma_{t_k} \to 0, \quad \|u_{t_k} - z_{t_k}\| \le M\tau_{t_k} \to 0. \tag{3.15}$$

From condition (i) of Assumption 2, it turns out that

$$\|q_{t_{k+1}} - u_{t_k}\| \le \beta_{t_k} \|\phi(w_{t_k}) - u_{t_k}\| \to 0, \; k \to \infty. \tag{3.16}$$

From (3.13), (3.14) and (3.15), we obtain

$$\begin{cases} \lim_{k\to\infty} \|y_{t_k} - q_{t_k}\| = 0, \\ \lim_{k\to\infty} \|u_{t_k} - y_{t_k}\| = 0, \\ \lim_{k\to\infty} \|u_{t_k} - q_{t_k}\| = 0, \\ \lim_{k\to\infty} \|z_{t_k} - q_{t_k}\| = 0, \\ \lim_{k\to\infty} \|q_{t_{k+1}} - q_{t_k}\| = 0. \end{cases} \tag{3.17}$$

Since $\{q_t\}$ is bounded, there exists a subsequence $\{q_{t_k}\} \rightharpoonup x^* \in \Gamma$. Also, using the fact that $\{y_t\}$, $\{z_t\}$ and $\{w_t\}$ are bounded, there exist subsequences $\{y_{t_k}\}$ of $\{y_t\}$, $\{z_{t_k}\}$ of $\{z_t\}$ and $\{w_{t_k}\}$ of $\{w_t\}$ which converge weakly to $x^* \in \Gamma$. We will verify that $\Psi_i(x^*) = 0$ and $\Delta_j(B_j x^*) = 0$ for each fixed $1 \le i \le r$ and $1 \le j \le m$. To establish this, we apply (3.11) to get that $\Psi_i(w_{t_k}) \to 0$ in norm and $\Delta_j(B_j z_{t_k}) \to 0$ in norm (as $k \to \infty$). Since Ψ_i is λ_i- ism, we get

$$\langle \Psi_i w_{t_k} - \Psi_i x^*, w_{t_k} - x^* \rangle \ge \lambda_i \|\Psi_i w_{t_k} - \Psi_i x^*\|^2. \tag{3.18}$$

Now, since $\Psi_i w_{t_k} \to 0$ in norm and $w_{t_k} \rightharpoonup x^*$, by taking the limit as $k \to \infty$ in (3.18), we arrive at $\Psi_i(x^*) = 0$. Similarly, since Δ_j is θ_j- ism, we can repeat the argument in (3.18) (simply replacing Ψ_i with Δ_j) with $B_j z_{t_k} \rightharpoonup B_j x^*$. Hence, $0 =$

$\Delta_j(B_jx^*)$. In addition, using (3.12) and Lemma 2.3, we get that $x^* \in \bigcap_{i=1}^{r} Fix(U_i)$. Therefore, we conclude that $x^* \in \Gamma$.

Next, we show that $\{q_t\}$ converges strongly to p.

From Algorithm 3.1, (3.6) and (3.8), we have

$$\begin{aligned}
\|q_{t+1} - p\|^2 &= \|\beta_t(\phi(w_t) - p) + (1-\beta_t)(u_t - p)\|^2 \\
&= \beta_t^2\|\phi(w_t) - \phi(p) + \phi(p) - p\|^2 + (1-\beta_t)\|u_t - p\|^2 \\
&+ 2\beta_t(1-\beta_t)\langle \phi(w_t) - \phi(p) + \phi(p) - p, u_t - p\rangle \\
&\leq 2\beta_t^2(\kappa^2\|w_t - p\|^2 + \|\phi(p) - p\|^2) + (1-\beta_t)^2\|w_t - p\|^2 \\
&+ 2\beta_t(1-\beta_t)(\kappa\|w_t - p\|^2 + \langle \phi(p) - p, u_t - p\rangle) \\
&= (1 - \beta_t(2 - \beta_t(1+2\kappa^2) - 2\kappa(1-\beta_t)))\|w_t - p\|^2 \\
&+ 2\beta_t^2\|\phi(p) - p\|^2 + 2\beta_t(1-\beta_t)\langle \phi(p) - p, u_t - p\rangle \\
&\leq (1 - \beta_t(2 - \beta_t(1+2\kappa^2) - 2\kappa(1-\beta_t)))\|q_t - p\|^2 + 2M_2\varepsilon_t \\
&+ 2\beta_t^2\|\phi(p) - p\|^2 + 2\beta_t(1-\beta_t)\langle \phi(p) - p, u_t - p\rangle. \qquad (3.19)
\end{aligned}$$

By setting $a_t = \|q_t - p\|^2, \Upsilon_t = \beta_t(2 - \beta_t(1+2\kappa^2) - 2\kappa(1-\beta_t))$, and

$$\Phi_t := \frac{2(M_2\frac{\epsilon_t}{\beta_t} + \beta_t\|\phi(p) - p\|^2 + (1-\beta_t)\langle \phi(p) - p, u_t - p\rangle)}{2 - \beta_t(1+2\kappa^2) - 2\kappa(1-\beta_t)},$$

then (3.19) implies that

$$q_{t+1} \leq (1-\Upsilon_t)a_t + \Upsilon_t\Phi_t. \qquad (3.20)$$

Since $\sum_{t=0}^{\infty}\beta_t = \infty$, we have that $\sum_{t=0}^{\infty}\Upsilon_t = \infty$. Thus, $\{\Upsilon_t\}$ satisfies condition (i) of Lemma 2.4. Next, we show that $\limsup_{k\to\infty}\Phi_{t_k} \leq 0$. To establish this, we choose a subsequence $\{u_{t_k}\}$ of $\{u_t\}$ such that

$$\lim_{k\to\infty}\langle \phi(p) - p, u_{t_k} - p\rangle = \limsup_{t\to\infty}\langle \phi(p) - p, u_t - p\rangle.$$

Since $\{u_{t_k}\} \rightharpoonup x^*$, it follows that

$$\begin{aligned}
\limsup_{t\to\infty}\langle \phi(p) - p, u_t - p\rangle &= \lim_{k\to\infty}\langle \phi(p) - p, u_{t_k} - p\rangle \\
&= \langle \phi(p) - p, x^* - p\rangle \\
&\leq 0. \qquad (3.21)
\end{aligned}$$

Thus, $\limsup_{k\to\infty}\Phi_{t_k} \leq 0$. By substituting (3.21) into (3.20) and applying Lemma 2.4, we obtain that the sequence $\{q_t\}$ converges strongly to $p \in \Gamma$, which completes the proof.

□

Algorithm 3.2. Multi-step inertial method for split common null point and fixed point problem.

Initialization: Given $\epsilon > 0$, $\delta > 3, s \in \mathbb{N}^+$ let $q_1, q_0, \cdots, q_{1-s} \in \mathcal{H}_1$ and $\{\rho_t\} \in (0, \omega)$.

Step 1: Given $q_t, q_{t-1}, \cdots, q_{t-s}$ and compute

$$w_t = q_t + \sum_{i \in Q} \xi_{i,t}(q_{t-i} - q_{t-i-1}),$$

where $Q := \{0, 1, \cdots, s-1\}$ and $\xi_{i,t}$ satisfies $0 \leq |\xi_{i,t}| \leq \bar{\xi}_t$ with $\bar{\xi}_t$ defined by

$$\bar{\xi}_t = \begin{cases} \min\left\{\frac{t-1}{t+\delta-1}, \quad \frac{\varepsilon_t}{\sum_{i \in Q} \|q_{t-i} - q_{t-i-1}\|}\right\}, & \sum_{i \in Q} \|q_{t-i} - q_{t-i-1}\| \neq 0 \\ \frac{t-1}{t+\delta-1}, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$y_t = w_t - \gamma_t \sum_{i=1}^{r} \Psi_i(w_t),$$

where

$$\gamma_t = \begin{cases} \frac{\rho_t \sum_{i=1}^{r} h_i(w_t)}{\sigma^2(w_t)}, & \sigma^2(w_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: Compute

$$z_t = y_t + \sum_{i=1}^{r} \alpha_{t,i} \frac{1 - \varphi_i}{2}(U_i - I)y_t.$$

Step 4: Compute

$$u_t = z_t - \tau_t \sum_{j=1}^{m} B_j^* \Delta_j(B_j z_t),$$

where

$$\tau_t = \begin{cases} \frac{\rho_t \sum_{j=1}^{m} s_j(z_t)}{\sigma^2(z_t)}, & \sigma^2(z_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 5: Compute

$$q_{t+1} = \beta_t \phi(w_t) + (1 - \beta_t)u_t.$$

Step 6: If $\|q_{t+1} - q_t\| \leq \varepsilon$, then the iterative process stops. Otherwise, set $t := t+1$ and go to **Step 1.**

Theorem 3.2. *Suppose Assumptions 1 and Assumptions 2 hold. Then the sequence* $\{q_t\}$ *generated by Algorithm 3.1 converges in norm to* $p = P_\Gamma(0)$ *(i.e. the minimum norm element of the solution set* Γ*).*

Proof. Let $p \in \Gamma$. Then we have from Step 1 of Algorithm 3.2 that

$$\|y_t - p\| = \|q_t + \sum_{i \in Q} |\xi_{i,t}| \cdot \|q_{t-i} - q_{t-i-1}\| \\ \leq \|q_t - p\| + \bar{\xi}_t \sum_{i \in Q} \|q_{t-i} - q_{t-i-1}\|.$$

Using the approach in Algorithm 3.1, we can establish that $\{q_t\}, \{u_t\}, \{w_t\}, \{y_t\}$ and $\phi(w_t)$ are bounded.

Also, using Step 1 of Algorithm 3.2, we have

$$\begin{aligned} \|w_t - p\|^2 &= \|q_t + \sum_{i \in Q} \xi_{i,t}(q_{t-i} - q_{t-i-1}) - p\|^2 \\ &\leq \|q_t - p\|^2 + 2\langle q_t - p + \sum_{i \in Q} \xi_{i,t}(q_{t-i} - q_{t-i-1}), \sum_{i \in Q} \xi_{i,t}(q_{t-i} - q_{t-i-1})\rangle \\ &\leq \|q_t - p\|^2 + 2M_4|\xi_t| \sum_{i \in Q} \|q_{t-i} - q_{t-i-1}\| \\ &\leq \|q_t - p\|^2 + 2M_4\epsilon_t, \end{aligned}$$

where $M_4 = \sup_{t \geq 1} \big\{ \|q_t - p\| + |\xi_t| \sum_{i \in Q} \|q_{t-i} - q_{t-i-1}\| \big\}$.

The rest of the proof follows from the one of Theorem 3.1. This completes the proof. □

We state the consequence of our main result.

If U_i is a quasi-nonexpansive mapping, then we have

Corollary 3.1.

Algorithm 3.3. Extrapolation method for split common null point and fixed point problem.

Initialization: Given $\epsilon > 0$, $\delta > 3$, $\sum_{i=1}^{r} \alpha_{t,i} = 1$ and let $q_0, q_1 \in \mathcal{H}_1$ and $\{\rho_t\} \in (0, \omega)$.

Step 1: Given q_{t-1}, q_t and compute

$$w_t = q_t + \xi_t(q_t - q_{t-1}),$$

where ξ_t satisfies $0 \leq |\xi_t| \leq \bar{\xi}_t$ with $\bar{\xi}_t$ defined by

$$\bar{\xi}_t = \begin{cases} \min\left\{ \frac{t-1}{t+\delta-1}, \quad \frac{\varepsilon_t}{||q_t - q_{t-1}||} \right\}, & q_t \neq q_{t-1} \\ \frac{t-1}{t+\delta-1}, & q_t = q_{t-1}. \end{cases}$$

Step 2: Compute

$$y_t = w_t - \gamma_t \sum_{i=1}^{r} \Psi_i(w_t),$$

where

$$\gamma_t = \begin{cases} \frac{\rho_t \sum_{i=1}^{r} h_i(w_t)}{\sigma^2(w_t)}, & \sigma^2(w_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: Compute

$$z_t = \alpha_{t,0} y_t + \sum_{i=1}^{r} \alpha_{t,i} U_i y_t.$$

Step 4: Compute

$$u_t = z_t - \tau_t \sum_{j=1}^{m} B_j^* \Delta_j (B_j z_t),$$

where

$$\tau_t = \begin{cases} \frac{\rho_t \sum\limits_{j=1}^{m} s_j(z_t)}{\sigma^2(z_t)}, & \sigma^2(z_t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 5: Compute

$$q_{t+1} = \beta_t \phi(w_t) + (1 - \beta_t) u_t.$$

Step 6: If $\|q_{t+1} - q_t\| \leq \varepsilon$, then the iterative process stops. Otherwise, set $t := t+1$ and go to **Step 1.**

4. Numerical example

In this section, we present a numerical example to demonstrate the performance of our iterative method in comparison with the un-accelerated iterative method.

Example 4.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2[0, 2\pi]$. Define the mappings Ψ_1, Δ_1 and $B_1 x(t)$, $\Psi_1 x(t) := \frac{x(t)}{2}, \Delta_1(x)(t) := \frac{2x(t)}{3}$ and $U_1 x(t) := \frac{-5x(t)}{3}$ for all $x \in L^2[0, 2\pi]$. Then it can be shown that Ψ_1 and Δ_1 are $\frac{1}{2}-$ inverse strongly monotone mappings. In addition, it is easy to observe that U_1 is $\frac{1}{4}-$ strict pseudo-contraction. Put $\Upsilon(x) = \frac{1}{100}x, \epsilon_t = \frac{\beta_t}{t^{0.01}}$ with $\beta_t = \frac{1}{10t}, \omega = 2, \rho_t = 2 - (\frac{1}{(t+1)})$ and $\alpha_{t,1} = \frac{1}{2}$ for all $t \geq 1$. We use $\|q_{t+1} - q_t\| < \varepsilon$ as the stopping criteria. Take $\varepsilon = 10^{-5}$, we display the numerical result in Figure 1.

(Case 1) $q_0(t) = (\sin(-3t) + \cos(-10t))/600$ and $q_1(t) = (\sin(-5t) + \cos(-7t))/500$,

(Case 2) $q_0(t) = \frac{t^2}{100}$ and $q_1(t) = \frac{2t^2}{300}$,

(Case 3) $q_0(t) = \frac{(t^2 - e^{-t})}{100}$ and $q_1(t) = \frac{(t^2 - e^{-t})}{300}$.

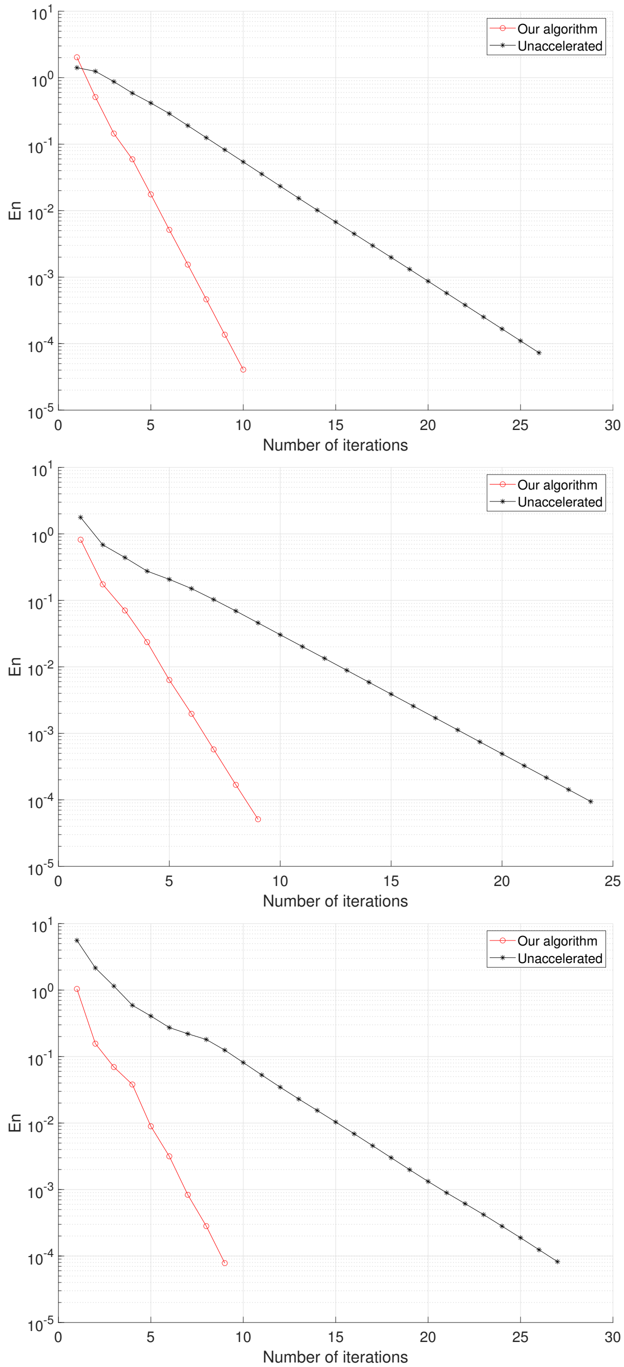

Figure 1. Example 4.1. Top : Case 1, Middle: Case 2, Bottom: Case 3

5. Conclusion

In this manuscript, we propose two different iterative methods for solving SCNPP and fixed point problem of finite family of a demimetric mappings. We establish strong convergence results for both iterative methods under the assumption that the operators are inverse strongly monotone. Our algorithms have two advantages: (i) they are forward (hence less computational cost), which do not involve any computation of any resolvent of a monotone operator as opposed by several backward algorithms in the literature, and (ii) they do not require any prior knowledge of the operator norms. These advantages makes both algorithms easily implementable. Lastly, the class of mappings employed in this manuscript generalizes the class of pseudo-contractive, quasi-nonexpansive and nonexpansive mappings.
Competing interests: The authors declare that there is no conflicting interests.

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