Journal of Nonlinear Modeling and Analysis
Volume 7, Number 4, August 2025, 1307–1331
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1307

Strong Convergence Theorem Involving Two-Step Inertial Technique Without On-Line Rule for Split Feasibility Problem

Chibueze C. Okeke[1], Abubakar Adamu[2,3,4,†] and Kalu O. Okorie[1]

Abstract This work presents an approach for solving the split feasibility problem in an efficient manner. For solving the split feasibility problem, we present a method with a two-step inertial extrapolation and self-adaptive stepsize. The adjustable stepsize and two-step inertial extrapolation both contribute to the proposed method's improved rate of convergence and decreased computational complexity. The strong convergence results are obtained without on-line rule of the inertial parameters and the iterates. This makes our proof arguments different from what is obtainable in the literature where on-line rule is imposed on algorithms involving inertial extrapolation step. As far as we know, no strong convergence result has been obtained before now for algorithms with two step inertial for solving split feasibility problems in the literature. To demonstrate the viability of our suggested strategy, numerical results are provided at the end.

Keywords Split feasibility problems, two-step inertial technique, CQ methods, strong convergence

MSC(2010) 47H05, 47H10, 49J20, 47J25.

1. Introduction

The split feasibility problem (SFP) is defined as the problem of finding a point $\hat{x} \in \mathcal{C}$ such that

$$A\hat{x} \in \mathcal{Q}, \tag{1.1}$$

where $\mathcal{C} \subseteq \mathcal{H}_1$ and $\mathcal{Q} \subseteq \mathcal{H}_2$ are nonempty, closed and convex sets, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded and linear operator. We denote the solution set of the problem (1.1) by Ω. For the purpose of resolving inverse problems associated with phase retrievals and medical image recovery, Censor and Elfving [10] first presented the SFP in

†The corresponding author.
Email address: chibueze.okeke87@yahoo.com, chibueze.okeke@wits.ac.za (C. C. Okeke), abubakar.adamu@neu.edu.tr, aadamu@aust.edu.ng (A. Adamu), 2710937@students.wits.ac.za (K.O. Okorie)
[1]School of Mathematical Sciences, University of the Witwatersrand, Johannesburg, South Africa
[2]School of Mathematics, Chongqing Normal University, Chongqing 400047, China
[3]Operational Research Center in Healthcare, Near East University, TRNC Mersin 10, Nicosia 99138, Turkey
[4]Mathematics Institute, African University of Science and Technology, Abuja 900107, Nigeria

finite dimensional Hilbert space. Numerous studies have demonstrated the SFP's versatility in a variety of fields, including computer tomography, picture restoration, and data reduction (see [11–14, 33, 39] and other references therein).

Various iterative strategies for solving the SFP have been investigated and introduced by a number of researchers (see [9, 15–17, 20, 36, 41] and other references therein). The CQ algorithm, developed by Bryne [8], is a well-known algorithm for finding the solution to the SFP. Iteratively, this algorithm generates the sequence $\{x_k\}$:

$$x_{k+1} = P_{\mathcal{C}}(x_k - \lambda A^*(I - P_{\mathcal{Q}})Ax_k), \quad \forall\ k \geq 1, \tag{1.2}$$

where $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$, $P_{\mathcal{C}},\ P_{\mathcal{Q}}$ are the orthogonal projections onto $\mathcal{C}$ and $\mathcal{Q}$ respectively. A weak convergence result was established by the author. The drawback of this approach is that it requires the calculation of the spectral radius of the matrix A^*A or the norm estimate of the linear operator A, both of which are challenging and occasionally impossible to do in an infinite dimensional setting. To overcome this drawback, Byrne [8] presented a method for estimating matrix norms (see [8], Proposition 4.1). The condition on this method is highly stringent. In order to overcome this drawback, López et al. [27] substituted an adaptive stepsize for the stepsize in (1.2) and defined it as follows:

$$\eta_k = \frac{\sigma_k F(x_k)}{\|\nabla F(x_k)\|^2}, \quad k \geq 1,$$

where $\eta_k \in (0, 4)$, $F(x_k) = \frac{1}{2}\|(I - P_{\mathcal{Q}})Ax_k\|^2$ and $\nabla F(x_k) = A^*(I - P_{\mathcal{Q}}Ax_k)$ for all $k \geq 1$. Several authors have adopted this adaptive stepsize for solving the SFP (see [19, 20, 25]).

Iterative methods for approximating solutions of the SFP are known to have slow convergence properties. In recent years, a host of researchers have invested considerable effort into enhancing the convergence properties of these iterative algorithms. Among the prominent strategies for getting acceleration is the inertial extrapolation technique, which traces its roots back to Polyak's early work on smooth convex minimization problems. In essence, the inertial acceleration strategy entails forming a nonconvex combination of two previous terms to derive the subsequent iterate. For further insights into this technique and its applications to iterative methods tailored for solving the SFP (1.1), interested readers may refer to [2, 3, 5, 6, 32], along with the additional references cited therein.

Dang et al. [18] recently proposed one-step inertial relaxed CQ techniques for finding the solution of (1.1) and they proposed them as follows:

$$x_{k+1} = P_{\mathcal{C}_k}(y_k - \lambda A^*(I - P_{\mathcal{Q}_k})Ay_k) \tag{1.3}$$

and

$$x_{k+1} = (1 - \alpha_k)y_k + \alpha_k P_{\mathcal{C}_k}(y_k - \lambda A^*(I - P_{\mathcal{Q}_k})Ay_k), \tag{1.4}$$

where $y_k = x_k + \theta_k(x_k - x_{k-1})$, $\alpha_k \in (0, 1)$, $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$ and $0 \leq \theta_k \leq \bar{\theta}_k$ with

$$\bar{\theta}_k := \min\left\{\theta, \left(\max\frac{1}{\{k^2\|x_k - x_{k-1}\|^2,\ k^2\|x_k - x_{k-1}\|\}}\right)\right\}, \quad \theta \in [0, 1).$$

The authors proved that $\{x_k\}$ generated by algorithms (1.3) and (1.4) converges weakly to a point in Ω.

Remark 1.1. The condition on $\bar{\theta}_k$ is what is called the on-line rule. Observe that the step size λ depends on the knowledge of the operator norm. It is well-known that computing norm of operators is not an easy task in practice.

However, it has been discovered that in certain scenarios, the one-step inertial technique fails to give the desired acceleration effect. In [26] it was remarked that acceleration could be obtained using more than two points x_k, x_{k-1}. For example, acceleration may be obtained using the two-step inertial method defined as follows:

$$y_k = x_k + \theta(x_k - x_{k-1}) + \delta(x_{k-1} - x_{k-2}),$$

where $\theta > 0$ and $\delta < 0$. Poon and Liang [31] discussed the limitation of the one-step inertial acceleration of ADMM and they proposed an adaptive acceleration for ADMM. In [30], Polyak discussed that the multi-step inertial methods can boost the speed of optimization methods even though no convergence results of such multi-step inertial methods were given by Polyak [30]. Recently, the multi-step inertial methods have been studied by some researchers (see [21, 23, 28, 29]).

Inspired by the mentioned works above and other related works in literature (without on-line rule see, e.g., [24, 40]), our contributions are the following:

- we introduce a new Halpern-type CQ method with a two-step inertial extrapolation and self-adaptive stepsize for finding the solution of the SFP (1.1). The strong convergence results of the sequence generated by our proposed method are presented.
- Our approach includes two-step inertial (which accelerates convergence) and a self-adaptive step size (which reduces computational complexity). Consequently, our approach overcomes the restrictions of the one-step inertial with on-line rule studied in [1, 4, 5, 7, 18] and also the limitation of estimating the linear operator or the spectral radius of a matrix used in [8].
- We give numerical results of our proposed method to demonstrate the applicability of our method.

2. Preliminaries

To obtain our strong convergence, we present some basic results, lemmas and definitions in this section.

Definition 2.1. A mapping $S : \mathcal{H}_1 \to \mathcal{H}_1$ is

(i.) nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall\, x, y \in \mathcal{H}_1;$$

(ii.) firmly nonexpansive if

$$\|Sx - Sy\|^2 \le \|x - y\|^2 - \|(I - S)x - (I - S)y\|^2, \quad \forall\, x, y \in \mathcal{H}_1.$$

Equivalently, the firmly nonexpansive mapping is given by

$$\|Sx - Sy\|^2 \le \langle x - y, Sx - Sy\rangle, \quad \forall\, x, y \in \mathcal{H}_1.$$

It can be seen in [22] that T is firmly nonexpansive if and only if $I - S$ is firmly nonexpansive.

Recall that for a nonempty, closed and convex subset $\mathcal{C}$ of $\mathcal{H}_1$, the metric projection denoted by $P_{\mathcal{C}}$, is a map defined on $\mathcal{H}_1$ onto $\mathcal{C}$ which assigns to each $x \in \mathcal{H}_1$, the unique point in $\mathcal{C}$, denoted by $P_{\mathcal{C}}x$ such that

$$||x - P_{\mathcal{C}}x|| \leq ||x - y||, \forall \ \ y \in \mathcal{C}.$$

Lemma 2.1. *Let $\mathcal{C}$ be a closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and $x, y \in \mathcal{H}_1$. Then*

(i) $\|P_{\mathcal{C}}x - P_{\mathcal{C}}y\|^2 \leq \langle P_{\mathcal{C}}x - P_{\mathcal{C}}y,\ x - y\rangle$;

(ii) $\|P_{\mathcal{C}}x - y\|^2 \leq \|x - y\|^2 - \|x - P_{\mathcal{C}}x\|^2$.

Lemma 2.2. *If u and v are non-negative numbers, then*

(i) $uv \leq \frac{u^2}{2p} + \frac{pv^2}{2}, \quad \forall p > 0,$ **(Peter-Paul Inequality)**

(ii) $(u + v)^2 \leq (2 + \sqrt{2})u^2 + \sqrt{2}v^2$.

Lemma 2.3. *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}_1$. For any $x \in \mathcal{H}_1$ and $z \in \mathcal{C}$, we have*

$$z = P_{\mathcal{C}}x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \textit{for all} \ \ y \in \mathcal{C}.$$

Lemma 2.4. *The following assertions hold in $\mathcal{H}_1$:*

(1) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in \mathcal{H}_1;$

(2) $\|\alpha x + \beta y\|^2 = \alpha(\alpha + \beta)\|x\|^2 + \beta(\alpha + \beta)\|y\|^2 - \alpha\beta(1 - \alpha)\|x - y\|^2, \quad \forall x, y \in \mathcal{H}_1,\ \alpha, \beta \in \mathbb{R};$

(3) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall\ x, y \in \mathcal{H}_1.$

Lemma 2.5. *Let $x, y, z \in \mathcal{H}_1$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\begin{aligned} &\|(1 + \alpha)x - (\alpha - \beta)y - \beta z\|^2 \\ =&(1 + \alpha)\|x\|^2 - (\alpha - \beta)\|y\|^2 - \beta\|z\|^2 + (1 + \alpha)(\alpha - \beta)\|x - y\|^2 \\ &+ \beta(1 + \alpha)\|x - z\|^2 - \beta(\alpha - \beta)\|y - z\|^2. \end{aligned}$$

Definition 2.2. A function $F : \mathcal{H}_1 \to \mathbb{R}$ is called convex, if for all $v \in [0, 1]$ and $x, y \in \mathcal{H}_1$,

$$F(vx + (1 - v)y) \leq vF(x) + (1 - v)F(y).$$

Remark 2.1. If F is convex on $\mathcal{H}_1$ and differentiable then

$$F(y) \geq F(x) + \langle y - x,\ \nabla F(x)\rangle, \quad \forall\ y \in \mathcal{H}_1.$$

Definition 2.3. A convex function $F : \mathcal{H}_1 \to \mathbb{R}$ is said to be subdifferentiable at a point $x \in \mathcal{H}_1$ if the set

$$\partial F(x) = \{u \in \mathcal{H}_1 \mid F(y) \geq F(x) + \langle u, y - x\rangle,\ \forall y \in \mathcal{H}_1\} \tag{2.1}$$

is nonempty, where each element in $\partial F(x)$ is called a subgradient of F at x, $\partial F(x)$ is called the subdifferential of F at x and the inequality in (2.1) is called the subdifferential inequality of F at x.

Remark 2.2. If F is convex and differential, then its gradient and subgradient coincide.

Definition 2.4. A function $F : \mathcal{H}_1 \to \mathbb{R}$ is said to be lower semicontinuous at x if

$$x_n \to x \quad \text{implies} \quad F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Note that F is lower semicontinuous on $\mathcal{H}_1$ if it is lower semicontinuous at every point $x \in \mathcal{H}_1$.

Lemma 2.6. *[9] Let* $F(x) := \frac{1}{2}\|(I - P_{\mathcal{Q}})Ax\|^2, \quad x \in \mathcal{C}$. *Then*

(a) F is convex and differentiable.

(b) $\nabla F(x) = A^*(I - P_{\mathcal{Q}})Ax, \quad x \in \mathcal{H}_1$.

(c) F is lower semicontinuous on $\mathcal{H}_1$.

(d) ∇F Lipschitz continuous with Lipschitz constant $\|A\|^2$.

Lemma 2.7. *Suppose that $\{\Upsilon_k\}$ and $\{r_k\}$ are sequences of nonnegative real numbers such that*

$$\Upsilon_{k+1} \leq (1 - \alpha_k)\Upsilon_k + s_k + r_k, \quad n \geq 0,$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{s_k\}$ is a real sequence. Let $\sum_{t=1}^{\infty} r_k < \infty$ and $s_k \leq \alpha_k M$ for some $M \geq 0$. Then, $\{\Upsilon_k\}$ is bounded.

3. Proposed method

Assumption 1. The following assumptions will be used in the convergence analysis of our proposed Algorithm.

(a) $\sigma \in (0, 4)$;

(b) θ and δ lie in the region

$$\text{choose } 0 < \theta \leq \frac{1}{3}, \ \ \tau \in (0,1), \ \chi = -\frac{1}{\tau} \quad \text{and then choose } \delta \text{ such that}$$
$$\max\left\{-\frac{\theta}{2}, \frac{2\theta - \frac{2}{3} - 8\theta^2\chi}{3 + 5\theta}, \theta - \frac{1}{3} - 2\chi\right\} < \delta \leq 0 \text{ and } \quad \delta^2 \leq \theta^2;$$

(c) with the choices of δ and θ in (b) above, compute

$$\frac{\delta(3\theta + 2 + 4\delta\chi)}{2(\theta - \theta\delta - 2\theta^2\chi - \frac{1}{3}) - \delta}$$

and choose

$$\alpha_k \in \left(0, 1 - \frac{\delta(3\theta + 2 + 4\delta\chi)}{2(\theta - \theta\delta - 2\theta^2\chi - \frac{1}{3}) - \delta}\right)$$

such that $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$.

Algorithm 1. (Strong convergent 2-step inertial CQ method)

Initialization: Let $x_0, x_1, x_2 \in \mathcal{H}_1$ be chosen arbitrarily. Set $k := 2$.

Step 1: Given the iterates $x_{k-2},\ x_{k-1}, x_k$ for each $k \geq 2$, choose $\{\alpha_k\}$, δ and θ satisfying Assumption (1).

Step 2: Compute

$$\begin{cases} w_k = x_k + \theta(x_k - x_{k-1}) + \delta(x_{k-1} - x_{k-2}), \\ y_k = w_k - \eta_k \nabla F_k(w_k), \end{cases} \tag{3.1}$$

$$x_{k+1} = P_{\mathcal{C}_k}(\alpha_k x_0 + (1-\alpha_k) y_k),$$

where

$$F_k(w_k) := \frac{1}{2}\left\|\Big(I - P_{\mathcal{Q}_k}\Big) A w_k\right\|^2, \quad \nabla F_k(w_k) := A^*(I - P_{\mathcal{Q}_k}) A w_k$$

and

$$\eta_k := \begin{cases} \frac{\sigma F_k(w_k)}{\|\nabla F_k(w_k)\|^2}, & \|\nabla F_k(w_k)\| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Set $k \leftarrow k+1$, and go to **Step 2**.

4. Convergence analysis

Lemma 4.1. *Assume that the solution set Ω of (1.1) is nonempty. Then the sequence $\{x_k\}$ generated by Algorithm 1 satisfying Assumption 1 is bounded.*

Proof. Let $p \in \Omega$. Since $\nabla F_k(w_k) = A^*(I - P_{\mathcal{Q}_k})Aw_k$, we obtain from the firmly nonexpansivity of $I - P_{\mathcal{Q}_k}$ and the definition of $F_k(w_k)$ that

$$\begin{aligned} \langle \nabla F_k(w_k),\ w_k - p\rangle &= \langle A^*(I - P_{\mathcal{Q}_k})Aw_k,\ w_k - p\rangle \\ &= \langle (I - P_{\mathcal{Q}_k})Aw_k,\ Aw_k - Ap\rangle \\ &= \langle (I - P_{\mathcal{Q}_k})Aw_k - (I - P_{\mathcal{Q}_k})Ap,\ Aw_k - Ap\rangle \\ &\geq \|(I - P_{\mathcal{Q}_k})Aw_k\|^2 \\ &= 2F_k(w_k). \end{aligned} \tag{4.1}$$

From the definition of y_k and η_k we have

$$\begin{aligned} \|y_k - w_k\| &= \|w_k - \eta_k \nabla F_k(w_k) - w_k\| \\ &= \eta_k \|\nabla F_k(w_k)\| \\ &= \sigma \frac{F_k(w_k)}{\|\nabla F_k(w_k)\|}. \end{aligned} \tag{4.2}$$

From the definition of y_k in Step 2, (4.1) and (4.2), we have

$$\begin{aligned} \|y_k - p\|^2 &= \|w_k - p - \eta_k \nabla F_k(w_k)\|^2 \\ &= \|w_k - p\|^2 + (\eta_k)^2 \|\nabla F_k(w_k)\|^2 - 2\eta_k \langle \nabla F_k(w_k),\ w_k - p\rangle \\ &\leq \|w_k - p\|^2 + (\eta_k)^2 \|\nabla F_k(w_k)\|^2 - 4\eta_k F_k(w_k) \end{aligned}$$

$$
\begin{aligned}
&= \|w_k - p\|^2 - 4\eta_k F_k(w_k) + (\eta_k)^2 \|\nabla F_k(w_k)\|^2 \\
&= \|w_k - p\|^2 - \sigma(4-\sigma)\frac{(F_k(w_k))^2}{\|\nabla F_k(w_k)\|^2} \\
&\le \|w_k - p\|^2 - \frac{4-\sigma}{\sigma}\|y_k - w_k\|^2. \qquad (4.3)
\end{aligned}
$$

Also, by the definition of w_k, we have

$$
\begin{aligned}
\|w_k - p\|^2 &= \|(1+\theta)(x_k - p) - (\theta-\delta)(x_{k-1} - p) - \delta(x_{k-2} - p)\|^2 \\
&= (1+\theta)\|x_k - p\|^2 - (\theta-\delta)x_{k-1} - p\|^2 - \delta\|x_{k-2} - p\|^2 \qquad (4.4) \\
&\quad +(1+\theta)(\theta-\delta)\|x_k - x_{k-1}\|^2 \\
&\quad +\delta(1+\theta)\|x_k - x_{k-2}\|^2 - \delta(\theta-\delta)\|x_{k-1} - x_{k-2}\|^2.
\end{aligned}
$$

From the definition of w_k and applying the Cauchy Schwartz inequality, we have

$$
\begin{aligned}
\|x_{k+1} - w_k\|^2 &= \|x_{k+1} - (x_k + \theta(x_k - x_{k-1}) + \delta(x_{k-1} - x_{k-2}))\|^2 \\
&= \|x_{k+1} - x_k\|^2 - 2\theta\langle x_{k+1} - x_k,\ x_k - x_{k-1}\rangle \\
&\quad + 2\delta\langle x_k - x_{k+1},\ x_{k-1} - x_{k-2}\rangle + \theta^2\|x_k - x_{k-1}\|^2 \\
&\quad + 2\delta\theta\langle x_k - x_{k-1},\ x_{k-1} - x_{k-2}\rangle + \delta^2\|x_{k-1} - x_{k-2}\|^2 \\
&\ge \|x_{k+1} - x_k\|^2 - 2\theta\|x_{k+1} - x_k\|\|x_k - x_{k-1}\| \\
&\quad - 2|\delta|\|x_{k+1} - x_k\|\|x_{k-1} - x_{k-2}\| + \theta^2\|x_k - x_{k-1}\|^2 \\
&\quad - 2|\delta|\theta\|x_{k-1} - x_k\|\|x_{k-1} - x_{k-2}\| + \delta^2\|x_{k-1} - x_{k-2}\|^2 \\
&\ge \|x_{k+1} - x_k\|^2 - \theta\Big[\|x_{k+1} - x_k\|^2 + \|x_k - x_{k-1}\|^2\Big] \\
&\quad - |\delta|\Big[\|x_{k+1} - x_k\|^2 + \|x_{k-1} - x_{k-2}\|^2\Big] \\
&\quad + \theta^2\|x_k - x_{k-1}\|^2 - |\delta|\theta\Big[\|x_{k-1} - x_k\|^2 + \|x_{k-1} - x_{k-2}\|^2\Big] \\
&\quad + \delta^2\|x_{k-1} - x_{k-2}\|^2 \\
&= (1 - |\delta| - \theta)\|x_{k+1} - x_k\|^2 + (\theta^2 - \theta - |\delta|\theta)\|x_k - x_{k-1}\|^2 \\
&\quad + (\delta^2 - |\delta| - |\delta|\theta)\|x_{k-1} - x_{k-2}\|^2. \qquad (4.5)
\end{aligned}
$$

Let $a_k := \alpha_k x_0 + (1-\alpha_k)y_k$. By Lemma 2.4, we have

$$
\begin{aligned}
\|a_k - p\|^2 &= \|\alpha_k x_0 + (1-\alpha_k)y_k - p\|^2 \\
&= \|(y_k - p) - \alpha_k(y_k - x_0)\|^2 \\
&= \|y_k - p\|^2 + (\alpha_k)^2\|y_k - x_0\|^2 - 2\alpha_k\langle y_k - p, y_k - x_0\rangle \\
&= \|y_k - p\|^2 + (\alpha_k)^2\|y_k - x_0\|^2 - \alpha_k\|y_k - x_0\|^2 \\
&\quad -\alpha_k\|y_k - p\|^2 + \alpha_k\|x_0 - p\|^2. \qquad (4.6)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|x_{k+1} - a_k\|^2 &= \|a_k - x_{k+1}\|^2 \\
&= \|y_k - x_{k+1}\|^2 + (\alpha_k)^2\|y_k - x_0\|^2 - \alpha_k\|y_k - x_0\|^2 \\
&\quad -\alpha_k\|y_k - x_{k+1}\|^2 + \alpha_k\|x_0 - x_{k+1}\|^2. \qquad (4.7)
\end{aligned}
$$

Also, from (3.1) and Lemma 2.1, we get

$$\|x_{k+1}-p\|^2 \leq \|a_k-p\|^2 - \|x_{k+1}-a_k\|^2. \tag{4.8}$$

Putting (4.6) and (4.7) in (4.8), we have

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
&\leq (1-\alpha_k)\|y_k-p\|^2+\alpha_k\|x_0-p\|^2-(1-\alpha_k)\|w_k-x_{k+1}\|^2-\alpha_k\|x_{k+1}-x_0\|^2\\
&\leq (1-\alpha_k)\|w_k-p\|^2+\alpha_k\|x_0-p\|^2-(1-\alpha_k)\|y_k-x_{k+1}\|^2.
\end{aligned}\tag{4.9}$$

From (3.1), we have $y_k = w_k - \eta_k \nabla F_k(w_k)$. Using this and Lemma 2.2(i), we obtain the following:

$$\begin{aligned}
\|y_k-x_{k+1}\|^2 &= \|w_k-x_{k+1}-\eta_k\nabla F_k(w_k)\|^2\\
&= \|w_k-x_{k+1}\|^2-2\langle \eta_k\nabla F_k(w_k), w_k-x_{k+1}\rangle+\|\eta_k\nabla F_k(w_k)\|^2\\
&\geq \|w_k-x_{k+1}\|^2-2\|\eta_k\nabla F_k(w_k)\|\|w_k-x_{k+1}\|+\|\eta_k\nabla F_k(w_k)\|^2\\
&\geq \|w_k-x_{k+1}\|^2-\tau\|\eta_k\nabla F_k(w_k)\|^2-\frac{1}{\tau}\|w_k-x_{k+1}\|^2+\|\eta_k\nabla F_k(w_k)\|^2\\
&\geq (1+\chi)\|w_k-x_{k+1}\|^2,
\end{aligned}\tag{4.10}$$

where the last inequality follows from the fact that $\tau \in (0,1)$ and $\chi = -\frac{1}{\tau} < -1$. Using (4.10) in (4.9), we get

$$\begin{aligned}
\|x_{k+1}-p\|^2 &\leq (1-\alpha_k)\|w_k-p\|^2+\alpha_k\|x_0-p\|^2-(1-\alpha_k)(1+\chi)\|w_k-x_{k+1}\|^2\\
&\leq (1-\alpha_k)\|w_k-p\|^2-(1-\alpha_k)\|w_k-x_{k+1}\|^2-\chi(1-\alpha_k)\|w_k-x_{k+1}\|^2\\
&\quad +\alpha_k\|x_0-p\|^2.
\end{aligned}\tag{4.11}$$

Now, we estimate $-\chi(1-\alpha_k)\|w_k-x_{k+1}\|^2$ as follows:

$$\|w_k-x_{k+1}\|^2 \leq 2\|w_k-x_k\|^2+2\|x_k-x_{k+1}\|^2. \tag{4.12}$$

Observe that

$$\begin{aligned}
2\|w_k-x_k\|^2 &= 2\|x_k+\theta(x_k-x_{k-1})+\delta(x_{k-1}-x_{k-2})-x_k\|^2\\
&= 2\|\theta(x_k-x_{k-1})+\delta(x_{k-1}-x_{k-2})\|^2\\
&= 4\theta^2\|x_k-x_{k-1}\|^2+4\delta^2\|x_{k-1}-x_{k-2}\|^2.
\end{aligned}\tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\begin{aligned}
-\chi(1-\alpha_k)\|w_k-x_{k+1}\|^2 &\leq -2\chi(1-\alpha_k)\|x_k-x_{k+1}\|^2-4\theta^2\chi(1-\alpha_k)\|x_k-x_{k-1}\|^2\\
&\quad -4\delta^2\chi(1-\alpha_k)\|x_{k-1}-x_{k-2}\|^2.
\end{aligned}\tag{4.14}$$

Using (4.4), (4.5) and (4.14) in (4.11), we obtain

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
\leq&(1-\alpha_k)\left[(1+\theta)\|x_k-p\|^2-(\theta-\delta)\|x_{k-1}-p\|^2-\delta\|x_{k-2}-p\|^2\right.\\
&\left.+(1+\theta)(\theta-\delta)\|x_k-x_{k-1}\|^2+\delta(1+\theta)\|x_k-x_{k-2}\|^2-\delta(\theta-\delta)\|x_{k-1}-x_{k-2}\|^2\right]\\
&-(1-\alpha_k)\left[(1-|\delta|-\theta)\|x_{k+1}-x_k\|^2+(\theta^2-\theta-|\delta|\theta)\|x_k-x_{k-1}\|^2\right.\\
&\left.+(\delta^2-|\delta|-|\delta|\theta)\|x_{k-1}-x_{k-2}\|^2\right]+\alpha_k\|x_0-p\|^2
\end{aligned}$$

$$
\begin{aligned}
&- 2\chi(1-\alpha_k)\|x_k - x_{k+1}\|^2 - 4\theta^2\chi(1-\alpha_k)\|x_k - x_{k-1}\|^2\\
&- 4\delta^2\chi(1-\alpha_k)\|x_{k-1} - x_{k-2}\|^2,
\end{aligned} \tag{4.15}
$$

which implies the following:

$$
\begin{aligned}
&\|x_{k+1}-p\|^2 - \theta\|x_k-p\|^2 - \delta\|x_{k-1}-p\|^2\\
\le&(1-\alpha_k)\left[\|x_k-p\|^2 - \theta\|x_{k-1}-p\|^2 - \delta\|x_{k-2}-p\|^2\right] - \theta\alpha_k\|x_k-p\|^2\\
&- \delta\alpha_k\|x_{k-1}-p\|^2 + (1-\alpha_k)\left[(1+\theta)(\theta-\delta) - (\theta^2-\theta-|\delta|\theta) - 4\theta^2\chi\right]\|x_k-x_{k-1}\|^2\\
&- (1-\alpha_k)\left[\delta(\theta-\delta) + (\delta^2 - |\delta| - |\delta|\theta + 4\delta^2\chi)\right]\|x_{k-1}-x_{k-2}\|^2 + \alpha_k\|x_0-p\|^2\\
&- (1-\alpha_k)\left[(1-|\delta|-\theta) + 2\chi\right]\|x_k-x_{k+1}\|^2 + (1-\alpha_k)\delta(1+\theta)\|x_k-x_{k-2}\|^2.
\end{aligned} \tag{4.16}
$$

We estimate $(1-\alpha_k)\delta(1+\theta)\|x_k-x_{k-2}\|^2$ as follows:

$$
\begin{aligned}
\|x_k-x_{k-2}\|^2 =&(\|x_k-x_{k-1}-(x_{k-2}-x_{k-1})\|)^2\\
\ge&(\|x_k-x_{k-1}\| - \|x_{k-2}-x_{k-1}\|)^2\\
=&\|x_k-x_{k-1}\|^2 - 2\|x_k-x_{k-1}\|\|x_{k-2}-x_{k-1}\| + \|x_{k-2}-x_{k-1}\|^2\\
\ge&\|x_k-x_{k-1}\|^2 - \frac{1}{2}\|x_k-x_{k-1}\|^2 - 2\|x_{k-2}-x_{k-1}\|^2 + \|x_{k-2}-x_{k-1}\|^2\\
=&\frac{1}{2}\|x_k-x_{k-1}\|^2 - \|x_{k-2}-x_{k-1}\|^2.
\end{aligned} \tag{4.17}
$$

From (4.17) and the fact that $\delta \le 0$, we obtain

$$
\begin{aligned}
(1-\alpha_k)\delta(1+\theta)\|x_k-x_{k-2}\|^2 \le& \frac{1}{2}(1-\alpha_k)\delta(1+\theta)\|x_k-x_{k-1}\|^2\\
&-(1-\alpha_k)\delta(1+\theta)\|x_{k-2}-x_{k-1}\|^2.
\end{aligned} \tag{4.18}
$$

Using (4.18) in (4.16) and the fact that $-|\delta| = \delta$, we get

$$
\begin{aligned}
&\|x_{k+1}-p\|^2 - \theta\|x_k-p\|^2 - \delta\|x_{k-1}-p\|^2\\
\le&(1-\alpha_k)\left[\|x_k-p\|^2 - \theta\|x_{k-1}-p\|^2 - \delta\|x_{k-2}-p\|^2\right] + (-\theta\alpha_k - 2\delta\alpha_k)\|x_k-p\|^2\\
&+ (1-\alpha_k)\left[2\theta - 2\delta\theta - \delta - 4\theta^2\chi\right]\|x_k-x_{k-1}\|^2\\
&- (1-\alpha_k)\left[3\delta\theta + 2\delta + 4\delta^2\chi\right]\|x_{k-1}-x_{k-2}\|^2 + \alpha_k\|x_0-p\|^2\\
&- (1-\alpha_k)\left[(1-|\delta|-\theta) + 2\chi\right]\|x_k-x_{k+1}\|^2\\
&+ \left[\frac{1}{2}(1-\alpha_k)\delta(1+\theta) - 2\delta\alpha_k\right]\|x_k-x_{k-1}\|^2.
\end{aligned} \tag{4.19}
$$

The inequality above implies the following:

$$
\begin{aligned}
&\|x_{k+1}-p\|^2 - \theta\|x_k-p\|^2 - \delta\|x_{k-1}-p\|^2 + \frac{2}{3}\|x_k-x_{k+1}\|^2\\
\le&(1-\alpha_k)\left[\|x_k-p\|^2 - \theta\|x_{k-1}-p\|^2 - \delta\|x_{k-2}-p\|^2 + \frac{2}{3}\|x_k-x_{k-1}\|^2\right]\\
&+ (1-\alpha_k)\left[2\theta - 2\theta\delta - \delta - 4\theta^2\chi - \frac{2}{3}\right]\|x_k-x_{k-1}\|^2 + (-\theta\alpha_k - 2\delta\alpha_k)\|x_k-p\|^2\\
&- (1-\alpha_k)\left[3\delta\theta + 2\delta + 4\delta^2\chi\right]\|x_{k-1}-x_{k-2}\|^2 + \alpha_k\|x_0-p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \left[(1-\alpha_k)(1-|\delta|-\theta+2\chi) - \frac{2}{3}\right] \|x_k - x_{k+1}\|^2 \\
&+ \left[\frac{1}{2}(1-\alpha_k)\delta(1+\theta) - 2\delta\alpha_k\right] \|x_k - x_{k-1}\|^2. \qquad (4.20)
\end{aligned}
$$

Observe that

$$(-\theta\alpha_k - 2\delta\alpha_k) \le 0, \qquad (4.21)$$

since $-\frac{\theta}{2} \le \delta$. We note that

$$\liminf_{k\to\infty} \left((1-\alpha_k)(1+\delta-\theta+2\chi) - \frac{2}{3}\right) = 1+\delta-\theta+2\chi - \frac{2}{3} > 0, \qquad (4.22)$$

since $\theta - \frac{1}{3} - 2\chi < \delta$. Therefore, there exists $k_1 \in \mathbb{N}$ such that $\forall k \ge k_1$, and we have

$$(1-\alpha_k)(1+\delta-\theta+2\chi) - \frac{2}{3} > 0. \qquad (4.23)$$

Also,

$$\lim_{k\to\infty} \left[\frac{1}{2}(1-\alpha_k)\delta(1+\theta) - 2\delta\alpha_k\right] = \frac{1}{2}\delta(1+\theta) < 0. \qquad (4.24)$$

Therefore, there exists $k_2 \in \mathbb{N}$ such that $\forall k \ge k_2 \ge k_1$.

$$\frac{1}{2}(1-\alpha_k)\delta(1+\theta) - 2\delta\alpha_k < 0. \qquad (4.25)$$

We obtain from (4.21) to (4.25) that

$$
\begin{aligned}
&\|x_{k+1} - p\|^2 - \theta\|x_k - p\|^2 - \delta\|x_{k-1} - p\|^2 + \frac{2}{3}\|x_k - x_{k+1}\|^2 \\
&\quad -(1-\alpha_k)\left[2\theta - 2\theta\delta - \delta - 4\theta^2\chi - \frac{2}{3}\right] \|x_k - x_{k-1}\|^2 \\
&\le (1-\alpha_k)\left[\|x_k - p\|^2 - \theta\|x_{k-1} - p\|^2 - \delta\|x_{k-2} - p\|^2 + \frac{2}{3}\|x_k - x_{k-1}\|^2\right] \\
&\quad -(1-\alpha_k)\left[3\delta\theta + 2\delta + 4\delta^2\chi\right] \|x_{k-1} - x_{k-2}\|^2 + \alpha_k\|x_0 - p\|^2. \qquad (4.26)
\end{aligned}
$$

Since $\alpha_k < 1 - \frac{\delta(3\theta+2+4\delta\chi)}{2(\theta-\theta\delta-2\theta^2\chi-\frac{1}{3})-\delta}$, we have that

$$-(3\delta\theta + 2\delta + 4\delta^2\chi) < -(1-\alpha_k)\left(2(\theta - \theta\delta - 2\theta^2\chi - \frac{1}{3}) - \delta\right).$$

From (4.26),we get

$$
\begin{aligned}
&\|x_{k+1} - p\|^2 - \theta\|x_k - p\|^2 - \delta\|x_{k-1} - p\|^2 + \frac{2}{3}\|x_{k+1} - x_k\|^2 \\
&\quad -(3\delta\theta + 2\delta + 4\delta^2\chi)\|x_k - x_{k-1}\|^2 \\
&\le (1-\alpha_k)\Big[\|x_k - p\|^2 - \theta\|x_{k-1} - p\|^2 - \delta\|x_{k-2} - p\|^2 + \frac{2}{3}\|x_k - x_{k-1}\|^2 \\
&\quad -(3\delta\theta + 2\delta + 4\delta^2\chi)\|x_{k-1} - x_{k-2}\|^2\Big] + \alpha_k\|x_0 - p\|^2. \qquad (4.27)
\end{aligned}
$$

Define for each $k \geq k_2$,

$$\Upsilon_{k+1} := \|x_{k+1}-p\|^2 - \theta\|x_k - p\|^2 - \delta\|x_{k-1}-p\|^2 + \frac{2}{3}\|x_{k+1}-x_k\|^2 \\ -(3\delta\theta + 2\delta + 4\delta^2\chi)\|x_k - x_{k-1}\|^2. \tag{4.28}$$

We now show that $\Upsilon_k \geq 0, \ \forall k \geq k_2$. Observe that

$$\|x_{k-1}-p\|^2 \leq 2\|x_k - x_{k-1}\|^2 + 2\|x_k - p\|^2.$$

So,

$$\begin{aligned}\Upsilon_{k+1} &= \|x_{k+1}-p\|^2 - \theta\|x_k - p\|^2 - \delta\|x_{k-1}-p\|^2 + \frac{2}{3}\|x_{k+1}-x_k\|^2 \\ &\quad -(3\delta\theta + 2\delta + 4\delta^2\chi)\|x_k - x_{k-1}\|^2 \\ &\geq \|x_k - p\|^2 - 2\theta\|x_k - x_{k-1}\|^2 - 2\theta\|x_k - p\|^2 - \delta\|x_{k-2}-p\|^2 \\ &\quad + \frac{2}{3}\|x_k - x_{k-1}\|^2 - (3\delta\theta + 2\delta + 4\delta^2\chi)\|x_{k-1}-x_{k-2}\|^2 \\ &= (1-2\theta)\|x_k - p\|^2 + \left(\frac{2}{3} - 2\theta\right)\|x_k - x_{k-1}\|^2 - \delta\|x_{k-2}-p\|^2 \\ &\quad -(3\delta\theta + 2\delta + 4\delta^2\chi)\|x_{k-1}-x_{k-2}\|^2 \\ &\geq 0, \end{aligned} \tag{4.29}$$

since $0 \leq \theta < \frac{1}{3}$ and $\delta \leq 0$. We obtain from (4.27) that

$$\Upsilon_{k+1} \leq (1-\alpha_k)\Upsilon_k + \alpha_k\|x_0 - p\|^2. \tag{4.30}$$

By Lemma 2.7, we have that the sequence $\{\Upsilon_k\}$ is bounded. Consequently, from (4.29) that $\{x_k\}$ is bounded. □

We now give our strong convergence result.

Theorem 4.1. *Suppose that $\{x_k\}$ is generated by Algorithm 1. Then $\{x_k\}$ converges strongly to $P_\Omega(x_0)$ when Assumption 1 is satisfied.*

Proof. Suppose $p = P_\Omega(x_0)$. Then by Lemma 2.4, we have

$$\begin{aligned}\|a_k - p\|^2 &= \|\alpha_k x_0 + (1-\alpha_k)y_k - p\|^2 \\ &= \|\alpha_k(x_0 - p) + (1-\alpha_k)(y_k - p)\|^2 \\ &= \alpha_k\|x_0 - p\|^2 + (1-\alpha_k)^2\|y_k - p\|^2 + 2\alpha_k(1-\alpha_k)\langle x_0 - p, y_k - p\rangle \\ &\leq \alpha_k\|x_0 - p\|^2 + (1-\alpha_k)^2\|w_k - p\|^2 + 2\alpha_k(1-\alpha_k)\langle x_0 - p, y_k - p\rangle, \end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\|x_{k+1} - a_k\|^2 &= (\alpha_k)^2\|x_0 - x_{k+1}\|^2 + (1-\alpha_k)^2\|y_k - x_{k+1}\|^2 \\ &\quad + 2\alpha_k(1-\alpha_k)\langle x_0 - x_{k+1}, y_k - x_{k+1}\rangle \\ &\geq (\alpha_k)^2\|x_0 - x_{k+1}\|^2 + (1-\alpha_k)^2\|y_k - x_{k+1}\|^2 \\ &\quad - 2\alpha_k(1-\alpha_k)\|x_0 - x_{k+1}\|\|y_k - x_{k+1}\| \\ &\geq (\alpha_k)^2\|x_{k+1} - x_0\|^2 + (1-\alpha_k)^2\|x_{k+1} - y_k\|^2 \end{aligned}$$

$$-2\alpha_k(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|, \tag{4.32}$$

where $\mathcal{M} := \sup_{k\geq k_1}\|x_{k+1}-x_0\| < \infty$, since $\{x_k\}$ is bounded. Putting (4.31) and (4.32) in (4.8)

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
\leq&(\alpha_k)^2\|x_0-p\|^2+(1-\alpha_k)^2\|w_k-p\|^2+2\alpha_k(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&-\alpha^2\|x_{k+1}-x_0\|^2-(1-\alpha_k)^2\|x_{k+1}-y_k\|^2+2\alpha_k(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\\
\leq&(1-\alpha_k)\|w_k-p\|^2+\alpha_k\Big(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\Big)-(1-\alpha_k)^2\|x_{k+1}-y_k\|^2.
\end{aligned} \tag{4.33}$$

Using (4.10) in (4.33), we obtain

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
\leq&(\alpha_k)^2\|x_0-p\|^2+(1-\alpha_k)^2\|w_k-p\|^2+2\alpha_k(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&-\alpha^2\|x_{k+1}-x_0\|^2-(1-\alpha_k)^2\|x_{k+1}-y_k\|^2+2\alpha_k(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\\
\leq&(1-\alpha_k)\|w_k-p\|^2+\alpha_k\Big(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\Big)-(1-\alpha_k)^2(1+\chi)\|w_k-x_{k+1}\|^2\\
=&(1-\alpha_k)\|w_k-p\|^2+\alpha_k\Big(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\Big)-(1-\alpha_k)^2\|w_k-x_{k+1}\|^2\\
&-\chi(1-\alpha_k)^2\|w_k-x_{k+1}\|^2.
\end{aligned} \tag{4.34}$$

Substituting (4.4), (4.5) and (4.14) in (4.34) gives us

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
\leq&(1-\alpha_k)\Big[(1+\theta)\|x_k-p\|^2-(\theta-\delta)\|x_{k-1}-p\|^2+(1+\theta)(\theta-\delta)\|x_k-x_{k-1}\|^2\\
&+\delta(1+\theta)\|x_k-x_{k-2}\|^2-\delta(\theta-\delta)\|x_{k-1}-x_{k-2}\|^2-\delta\|x_{k-2}-p\|^2\Big]\\
&+\alpha_k\left(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\right)\\
&-(1-\alpha_k)^2\Big[(1+\delta-\theta)\|x_{k+1}-x_k\|^2+(\theta^2-\theta+\delta\theta)\|x_k-x_{k-1}\|^2\\
&+(\delta^2+\delta+\delta\theta)\|x_{k-1}-x_{k-2}\|^2\Big]-2\chi(1-\alpha_k)^2\|x_k-x_{k+1}\|^2\\
&-4\theta^2\chi(1-\alpha_k)^2\|x_k-x_{k-1}\|^2-4\delta^2\chi(1-\alpha_k)^2\|x_{k-1}-x_{k-2}\|^2.
\end{aligned}$$

Since $\delta\leq 0$, the inequality above implies the following:

$$\begin{aligned}
&\|x_{k+1}-p\|^2\\
\leq&(1-\alpha_k)\Big[(1+\theta)\|x_k-p\|^2-(\theta-\delta)\|x_{k-1}-p\|^2-\delta\|x_{k-2}-p\|^2\Big]\\
&+(1-\alpha_k)\Big[(1+\theta)(\theta-\delta)-(1-\alpha_k)(\theta^2-\theta+\delta\theta+4\theta^2\chi)\Big]\|x_k-x_{k-1}\|^2\\
&-(1-\alpha_k)\big[\delta(\theta-\delta)+(1-\alpha_k)(\delta^2+\delta+\delta\theta+4\delta^2\chi)\big]\|x_{k-1}-x_{k-2}\|^2
\end{aligned}$$

$$+\left[-(1-\alpha_k)^2(1+\delta-\theta+2\chi)\right]\|x_k-x_{k+1}\|^2$$
$$+\alpha_k\left(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|\right).$$

Therefore,

$$\begin{aligned}
&\|x_{k+1}-p\|^2-\theta\|x_k-p\|^2-\delta\|x_{k-1}-p\|^2+\frac{2}{3}\|x_{k+1}-x_k\|^2\\
&-(3\delta\theta+2\delta+4\delta^2\chi)\|x_k-x_{k-1}\|^2\\
\leq&(1-\alpha_k)\big[\|x_k-p\|^2-\theta\|x_{k-1}-p\|^2-\delta\|x_{k-2}-p\|^2+\frac{2}{3}\|x_k-x_{k-1}\|^2\\
&-(3\delta\theta+2\delta+4\delta^2\chi)\|x_{k-1}-x_{k-2}\|^2\big]\\
&+(1-\alpha_k)\Big[\left((1+\theta)(\theta-\delta)-\frac{2}{3}\right)-(3\delta\theta+2\delta+4\delta^2\chi)\\
&-(1-\alpha_k)(\theta^2-\theta+\delta\theta+4\theta^2\chi)\Big]\|x_k-x_{k-1}\|^2\\
&+(1-\alpha_k)\big[(3\delta\theta+2\delta+4\delta^2\chi)-\delta(\theta-\delta)\\
&-(1-\alpha_k)(\delta^2+\delta+\delta\theta+4\delta^2\chi)\big]\|x_{k-1}-x_{k-2}\|^2\\
&+\left[\frac{2}{3}-(1-\alpha_k)^2(1+\delta-\theta+2\chi)\right]\|x_k-x_{k+1}\|^2-\theta\alpha_k\|x_k-p\|^2\\
&-\delta\alpha_k\|x_{k-1}-p\|^2+\alpha_k(\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle\\
&+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|)\\
\leq&(1-\alpha_k)\Upsilon_k+\alpha_k\Phi_k+(-\theta\alpha_k-2\alpha_k\delta)\|x_k-p\|^2\\
&+\Big[(1-\alpha_k)\big[\left((1+\theta)(\theta-\delta)-\frac{2}{3}\right)-(3\delta\theta+2\delta+4\delta^2\chi)\\
&-(1-\alpha_k)(\theta^2-\theta+\delta\theta+4\theta^2\chi)\big]-2\delta\alpha_k\Big]\|x_k-x_{k-1}\|^2\\
&+(1-\alpha_k)\big[(3\delta\theta+2\delta+4\delta^2\chi)-\delta(\theta-\delta)-(1-\alpha_k)(\delta^2+\delta\\
&+\delta\theta+4\delta^2\chi)\big]\|x_{k-1}-x_{k-2}\|^2\\
&+\left[\frac{2}{3}-(1-\alpha_k)^2(1+\delta-\theta+2\chi)\right]\|x_k-x_{k+1}\|^2,
\end{aligned}\tag{4.35}$$

where

$$\Phi_k:=\alpha_k\|x_0-p\|^2+2(1-\alpha_k)\langle x_0-p,y_k-p\rangle+2(1-\alpha_k)\mathcal{M}\|x_{k+1}-y_k\|^2.$$

Since $\delta\geq-\frac{\theta}{2}$, we have

$$-2\delta\alpha_k-\theta\alpha_k\leq 0,\quad \forall k\geq k_2.\tag{4.36}$$

Observe that

$$\lim_{k\to\infty}\left[\frac{2}{3}-(1-\alpha_k)^2(1+\delta-\theta+2\chi)\right]=\frac{2}{3}-(1+\delta-\theta-2\chi)\leq 0,$$

since $2\chi+\theta-\frac{1}{3}\leq\delta$. Observe that $\theta-\frac{1}{3}+2\chi\leq\theta-\frac{1}{3}-2\chi\leq\delta$. Also,

$$\lim_{k\to\infty}\left[(3\delta\theta+2\delta+4\delta^2\chi)-\delta(\theta-\delta)-(1-\alpha_k)(\delta^2+\delta+\delta\theta+4\delta^2\chi)\right]$$

$$= \delta(\theta+1) \leq 0. \tag{4.37}$$

Furthermore,

$$\lim_{k\to\infty}\Big[(1-\alpha_k)\big[\Big((1+\theta)(\theta-\delta)-\frac{2}{3}\Big)-(3\delta\theta+2\delta+4\delta^2\chi) \\ -(1-\alpha_k)(\theta^2-\theta+\delta\theta+4\theta^2)\big]-2\delta\alpha_k\Big]<0, \tag{4.38}$$

since $\frac{2\theta-\frac{2}{3}-8\theta^2\chi}{3+5\theta}<\delta$ and $\delta^2\leq\theta^2$. Using (4.36)-(4.38) in (4.35), there exists $k_2\geq k_1\in\mathbb{N}$ such that for all $k\geq k_2\geq k_1$,

$$\Upsilon_{k+1}\leq(1-\alpha_k)\Upsilon_k+\alpha_k\Phi_k+\Big[(1-\alpha_k)\big[\Big((1+\theta)(\theta-\delta)-\frac{2}{3}\Big)-(3\delta\theta+2\delta+4\delta^2\chi) \\ -(1-\alpha_k)(\theta^2-\theta+\delta\theta+4\theta^2\chi)\big]-2\delta\alpha_k\Big]\|x_k-x_{k-1}\|^2. \tag{4.39}$$

To conclude, it suffices to show, in view of Lemma 2.7 that $\limsup_{j\to\infty}\Phi_{k_j}\leq 0$ for each subsequence $\{\Upsilon_{k_j}\}\subset\{\Upsilon_k\}$ such that $\liminf_{j\to\infty}(\Upsilon_{k_j+1}-\Upsilon_{k_j})\geq 0$. Now, let $\{\Upsilon_{k_j}\}$ be a subsequence of $\{\Upsilon_k\}$ such that $\liminf_{j\to\infty}(\Upsilon_{k_j+1}-\Upsilon_{k_j})\geq 0$. From (4.39), we obtain

$$\limsup_{j\to\infty}\Big[-\Big[(1-\alpha_{k_j})\big[\Big((1+\theta)(\theta-\delta)-\frac{2}{3}\Big)-(3\delta\theta+2\delta+4\delta^2\chi) \\ -(1-\alpha_{k_j})(\theta^2-\theta+\delta\theta+4\theta^2\chi)\big]-2\delta\alpha_{k_j}\Big]\Big]\|x_{k_j}-x_{k_j-1}\|^2 \\ \leq\limsup_{j\to\infty}\big[(\Upsilon_{k_j}-\Upsilon_{k_j+1})+\alpha_{k_j}(\Phi_{k_j}-\Upsilon_{k_j})\big] \\ \leq-\liminf_{j\to\infty}(\Upsilon_{k_j+1}-\Upsilon_{k_j})\leq 0.$$

Since

$$\lim_{j\to\infty}\Big[-\Big[(1-\alpha_{k_j})\big[\Big((1+\theta)(\theta-\delta)-\frac{2}{3}\Big)-(3\delta\theta+2\delta+4\delta^2\chi) \\ -(1-\alpha_{k_j})(\theta^2-\theta+\delta\theta+4\theta^2\chi)\big]-2\delta\alpha_{k_j}\Big]\Big]>0,$$

we obtain

$$\lim_{j\to\infty}\|x_{k_j}-x_{k_j-1}\|=0. \tag{4.40}$$

Consequently,

$$\|w_{k_j}-x_{k_j}\|^2\leq\theta\|x_{k_j}-x_{k_j-1}\|+|\delta|\|x_{k_j-1}-x_{k_j-2}\|\to 0,\quad j\to\infty. \tag{4.41}$$

Also, we obtain

$$\lim_{j\to\infty}\|x_{k_j+1}-w_{k_j}\|=0. \tag{4.42}$$

By Lemma 4.1, $\{x_{k_j}\}$ is bounded. Therefore, we can choose a subsequence $\{x_{k_{j_i}}\}\subset\{x_{k_j}\}$ which converges weakly to some $u\in\mathcal{H}_1$ such that

$$\limsup_{j\to\infty}\langle x_0-p,x_{k_j}-p\rangle=\lim_{j\to\infty}\langle x_0-p,x_{k_{j_i}}-p\rangle$$

$$= \langle x_0 - p, u - p\rangle. \tag{4.43}$$

From (4.2) and (4.41), we have

$$\lim_{j\to\infty} \frac{F_k(w_{k_j})}{\|\nabla F_k(w_{k_j})\|} = 0. \tag{4.44}$$

Also, we note that

$$\begin{aligned}\|\nabla F_k(w_k)\| &= \|\nabla F_k(w_k) - \nabla F(p)\| \\ &\le \|A\|^2\|w_k - p\|, \quad \forall\ p \in \Omega.\end{aligned}$$

Hence, $\{\nabla F_k(x_k)\}$ is bounded. Therefore, from (4.2), we have

$$\lim_{j\to\infty} F_{k_j}(x_{k_j}) = 0. \tag{4.45}$$

Or equivalently,

$$\lim_{j\to\infty} \|(I - P_{\mathcal{Q}_{k_j}})Aw_{k_j}\|^2 = 0.$$

Since $\{w_{k_j}\}$ is bounded, there exists a subsequence $\{w_{k_{j_i}}\}$ of $\{w_{k_j}\}$ which converges weakly to u. Without loss of generalities, we can assume from (4.41) that $w_{k_j} \rightharpoonup u$. Since $P_{\mathcal{Q}_{k_j}}Aw^{k_j} \in Q_{k_j}$, we have

$$q(Aw_{k_j}) \le \langle \xi_{k_j}, Aw_{k_j} - P_{Q_{k_j}}Aw_{k_j}\rangle, \tag{4.46}$$

where $\xi_{k_j} \in \partial q(Aw_{k_j})$. From the boundedness assumption of ξ_{k_j} and (4.45), we have

$$q(Ax_{k_j}) \le \|\xi_{k_j}\|\|Aw_{k_j} - P_{\mathcal{Q}}Aw_{k_j}\| \to 0. \tag{4.47}$$

From the weak lower semicontinuity of the convex function $q(\hat{x})$ and since $w_{k_j} \rightharpoonup u$, it follows from (4.47) that

$$q(Au) \le \liminf_{j\to\infty} q(Aw_{k_j}) \le 0,$$

which means that $Au \in \mathcal{Q}$. Further, using the fact that $x_{k_j+1} \in \mathcal{C}_{k_j}$ and by the definition of $\mathcal{C}_{k_j}$, we get

$$c(w_{k_j}) \le \langle \psi_{k_j}, w_{k_j} - x_{k_j+1}\rangle,$$

where $\psi_{k_j} \in \partial c(w_{k_j})$. Due to the boundedness of ψ_{k_j} and (4.42), we have

$$c(w_{k_j}) \le \|\psi_{k_j}\|\|w_{k_j} - x_{k_j+1}\| \to 0, \quad j \to \infty.$$

Similarly, we obtain that $c(u) \le 0$, i.e., $u \in \mathcal{C}$, which implies that $u \in \Omega$. Since $p = P_\Omega(x_0)$, we have from (4.43) that

$$\limsup_{j\to\infty}\langle x_0 - p, x_{k_j} - p\rangle = \lim_{j\to\infty}\langle x_0 - p, u - p\rangle \le 0. \tag{4.48}$$

Therefore,

$$\limsup_{j\to\infty}\langle x_0 - p, y_{k_j} - p\rangle \le 0$$

by (4.41) and (4.48). Hence,

$$\limsup_{j\to\infty} \Phi_{k_j} \le 0.$$

Since $\sum_{k=1}^{\infty} \alpha_k = \infty$, we obtain by Lemma 2.7 in (4.35) that $\lim_{k\to\infty} \Upsilon_k = 0$. Using (4.29), we have that $\{x_k\}$ converges strongly to $P_\Omega(x_0)$. □

5. Numerical illustrations and applications to compressed sensing

In this section we compare the performance of our proposed Algorithm 1 with three recent algorithms in the literature. In the first example, we perform a sensitivity analysis on the parameters given in Assumption 1. This will give several sets of parameters that satisfy Assumption 1 and possibly give an idea to choose the optimal sets of control parameters for Algorithm 1.

Example 5.1. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix defined by

$$Ax := \begin{pmatrix} -1 & 3 & 5 \\ 5 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let $\mathcal{C} = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1^2 + x_2^2 - 4 \leq 0\}$ and $\mathcal{Q} = \{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 + x_3^2 - 1 \leq 0\}$. Set $\eta_0 = 1.99$, $\alpha_k = \frac{1}{k+1}$ and $x_0 = (0,0,0)^T$, $x_1 = x_2 = (1,1,1)^T$. The simulation is terminated when $\|x_{k+1} - x_k\| < 10^{-6}$ or $n = 1001$. The results obtained for various sets of parameters that satisfy Assumption 1 are presented in Table 1.

Table 1. Performance of Algorithm 1 with Respect to Parameters

σ	θ	δ	**Number of Iterations**	**CPU Time (secs)**
0.1	0.32	−0.15	1073	0.0688
0.1	0.32	−0.1	947	0.0662
0.1	0.32	−0.05	555	0.0405
0.1	0.22	−0.01	713	0.0487
0.1	0.22	−0.001	690	0.0466
0.1	0.22	0	687	0.0516
0.1	0.323	−0.05	485	0.0348
0.1	0.2	−0.05	876	0.0615
0.1	0.1	−0.05	1194	0.0817
0.1	0.01	−0.005	1493	0.0887
1.5	0.323	−0.05	141	0.0185
2	0.323	−0.05	138	0.0143
2.5	0.323	−0.05	135	0.0165
3.5	0.323	−0.05	160	0.0249

Discussion. All the sets of parameters considered in Table 1 satisfy the assumptions given in Assumption 1. From Table 1, we saw that the performance of our proposed Algorithm 1 is dependent on the choice of the control parameters σ, θ and δ. Again, from Table 1, we deduce that the best set of parameters for Algorithm 1 in this example are $\sigma = 2.5,\ \theta = 0.323$ and $\delta = -0.05$. However, even though there is no proof to guarantee convergence, for this particular example, there are sets of parameters that do not satisfy our Assumption 1 but give good approximation. For example $\sigma = 2.5,\ \theta = 0.82$ and $\delta = 0.05$. This perhaps might suggest that considering two-step inertial algorithm with both inertial parameters positive may provide acceleration.

Example 5.2. In this example, we will compare the performance of our proposed Algorithm 1 with the Algorithms proposed by Vinh et al. [37], Vinh et al. [38] and Shehu et al. [35]. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $Ax = Bx$ where B is a randomly generated matrix such that its entries $b_{ij} \in (0, 1)$.

$$\text{Let} \quad \mathcal{C} = \left\{x \in \mathbb{R}^n : \|x\| \leq \frac{3}{2}\right\} \text{ and } \mathcal{Q} = \left\{x \in \mathbb{R}^n : \|x\| \leq 2\right\}.$$

We will consider four dimensions $n = 50,\ 100,\ 300$ and 500 and study the behaviour of Algorithm 3.1 of Vinh et al. [37] (VHDC Alg 3.1), Algorithm 3.1 of Vinh et al. [38] (VCS Alg 3.1) and fully inertial versions of Algorithm 2 of Shehu et al. [35] (SDL Alg 2) with respect to these dimensions. In VHDC Algorithm 3.1, we choose $\rho_k = 0.1$, $\theta = 0.82$, $\epsilon_k = \frac{1}{(k+1)^2}$. In VCS Algorithm 3.1, we choose $\beta_k = \frac{1}{k+1}$ and $\rho_k = \frac{3.5k}{k+1}$. In SDL Alg 2, we choose $\theta_k = 0.9$ and $\sigma_k = 0$. The initial guess x_0 is set to be zeros, x_1 is generated randomly and we set $x_2 = x_1$, for all the algorithms. The simulation is terminated when $E_k = \|x_{k+1} - x_k\| < 10^{-6}$ or $k = 3001$. The results obtained for each dimension are presented in Table 2 and Figures 1 and 2.

Table 2. Performance of the Algorithms in Example 5.2

Iter and CPU Time for the Algorithms for Different Dimensions								
	Algorithm 1		VHDC Alg 3.1		VCS Alg 3.1		SDL Alg 2	
Dim. (n)	Iter	CPU Time	Iter	CPU Time	Iter	CPU Time	Iter	CPU Time
50	97	0.0142	123	0.0129	1580	0.1093	145	0.0165
100	98	0.0175	141	0.0282	1604	0.1332	213	0.0517
300	101	0.0429	348	0.1640	1623	0.3357	548	0.1853
500	117	0.1411	809	0.7493	1658	1.1713	959	0.9036

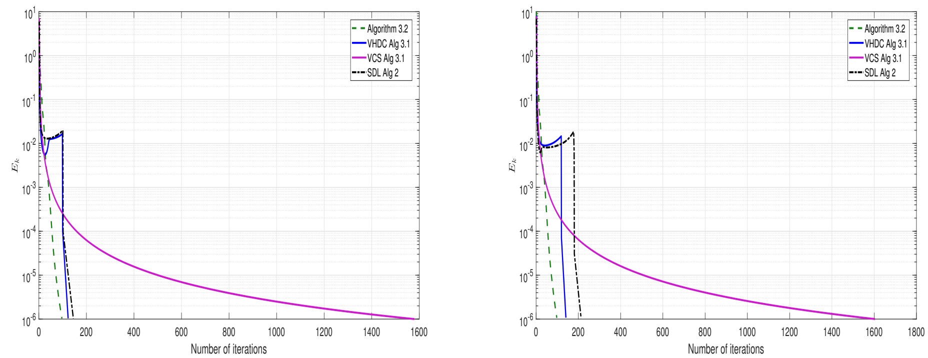

Figure 1. Graph of the Iterates: Top Left $n = 50$, Top Right= $n = 100$

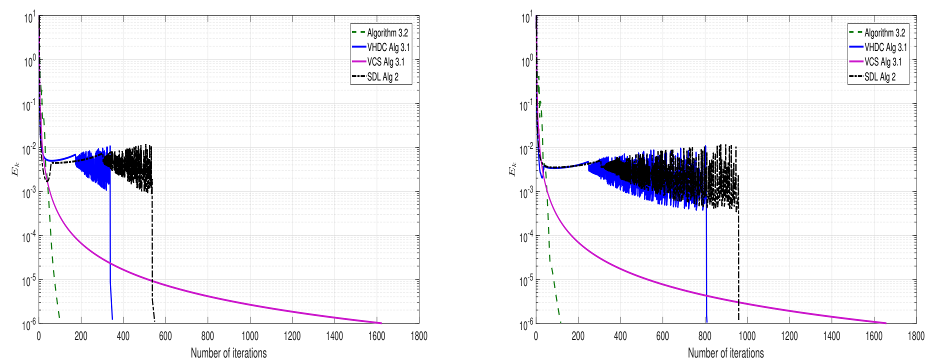

Figure 2. Graph of the Iterates: Top Left $n = 300$, Top Right= $n = 500$

Example 5.3. We consider the following LASSO problem

$$\min\left\{\frac{1}{2}\|Ax-b\|_2^2 : \ x \in \mathbb{R}^n, \ \|x\|_1 \le r\right\}, \tag{5.1}$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, $b \in \mathbb{R}^m$ and $r > 0$. We consider $n = 6144$ and $m = 1440$. A normal distribution with a standard deviation of zero and a unit variance serves as the basis for the matrix A. Additionally, the genuine spare signal x_* is formed by uniformly dispersing throughout the interval $[-1, 1]$ with spikes (nonzero entries) 90 and 180 while the rest are kept at zero. The sample data b is given as $b = Ax_*$. The solution of the minimization problem (5.1) under certain conditions on the matrix A is similar to the ℓ_1-norm solution of the undetermined linear system. For the problem under consideration (1.1), we define

$$\mathcal{C} = \left\{x : \|x\|_1 \le k\right\} \quad \text{and} \quad \mathcal{Q} = \{b\}.$$

We will use the subgradient projection since the projection onto the closed convex $\mathcal{C}$ does not have a closed form solution. Now, we define a convex function $d(x) :=$

$\|x\|_1 - k$ and define

$$\mathcal{C}_k = \Big\{x \in \mathbb{R}^k : d(w_k) + \langle \zeta_k,\ x - w_k\rangle \leq 0\},$$

where $\zeta \in \partial d(w_k)$. The orthogonal projection on $\mathcal{C}_k$ is given by

$$P_{\mathcal{C}_k}(\tilde{y}) = \begin{cases} y, & d(w_k) + \langle \zeta_k, \tilde{y} - w_k\rangle \leq 0, \\ y - \frac{d(w_k)+\langle \zeta_k, \tilde{y} - w_k\rangle \zeta_k}{\|\zeta_k\|^2}, & \text{otherwise.} \end{cases}$$

Therefore, at point x, the subdifferential ∂c is given by

$$\partial c(x) = \begin{cases} 1, & x > 0, \\ [-1, 1], & x = 0, \\ -1, & x < 0. \end{cases}$$

Now, we will compare the performance of our proposed Algorithm 1 with Algorithm 3.1 of Vinh et al. [37] (VHDC Alg 3.1), Algorithm 3.1 of Vinh et al. [38] (VCS Alg 3.1) and fully inertial version of Algorithm 2 of Shehu et al. [35] (SDL Alg 2) in the restoration process of the sparse signal. We use the same control parameters as in Example 5.2. Furthermore, we evaluate the mean square error (MSE) defined by:

$$MSE = \frac{1}{k}\|x_* - x\|^2 \tag{5.2}$$

to make sure that the restored signal has a good length and observation compared to the original signal, where x_* is an approximated signal of x. The initial points x_0, x_1, x_2 are chosen as zero vectors and $\|x_k - x_*\| < 10^{-5}$ or maximum number of iterations $k = 3000$ are used as stopping criterion. We present the results of the numerical simulations in Figures 3, 4, 5 and 6.

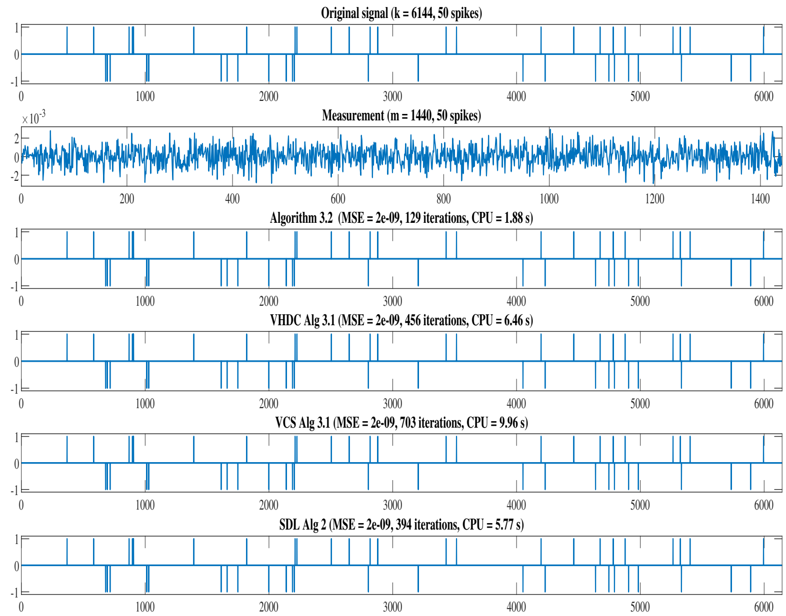

Figure 3. Restored Signal via the Algorithms for 50 spikes

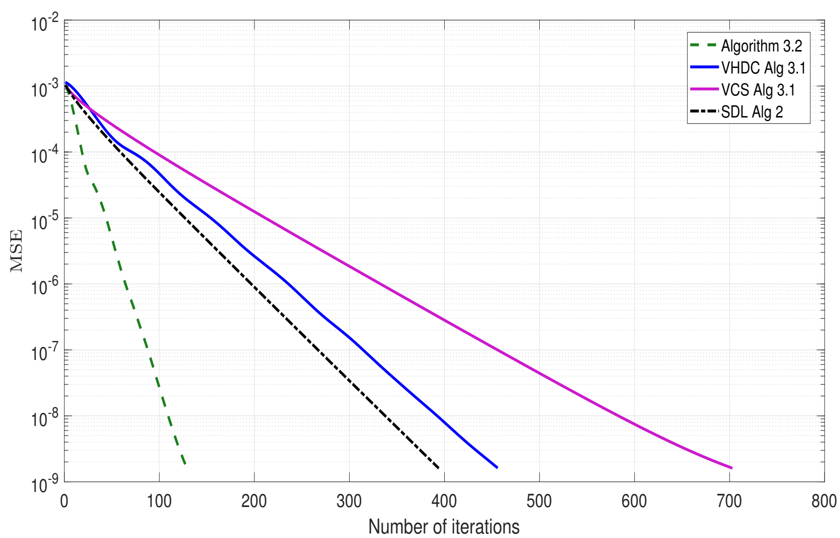

Figure 4. Graphical Presentation of the MES values of Algorithms for 50 spikes

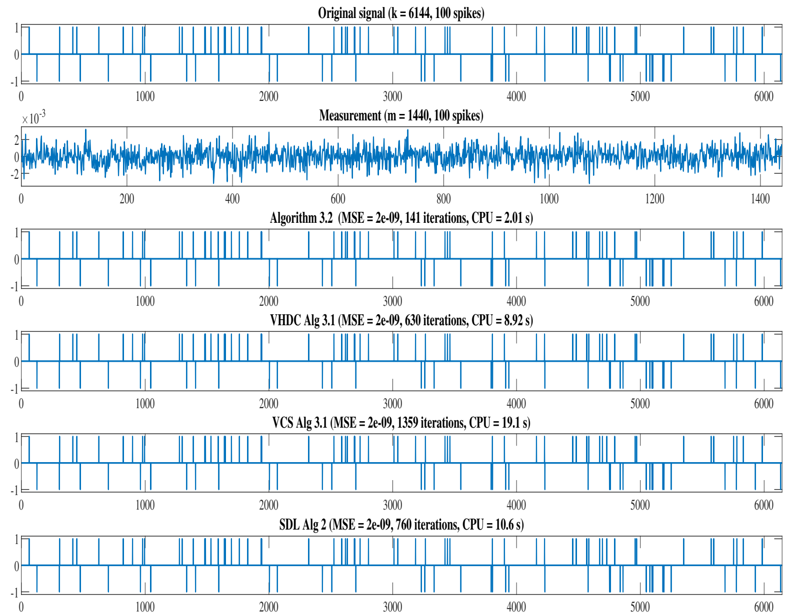

Figure 5. Restored Signal via the Algorithms for 50 spikes

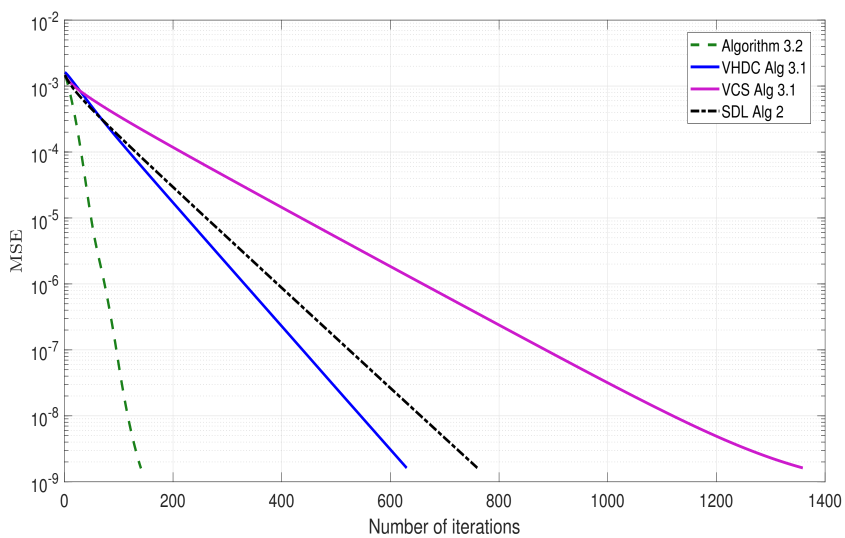

Figure 6. Graphical Presentation of the MES values of Algorithms for 50 spikes

6. Conclusion

In this work, we studied the split feasibility problem. We introduced the Halpern-type algorithm with two-step inertial extrapolation and self adaptive stepsize to solve the aforementioned problem. We proved that the sequence of the iterates generated by our proposed algorithm converges strongly to a solution of the SFP under some conditions on the iterative parameters without the on-line rule assumption. Finally, we presented numerical results of our proposed method to illustrate the applicability of our method.

Acknowledgment

The first and third authors acknowledge the School of Mathematics at University of the Witwatersrand for making their facilities available for the research. The authors thank the anonymous reviewers for their esteemed comments and suggestions that greatly improved the manuscript.

Funding

The first author is supported by the postdoctoral research grant from University of the Witwatersrand, South Africa.

Availability of data and material

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors worked equally on the results and approved the final manuscript.

References

[1] A. Adamu and A. A. Adam, *Approximation of solutions of split equality fixed point problems with applications*, Carpathian Journal of Mathematics, 2021, 37(3), 381–392.

[2] A. Adamu, P. Kumam, D. Kitkuan, A. Padcharoen, *A Tseng-type algorithm for approximating zeros of monotone inclusion and J-fixed-point problems with applications*, Fixed Point Theory and Algorithms for Science and Engineering, 2023, 1–23.

[3] A. Adamu, C.E. Chidume, D. Kitkuan, P. Kumam, *Geometric inequalities for solving variational inequality problems in certain Banach spaces*, Journal of Nonlinear Variational Analysis, 2023, 7, 267–278.

[4] H.A. Abass, K. O. Aremu, L. O. Jolaoso, O. T. Mewomo, *An inertial forward-backward splitting method for approximating solutions of certain optimization problems*, Journal of Nonlinear Functional Analysis, 2020, Article ID 6, 1–20.

[5] F. Alvarez, H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set Valued Analysis, 2001, 9(1-2), 3–11.

[6] R. I. Bot, E. R Csetnek, C. Hendrich, *Inertial Douglas-Rachford splitting for monotone inclusion*, Applied Mathematics and Computation, 2015, 256, 472–487.

[7] B. Ali, A.A. Adam, A. Adamu, *An accelerated algorithm involving quasi-ϕ-nonexpansive operators for solving split problems*, Journal of Nonlinear Modeling Analysis, 2023, 5(1), 54–72.

[8] C. L. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, 2002, 18, 441–453.

[9] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, 2004, 20(1), 103–120.

[10] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numerical Algorithms, 1994, 8(2-4), 221–239.

[11] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numerical Algorithms, 2012, 59, 301–323.

[12] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity modulated radiation therapy*, Physics in Medicine & Biology, 2006, 51, 2353–2365.

[13] C.E. Chidume, A.A. Adam, A. Adamu, *An iterative method involving a class of quasi-phi-nonexpansive mappings for solving split equality fixed point problems*, Creative Mathematics and Informatics, 2023, 32, 29–40.

[14] C.E. Chidume, A. Adamu, *On split generalized mixed equality equilibrium and split equality fixed point problems*, Applied Set-Valued Analysis and Optimization, 2020, 2, 273–283.

[15] C.E. Chidume, A. Adamu, *Solving split equality fixed point problem for quasi-phi-nonexpansive mappings*, Thai Journal of Mathematics, 2021, 19(4), 1699–1717.

[16] C.E. Chidume, A. Adamu, *A new iterative algorithm for split feasibility and fixed point problems*, Journal of Nonlinear Variational Analysis, 2021, 5(2), 201–210.

[17] Y. Dang, Y. Gao, *The strong convergence of a KM-CQ-like algorithm for a split feasibility problem*, Inverse Problems, 2011, 27(1), 015007, 1–9.

[18] Y. Dang, J. Sun, H. Xu, *Inertial accelerated algorithms for solving a split feasibility problem*, Journal of Industrial and Management Optimization, 2017, 13(3), 1383–1394.

[19] Q. L. Dong, Y. C. Tang, Y. J. Cho, ThM. Rassias, *Optimal choice of the step length of the projection and contraction methods for solving the split feasibility problem*, Journal of Global Optimization, 2018, 71, 341–360.

[20] A. Gibali, L. W. Liu, Y. C. Tang, *Note on the modified relaxation CQ algorithm for the split feasibility problem*, Optimization Letters, 2018, 12(4), 817–830.

[21] S. Gebregiorgis, P. Kumam, *Relaxed double inertia and viscosity algorithms with multiple-sets split feasibility problem with multiple-sets split feasibility problem with multiple output sets*, Bangmod International Journal of Mathematics and Computational Science, 2024, 10, 10–29.

[22] K. Goebel, W. A. Kirk, *On Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[23] C. Izuchukwu, Y. Shehu, Q.L. Dong, *Two-step inertial forward-reflected-backward splitting based algorithm for nonconvex mixed variational inequalities*, Journal of Computation and Applied Mathematics, 2023, 426, 115093.

[24] L.O. Jolaoso, Y. Shehu, J.C. Yao, *Strongly convergent inertial proximal point algorithm without on-line rule*, Journal of Optimization Theory Applications, 2024, 200, 555–584.

[25] R. Kraikaew, S. Saejung, *A simple look at the method for solving split feasibility problems in Hilbert spaces*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 2020, 114(3), 1–9.

[26] J. Liang, *Convergence rates of first-order operator splitting methods*, PhD thesis, Normandie Université; GREYC CNRS UMR 6072, 2016.

[27] G. López, V. Martín-Márquez, F. Wang, H. K. Xu, *Solving the split feasibility problem without prior knowledge of matrix norms*, Inverse Problems, 2012, 28(8), 085004, 18 pp.

[28] C.C. Okeke, A. Adamu, R. Promkam, P. Sunthrayuth, *Two-step inertial method for solving split common null point problem with multiple output sets in Hilbert spaces*, AIMS Mathematics, 2023, 8(9), 20201–20222.

[29] A.E. Ofem, J.A. Abuchu, G.C. Ugwunnadi, H.A. Nabwey, A. Adamu, O.K. Narain, *Double inertial steps extragradient-type methods for solving optimal control and image restoration problems*, AIMS Mathematics, 2024, 9, 12870–12905.

[30] B. T. Polyak, *Introduction to Optimization*, New York, Optimization Software, Publications Division, 1987.

[31] C. Poon, J. Liang, *Trajectory of Alternating Direction Method of Multipliers and Adaptive Acceleration*, In Advances In Neural Information Processing systems, 2019.

[32] Y. Nesterov, *A method for solving the convex programming problem with convergence rate $O(1/k^2)$*, Proceedings of the USSR Academy of Sciences, 1983, 269, 543–547.

[33] A. Taiwo, L. O. Jolaoso, O. T. Mewomo, *Viscosity approximation method for solving the multiple-set split equality common fixed-point problems for quasi-pseudocontractive mappings in Hilbert spaces*, Journal of Industrial and Management Optimization, 2021, 17(5), 2733–2759.

[34] A. Taiwo, L. O. Jolaoso, O. T. Mewomo, *Parallel hybrid algorithm for solving pseudomonotone equilibrium and split common fixed point problems*, Bulletin of the Malaysian Mathematical Sciences Society, 2020, 43, 1893–1918.

[35] Y. Shehu, Q.L. Dong, L.L. Lu, *Global and linear convergence of alternated inertial methods for split feasibility problems*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 2021, 115, 1–26.

[36] Y. Shehu, *Iterative methods for split feasibility problems in certain Banach spaces*, Journal of Nonlinear Convex Analysis, 2015, 16(12), 2351–2364.

[37] N. Vinh, P.T. Hoai, L.A. Dung, Y.J. Cho, *A new inertial self-adaptive gradient algorithm for the split feasibility problem and an application to the sparse recovery problem*, Acta Mathematica Sinica, English Series, 2023, 39(12), 2489–2506.

[38] N. Vinh, P. Cholamjiak, S. Suantai, *A new CQ algorithm for solving split feasibility problems in Hilbert spaces*, Bulletin of the Malaysian Mathematical Sciences Society, 2019, 42, 2517–2534.

[39] H. K. Xu, M. A. Alghamdi, N. Shahzad, *Regularization for the split feasibility problem*, Journal of Nonlinear Convex Analysis, 2016, 17, 513–525.

[40] Y. Yao, A. Adamu, Y. Shehu, *Strongly convergent inertial forward-backward-forward algorithm without on-line rule for variational inequalities*, Acta Mathematica Scientia, 2024, 44, 551–566.

[41] H. Yu, W. Zhan, F. Wang, *The ball-relaxed CQ algorithms for the split feasibility problem*, Optimization, 2018, 67(10), 1687–1699.