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Study of Certain Navier Problems in Sobolev Space with Weights

Y. Fadil[1,†], M. El Ouaarabi[2] and M. Oukessou[1]

Abstract In this paper, we study the following Navier problem

$$-\mathrm{div}\Big[v_1\mathcal{K}(z,\nabla w)+v_2\mathcal{L}(z,w,\nabla w)\Big]+\Delta\left[\phi_1|\Delta w|^{t-2}\Delta w+\phi_2|\Delta w|^{q-2}\Delta w\right]+ v_3 b(z,w)+v_4|w|^{p-2}w=h(z),$$

Here, $h\in L^{p'}(\mathcal{Q},v_1^{1-p'})$, $\mathcal{K}$, $\mathcal{L}$ and b are Carathéodory functions and ϕ_1,ϕ_2,v_1, v_2, v_3 and v_4 are A_p-weights functions. By using the theory of monotone operators (Browder–Minty Theorem), we demonstrate the existence and uniqueness of weak solution to the above problem.

Keywords Navier problem, degenerate quasilinear elliptic equations, weighted Sobolev spaces, weak solution

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1. Introduction

Nonlinear elliptic equations with perturbation in the sense of singularity and decay are useful problems arising from these differential equations in various applications, including non-Newtonian fluid mechanics, reaction-diffusion difficulties, flows in porous media and hydrology, (we refer to [3, 6, 19] where it is possible to find some examples of applications of degenerate elliptic equations).

In the so-called degenerate partial differential equations, which have different types of singularities in the coefficients, it is natural to find solutions in weighted Sobolev spaces [8–10, 13]. The weightless Sobolev spaces $W^{k,t}(\mathcal{Q})$, in general, appear as solution spaces for parabolic and elliptic partial differential equations. In particular when $t=q=2$ and $\phi_1=\phi_2\equiv 1, v_1=v_3=v_4=0$ and $v_2=1$ we have the equation

$$\Delta^2 w-\sum_{j=1}^{n}D_j\mathcal{L}_j(z,w,\nabla w)=h,$$

[†]the corresponding author.
Email address: yfadil447@gmail.com(Y. Fadil), mohamedelouaarabi93@gmail.com(M. El Ouaarabi), ouk_mohamed@yahoo.fr(M. Oukessou).
[1]Laboratory LMACS, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco.
[2]Fundamental and Applied Mathematics Laboratory, Faculty of Sciences Ain Chock, Hassan II University, BP 5366, 20100. Casablanca, Morocco.

where $\Delta^2 w$ is the biharmonic operator. Many real phenomena, such as radar imaging or incompressible flows, are the subject of mathematical models in which biharmonic equations are found.

There are a lot of examples of weight (see [13]). A well-established class of weights, introduced by B. Muckenhoupt [16] , is the class of A_p-weights (or Muckenhoupt class) . These weights have found many useful applications in harmonic analysis [17].

Our goal in this paper is to show the uniqueness and existence of a weak solution in the weighted Sobolev space. Consider $W_0^{1,t}(\mathcal{Q}, v)$ (see Definition 2.2) for the Navier problem associated with the degenerate elliptic equation

$$\begin{cases} \Delta\Big[\phi_1|\Delta w|^{t-2}\Delta w + \phi_2|\Delta w|^{q-2}\Delta w\Big] - \operatorname{div}\Big[v_1\mathcal{K}(z,\nabla w) + v_2\mathcal{L}(z,w,\nabla w)\Big] & \\ \qquad\qquad\qquad + v_3 b(z,w) + v_4|w|^{p-2}w = h & \text{in } \mathcal{Q}, \\ w(z) = \Delta w(z) = 0 & \text{on } \partial\mathcal{Q}, \end{cases} \tag{1.1}$$

where, $\mathcal{Q}$ is a bounded open set in $\mathbb{R}^d$,ϕ_1,ϕ_2, v_1, v_2, v_3 and v_4 are a weight functions, and the functions $\mathcal{L} : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\mathcal{K} : \mathcal{Q} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $b : \mathcal{Q} \times \mathbb{R} \longrightarrow \mathbb{R}$ are Caratéodory functions that satisfy the growth assumptions, monotonicity and ellipticity conditions. Problems like (1.1) have been studied by many authors in the unweighted and weighted case (see [2, 4, 22]).

The structure of this work is as follows: in Section 2, we give some basic results and some technical lemmas. In Section 3, we specify all the assumptions on $\mathcal{K}$, $\mathcal{L}$, b and we present the notion of weak solution for Problem (1.1). The main results will be proved in Section 4.

2. Preliminaries

To understand our findings, we must first review certain definitions and fundamental aspects which are used during this paper. Full presentations can be found in the monographs by A. Torchinsky [17] and J. Garcia-Cuerva et al. [11].

We will call a locally integrable function v by a weight on $\mathbb{R}^d$ such that $v(z) > 0$ for a.e. $z \in \mathbb{R}^d$. Each weight v gives rise to a measure on the measurable subsets of $\mathbb{R}^d$ by integration. This measure will be denoted v. Thus,

$$v(E) = \int_E v(z)dz \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^d.$$

For $0 < t < \infty$, we denote by $L^t(\mathcal{Q}, v)$ the space of measurable functions v on $\mathcal{Q}$ such that

$$||h||_{L^t(\mathcal{Q},v)} = \left(\int_{\mathcal{Q}} |h|^t v(z)dz\right)^{\frac{1}{t}} < \infty,$$

where h is a weight, and $\mathcal{Q}$, is open in $\mathbb{R}^d$. It is a widely known fact that the space $L^t(\mathcal{Q}, v)$, endowed with this norm is a Banach space. We also have that the dual space of $L^t(\mathcal{Q}, v)$ is the space $L^{t'}(\mathcal{Q}, v^{1-t'})$.

Let us now specify the conditions on the weight v that ensure that the functions in $L^t(\mathcal{Q}, v)$ are locally integrable on $\mathcal{Q}$.

Proposition 2.1. *([14, 15]). Let $1 \leq t < \infty$. If the weight v is such that*

$$v^{\frac{-1}{t-1}} \in L^1_{loc}(\mathcal{Q}) \qquad \text{if} \quad t > 1,$$

$$ess \sup_{z \in B} \frac{1}{v(z)} < +\infty \qquad \text{if} \quad t = 1,$$

for every ball $B \subset \mathcal{Q}$, then

$$L^t(\mathcal{Q}, v) \subset L^1_{loc}(\mathcal{Q}).$$

As a result, under the conditions of the Proposition 2.1, the convergence in $L^t(\mathcal{Q}, v)$ implies convergence in $L^1_{loc}(\mathcal{Q})$. In addition, every function in $L^t(\mathcal{Q}, v)$ has distributional derivatives. So it makes sense to talk about distributional derivatives of functions in $L^t(\mathcal{Q}, v)$.

Definition 2.1. Let $1 \leq t < \infty$. A weight v is said to be an A_t-weight, or v belongs to the Muckenhoupt class, if there exists a positive constant $\zeta = \zeta(t, v)$ such that, for every ball $B \subset \mathbb{R}^d$

$$\left(\frac{1}{|B|}\int_B v(z)dz\right)\left(\frac{1}{|B|}\int_B (v(z))^{\frac{-1}{t-1}}\, dz\right)^{t-1} \leqslant \zeta \qquad \text{if} \quad t > 1,$$

$$\left(\frac{1}{|B|}\int_B v(z)dz\right) ess \sup_{z \in B} \frac{1}{v(z)} \leqslant \zeta \qquad \text{if} \quad t = 1,$$

where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^d$.

The infimum over all such constants ζ is called the A_t constant of v. We denote by A_t, $1 \leq t < \infty$, the set of all A_t weights.

If $1 \leq q \leq t < \infty$, then $A_1 \subset A_q \subset A_t$ and the A_q constant of f equals the A_t constant of f (we refer to [12, 13, 18] for more informations about A_p-weights).

Proposition 2.2. *([19]). Let $f \in A_t$ with $1 \leqslant t < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^d$. Then*

$$\left(\frac{|E|}{|B|}\right)^t \leqslant C \frac{v(E)}{v(B)},$$

where C is the A_t constant of v.

The weighted Sobolev space $W^{k,t}(\mathcal{Q}, v)$ is defined as follows.

Definition 2.2. Let $\mathcal{Q} \subset \mathbb{R}^d$ be open, and let f be A_t-weights, $1 \leqslant t < \infty$. We define the weighted Sobolev space $W^{k,t}(\mathcal{Q}, v)$ as the set of functions $w \in L^t(\mathcal{Q}, v)$ with $D_k w \in L^t(\mathcal{Q}, v)$, for $k = 1, ..., n$. The norm of w in $W^{k,t}(\mathcal{Q}, v)$ is given by

$$||w||_{W^{k,t}(\mathcal{Q},v)} = \left(\int_{\mathcal{Q}} |w(z)|^t v dz + \int_{\mathcal{Q}} |\nabla w(z)|^t v dz\right)^{\frac{1}{t}}. \tag{2.1}$$

We also define $W^{1,t}_0(\mathcal{Q}, v)$ as the closure of $\mathcal{C}^\infty_0(\mathcal{Q})$ in $W^{1,t}(\mathcal{Q}, v)$ with respect to the norm (2.1).

Equipped with this norm, $W^{1,t}(\mathcal{Q},v)$ and $W_0^{1,t}(\mathcal{Q},v)$ are reflexive and separable Banach spaces (see [14, Proposition 2.1.2]). For more detail about the spaces $W^{1,t}(\mathcal{Q},v)$ see [13,15]. The dual of space $W_0^{1,t}(\mathcal{Q},v)$ is defined as

$$\left[W_0^{1,t}(\mathcal{Q},v)\right]^* = \left\{h - \sum_{i=1}^{n} D_i h_i/, \frac{h_i}{v} \in L^{t'}(\mathcal{Q},v), i = 1, ..., n\right\}.$$

To show the main reasoning of this paper, we rely on the following results .

Definition 2.3. We denote $\mathbb{H} = W_0^{1,p}(\mathcal{Q},v_1) \cap W^{2,t}(\mathcal{Q},\phi_1)$ with the norm

$$\|w\|_{\mathbb{H}} = \|\Delta w\|_{L^t(\mathcal{Q},\phi_1)} + \||\nabla w|\|_{L^p(\mathcal{Q},v_1)}.$$

Theorem 2.1. *([10]). Let $v \in A_t$, $1 \leqslant t < \infty$, and let $\mathcal{Q}$ be a bounded open set in $\mathbb{R}^d$. If $w_n \longrightarrow w$ in $L^t(\mathcal{Q},v)$, then there exists a subsequence (w_{n_m}) and $\psi \in L^t(\mathcal{Q},v)$ such that*

(i) *$w_{n_m}(z) \longrightarrow w(z)$, $n_m \longrightarrow \infty$, v-a.e. on $\mathcal{Q}$.*

(ii) *$|w_{n_m}(z)| \leqslant \psi(z)$, v-a.e. on $\mathcal{Q}$.*

Theorem 2.2. *([7]). Let $v \in A_t$, $1 < t < \infty$, and let $\mathcal{Q}$ be a bounded open set in $\mathbb{R}^d$. There exist constants $M_{\mathcal{Q}}$ and δ positive such that for all $\varphi \in W_0^{1,t}(\mathcal{Q},v)$ and all ν satisfying $1 \leqslant \nu \leqslant \frac{n}{n-1} + \delta$,*

$$||\varphi||_{L^{\nu t}(\mathcal{Q},v)} \leqslant M_{\mathcal{Q}} ||\nabla\varphi||_{L^t(\mathcal{Q},v)},$$

where $M_{\mathcal{Q}}$ depends only on n, t, the A_t constant of v and the diameter of $\mathcal{Q}$.

Proposition 2.3. *([5]). Let $1 < p < \infty$.*

(i) *There exists a positive constant M_p such that for all $\eta, \mu \in \mathbb{R}^d$, we have*

$$\Big||\mu|^{p-2}\mu - |\eta|^{p-2}\eta\Big| \leq M_p|\mu - \eta|\Big(|\mu| + |\eta|\Big)^{p-2}.$$

(ii) *There exist two positive constants β_p and τ_p such that for every $z, y \in \mathbb{R}^d$, it holds that*

$$\beta_p\Big(|z|+|y|\Big)^{p-2}|z-y|^2 \leq \Big\langle |z|^{p-2}z - |y|^{p-2}y, z-y\Big\rangle \leq \tau_p\Big(|z|+|y|\Big)^{p-2}|z-y|^2.$$

Theorem 2.3. *([21]). Let $\mathbb{S} : \mathbb{H} \longrightarrow \mathbb{H}^*$ be a coercive, hemi-continuous and monotone operator on the real, separable, reflexive Banach space $\mathbb{H}$. Then the following statements are valid:*

1- *The equation $\mathbb{S}w = T$ has a solution w in $\mathbb{H}$, for all $T \in \mathbb{H}^*$.*

2- *If the operator $\mathbb{S}$ is strictly monotone, then equation $\mathbb{S}w = T$ has a unique solution $w \in \mathbb{H}$.*

3. Basic assumptions and concept of solutions

3.1. Basic assumptions

Let us give the specific conditions of Problem (1.1). We assume the following assumptions: $\mathcal{Q}$ is a bounded open subset of $\mathbb{R}^d$($d \geq 2$); $1 < q, s < p < \infty$;

let v_1, v_2, v_3 and v_4 be a weights functions, and let $\mathcal{K}: \mathcal{Q} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$; $\mathcal{L}: \mathcal{Q} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, with $\mathcal{L}(z,\eta,\mu) = \Big(\mathcal{L}_1(z,\eta,\mu), ..., \mathcal{L}_n(z,\eta,\mu)\Big)$ and $\mathcal{K}(z,\mu) = \Big(\mathcal{K}_1(z,\mu), ..., \mathcal{K}_n(z,\mu)\Big)$ and $b: \mathcal{Q} \times \mathbb{R} \longrightarrow \mathbb{R}$, satisfying the following assumptions:

(I) $\mathcal{L}_k$, $\mathcal{K}_k$ (for $k = 1, ..., n$), and b are Caratéodory functions.

(II) There are positive functions h_1, h_2, h_3, $h_4 \in L^\infty(\mathcal{Q})$ and $\tau_1 \in L^{p'}(\mathcal{Q}, v_1)$ $\Big($with $\frac{1}{p} + \frac{1}{p'} = 1\Big)$, $\tau_2 \in L^{q'}(\mathcal{Q}, v_2)\Big($with $\frac{1}{q} + \frac{1}{q'} = 1\Big)$ and $\tau_3 \in L^{s'}(\mathcal{Q}, v_3)\Big($with $\frac{1}{s} + \frac{1}{s'} = 1\Big)$ such that :

$$|\mathcal{K}(z,\mu)| \le \tau_1(z) + h_1(z)|\mu|^{p-1},$$

$$|\mathcal{L}(z,\eta,\mu)| \le \tau_2(z) + h_2(z)|\eta|^{q-1} + h_3(z)|\mu|^{q-1},$$

and

$$|b(z,\eta)| \le \tau_3(z) + h_4(z)|\eta|^{s-1}.$$

(III) There exists a constant $\alpha > 0$ such that :

$$\Big\langle \mathcal{K}(z,\mu) - \mathcal{K}(z,\mu'), \mu - \mu' \Big\rangle \geqslant \alpha|\mu - \mu'|^p,$$

$$\Big\langle \mathcal{L}(z,\eta,\mu) - \mathcal{L}(z,\eta',\mu'), \mu - \mu' \Big\rangle \geqslant 0,$$

and

$$\Big(b(z,\eta) - b(z,\eta')\Big)\Big(\eta - \eta'\Big) \geqslant 0,$$

whenever $(\eta,\mu), (\eta',\mu') \in \mathbb{R} \times \mathbb{R}^n$ with $\eta \neq \eta'$ and $\mu \neq \mu'$ $\Big($where $\langle .,. \rangle$ denotes here the usual inner product in $\mathbb{R}^n\Big)$.

(IV) There are constants β_1, β_2, $\beta_3 > 0$ such that :

$$\Big\langle \mathcal{K}(z,\mu), \mu \Big\rangle \geqslant \beta_1|\mu|^p,$$

$$\Big\langle \mathcal{L}(z,\eta,\mu), \mu \Big\rangle \geqslant \beta_2|\mu|^q + \beta_3|\eta|^q,$$

and

$$b(z,\eta) \cdot \eta \geqslant 0.$$

3.2. Concept of solutions

The definition of a weak solution for Problem (1.1) is as follows

Definition 3.1. One says $w \in \mathbb{H}$ is a weak solution to Problem (1.1), provided that

$$\int_{\mathcal{Q}} |\Delta w|^{t-2}\Delta w \Delta v \phi_1 dz + \int_{\mathcal{Q}} |\Delta w|^{q-2}\Delta w \Delta v \phi_2 dz + \int_{\mathcal{Q}} \langle \mathcal{K}(z,\nabla w), \nabla v\rangle\, v_1\, dz$$
$$+ \int_{\mathcal{Q}} \langle \mathcal{L}(z,w,\nabla w), \nabla v\rangle\, v_2\, dz + \int_{\mathcal{Q}} b(z,w)\, v\, v_3\, dz + \int_{\mathcal{Q}} |w|^{p-2} w\, v\, v_4\, dz$$
$$= \int_{\mathcal{Q}} hv dz,$$

for all $v \in \mathbb{H}$.

Remark 3.1. We aim to establish a relationship between v_1, v_2 and v_3, in order to guarantee the existence and uniqueness of the solution to the problem (1.1). At first we notice, for all v_1, v_2, $v_3 \in A_p$:

(i) If $\frac{\phi_2}{\phi_1} \in L^{r_0}(\mathcal{Q}, \phi_1)$ where $r_0 = \frac{t}{t-q}$ and $(2 \leq q < t < \infty)$ then there exists a constant $M > 0$ such that

$$\|w\|_{L^q(\mathcal{Q},\phi_2)} \leq M_{t,q}\|w\|_{L^t(\mathcal{Q},\phi_1)}$$

where $M_{t,q} = \|\phi_2/\phi_1\|^{1/q}_{L^{r_0}(\mathcal{Q},\phi_1)}$. In fact, by Hölder's inequality

$$\begin{aligned}\|w\|^q_{L^q(\mathcal{Q},\phi_2)} &= \int_{\mathcal{Q}} |w|^q \phi_2 dz = \int_{\mathcal{Q}} |w|^q \frac{\phi_2}{\phi_1}\phi_1 dz \\ &\leq \left(\int_{\mathcal{Q}} |w|^{qt/q}\phi_1 dz\right)^{q/t} \left(\int_{\mathcal{Q}} \left(\frac{\phi_2}{\phi_1}\right)^{t/(t-q)} \phi_1 dz\right)^{(t-q)/t} \\ &= \|w\|^q_{L^t(\mathcal{Q},\phi_1)} \|\phi_2/\phi_1\|_{L^{r_0}(\mathcal{Q},\phi_1)}.\end{aligned}$$

(2i) If $\frac{v_2}{v_1} \in L^{r_1}(\mathcal{Q}, v_1)$ where $r_1 = \frac{p}{p-q}$ and $1 < q < p < \infty$, then, by Hölder inequality we obtain

$$||w||_{L^q(\mathcal{Q},v_2)} \leqslant M_{p,q}||w||_{L^p(\mathcal{Q},v_1)},$$

where $M_{p,q} = ||\frac{v_2}{v_1}||^{1/q}_{L^{r_1}(\mathcal{Q},v_1)}$.

(3i) Analogously, if $\frac{v_3}{v_1} \in L^{r_2}(\mathcal{Q}, v_1)$ where $r_2 = \frac{p}{p-s}$ and $1 < s < p < \infty$, then

$$||w||_{L^s(\mathcal{Q},v_3)} \leqslant M_{p,s}||w||_{L^p(\mathcal{Q},v_1)},$$

where $M_{p,s} = ||\frac{v_3}{v_1}||^{1/s}_{L^{r_2}(\mathcal{Q},v_1)}$.

(4i) Analogously, if $\frac{v_4}{v_1} \in L^{r_3}(\mathcal{Q}, v_1)$ where $r_3 = \frac{p}{p-p'}$ and $1 < p' < p < \infty$, then

$$||w||_{L^s(\mathcal{Q},v_4)} \leqslant M_{p,p'}||w||_{L^p(\mathcal{Q},v_1)},$$

where $M_{p,p'} = ||\frac{v_4}{v_1}||^{1/p'}_{L^{r_3}(\mathcal{Q},v_1)}$.

4. Main result

4.1. Result on the existence and uniqueness

The main result of this article is given in the following theorem.

Theorem 4.1. *Let* $v_i \in A_p (i = 1, 2, 3, 4)$ *and* $\phi_1, \phi_2 \in A_t$, $1 < q, s < p, t < \infty$ *and assume that the assumptions **(I) − (IV)** hold. If* $\frac{\phi_2}{\phi_1} \in L^{t/(t-q)}(\mathcal{Q}, \phi_1)$, $\frac{h}{v_1} \in L^{p'}(\mathcal{Q}, v_1)$, $\frac{v_2}{v_1} \in L^{p/(p-q)}(\mathcal{Q}, v_1)$ *and* $\frac{v_3}{v_1} \in L^{p/(p-s)}(\mathcal{Q}, v_1)$*, then Problem* (1.1) *has exactly one solution* $w \in \mathbb{H}$.

4.2. Proof of Theorem 4.1

The essence of our demonstration is to transform Problem (1.1) to an operator problem $\mathcal{A}w = \mathcal{G}$ and apply Theorem 2.3.

We define

$$\mathcal{F} : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$$

and

$$\mathcal{G} : \mathbb{H} \longrightarrow \mathbb{R},$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined below.

Then $w \in H$ is a weak solution of (1.1) if and only if

$$\mathcal{F}(w,v) = \mathcal{G}(v), \qquad \text{for all } \ v \in \mathbb{H}.$$

The proof of Theorem 4.1 is divided into several notes.

4.2.1. Equivalent operator equation

In this subsection, we prove that Problem (1.1) is equivalent to an operator equation $\mathcal{A}w = \mathcal{G}$.

Using Hölder inequality and Theorem 2.2, we obtain

$$\begin{aligned}
|\mathcal{G}(v)| &\leq \int_{\mathcal{Q}} \frac{|h|}{v_1} \, |v| v_1 \, dz \\
&\leq ||h/v_1||_{L^{p'}(\mathcal{Q},v_1)} ||v||_{L^p(\mathcal{Q},v_1)} \\
&\leq M_{\mathcal{Q}} ||h/v_1||_{L^{p'}(\mathcal{Q},v_1)} ||v||_{\mathbb{H}}.
\end{aligned}$$

Since $h/v_1 \in L^{p'}(\mathcal{Q}, v_1)$, then $\mathcal{G} \in \mathbb{H}^*$.

The operator $\mathbf{F}$ is broken down into the from

$$\mathcal{F}(w,v) = \mathcal{F}_1(w,v) + \mathcal{F}_2(w,v) + \mathcal{F}_3(w,v) + \mathcal{F}_4(w,v) + \mathcal{F}_5(w,v) + \mathcal{F}_6(w,v),$$

where $\mathcal{F}_i : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$, for $i = 1,2,3,4,5,6$, are defined as

$$\mathcal{F}_1(w,v) = \int_{\mathcal{Q}} \langle \mathcal{K}(z, \nabla w), \nabla v \rangle v_1 dz \qquad , \qquad \mathcal{F}_2(w,v) = \int_{\mathcal{Q}} \langle \mathcal{L}(z, w, \nabla w), \nabla v \rangle v_2 dz,$$

$$\mathcal{F}_3(w,v) = \int_{\mathcal{Q}} b(z,w) v \, v_3 dz \qquad , \qquad \mathcal{F}_4(w,v) = \int_{\mathcal{Q}} |w|^{p-2} w \, v \, v_4 \, dz,$$

$$\mathcal{F}_5(w,v) = \int_{\mathcal{Q}} |\Delta w|^{t-2} \Delta w \Delta v \phi_1 dz \qquad \text{and} \qquad \mathcal{F}_6(w,v) = \int_{\mathcal{Q}} |\Delta w|^{q-2} \Delta w \Delta v \phi_2 dz.$$

Then, we have

$$\begin{aligned}
|\mathcal{F}(w,v)| \leq |\mathcal{F}_1(w,v)| + |\mathcal{F}_2(w,v)| + |\mathcal{F}_3(w,v)| \\
+ |\mathcal{F}_4(w,v)| + |\mathcal{F}_5(w,v)| + |\mathcal{F}_6(w,v)|.
\end{aligned} \tag{4.1}$$

On the other hand, we get by using **(II)**, Hölder inequality, Remark 3.1 (i) and Theorem 2.2,

$$\begin{aligned}|\mathcal{F}_1(w,v)| &\leq \int_{\mathcal{Q}} |\mathcal{K}(z,\nabla w)||\nabla v|v_1 dz \\ &\leq \int_{\mathcal{Q}} \Big(\tau_1 + h_1|\nabla w|^{p-1}\Big)|\nabla v|v_1 dz \\ &\leq ||\tau_1||_{L^{p'}(\mathcal{Q},v_1)}||\nabla v||_{L^p(\mathcal{Q},v_1)} + ||h_1||_{L^\infty(\mathcal{Q})}||\nabla w||^{p-1}_{L^p(\mathcal{Q},v_1)}||\nabla v||_{L^p(\mathcal{Q},v_1)} \\ &\leq \Big(||\tau_1||_{L^{p'}(\mathcal{Q},v_1)} + ||h_1||_{L^\infty(\mathcal{Q})}||w||^{p-1}_{\mathbb{H}}\Big)\, ||v||_{\mathbb{H}},\end{aligned}$$

and

$$\begin{aligned}&|\mathcal{F}_2(w,v)| \\ &\leq \int_{\mathcal{Q}} |\mathcal{L}(z,w,\nabla w)||\nabla v|v_2 dz \\ &\leq \int_{\mathcal{Q}} \Big(\tau_2 + h_2|w|^{q-1} + h_3|\nabla w|^{q-1}\Big)|\nabla v|v_2 dz \\ &\leq ||\tau_2||_{L^{q'}(\mathcal{Q},v_2)}||\nabla v||_{L^q(\mathcal{Q},v_2)} + ||h_2||_{L^\infty(\mathcal{Q})}||w||^{q-1}_{L^q(\mathcal{Q},v_2)}||\nabla v||_{L^q(\mathcal{Q},v_2)} \\ &\quad + ||h_3||_{L^\infty(\mathcal{Q})}||\nabla w||^{q-1}_{L^q(\mathcal{Q},v_2)}||\nabla v||_{L^q(\mathcal{Q},v_2)} \\ &\leq ||\tau_2||_{L^{q'}(\mathcal{Q},v_2)}M_{p,q}||\nabla v||_{L^p(\mathcal{Q},v_1)} + ||h_2||_{L^\infty(\mathcal{Q})}M^{q-1}_{p,q}||w||^{q-1}_{L^p(\mathcal{Q},v_1)}M_{p,q}||\nabla v||_{L^p(\mathcal{Q},v_1)} \\ &\quad + ||h_3||_{L^\infty(\mathcal{Q})}M^{q-1}_{p,q}||\nabla w||^{q-1}_{L^p(\mathcal{Q},v_1)}M_{p,q}||\nabla v||_{L^p(\mathcal{Q},v_1)} \\ &\leq \Big[M^q_{p,q}\Big(M^{q-1}_{\mathcal{Q}}||h_2||_{L^\infty(\mathcal{Q})} + ||h_3||_{L^\infty(\mathcal{Q})}\Big)\,||w||^{q-1}_{\mathbb{H}} \\ &\quad + M_{p,q}||\tau_2||_{L^{q'}(\mathcal{Q},v_2)}\Big]||v||_{\mathbb{H}}.\end{aligned}$$

Analogously, using **(II)** and remark 3.1 (2i) , we obtain

$$\begin{aligned}|\mathcal{F}_3(w,v)| &\leq \int_{\mathcal{Q}} |b(z,w)||v|v_3 dz \\ &\leq \Big[M_{\mathcal{Q}}M_{p,s}||\tau_3||_{L^{s'}(f,v_3)} + M^s_{p,s}M^s_{\mathcal{Q}}||h_4||_{L^\infty(\mathcal{Q})}||w||^{s-1}_{\mathbb{H}}\Big]\, ||v||_{\mathbb{H}}.\end{aligned}$$

Next, by using Remark 3.1 (4i), we get

$$\begin{aligned}|\mathcal{F}_4(w,v)| &\leq \int_{\mathcal{Q}} |w|^{p-1}|v|v_4 dz \\ &\leq \Big(\int_{\mathcal{Q}} |w|^p v_4 dz\Big)^{1/p'}\Big(\int_{\mathcal{Q}} |v|^p v_4 dz\Big)^{1/p} \\ &= ||w||^{p-1}_{L^p(\mathcal{Q},v_4)}||v||_{L^p(\mathcal{Q},v_4)} \\ &\leq M^{p-1}_{\mathcal{Q}}M_{\mathcal{Q}}||\nabla w||^{p-1}_{L^p(\mathcal{Q},v_2)}||\nabla v||_{L^p(\mathcal{Q},v_2)} \\ &\leq M^p_{\mathcal{Q}}M^p_{p,p'}||w||^{p-1}_{\mathbb{H}}||v||_{\mathbb{H}}.\end{aligned}$$

We have

$$\begin{aligned}|\mathcal{F}_5(w,v)| \leq \int_{\mathcal{Q}} |\Delta w|^{t-1}|\Delta v|\phi_1 dz &\leq \Big(\int_{\mathcal{Q}} |\Delta w|^{(t-1)t'}\phi_1 dz\Big)^{1/t'}\Big(\int_{\mathcal{Q}} |\Delta v|^t\phi_1 dz\Big)^{1/t} \\ &= \|\Delta w\|^{t-1}_{L^t(\mathcal{Q},\phi_1)}\|\Delta v\|_{L^t(\mathcal{Q},\phi_1)} \\ &\leq \|w\|^{t-1}_{\mathbb{H}}\|v\|_{\mathbb{H}}.\end{aligned}$$

By Remark 3.1 (i), we get

$$
\begin{aligned}
|\mathcal{F}_6(w,v)| \le \int_{\mathcal{Q}} |\Delta w|^{q-1}|\Delta v|\phi_2 dz &\le \left(\int_{\mathcal{Q}} |\Delta w|^{(q-1)q'}\phi_2 dz\right)^{1/q'} \left(\int_{\mathcal{Q}} |\Delta v|^q \phi_2 dz\right)^{1/q} \\
&= \|\Delta w\|^{q-1}_{L^q(\mathcal{Q},\phi_2)} \|\Delta v\|_{L^q(\mathcal{Q},\phi_2)} \\
&\le M^{q-1}_{t,q} \|\Delta w\|^{q-1}_{L^t(\mathcal{Q},\phi_1)} M_{t,q} \|\Delta v\|_{L^t(\mathcal{Q},\phi_1)} \\
&\le M^q_{t,q} \|w\|^{q-1}_{\mathbb{H}} \|v\|_{\mathbb{H}}.
\end{aligned}
$$

Hence, in (4.1) we obtain, for all $w, v \in \mathbb{H}$

$$
\begin{aligned}
& |\mathcal{F}(w,v)| \\
\le & \Big[||\tau_1||_{L^{p'}(\mathcal{Q},v_1)} + ||h_1||_{L^\infty(\mathcal{Q})} ||w||^{p-1}_H + M_{\mathcal{Q}} M_{p,s} ||\tau_3||_{L^{s'}(\mathcal{Q},v_3)} \\
& + \ M_{p,q} ||\tau_2||_{L^{q'}(\mathcal{Q},v_2)} + M^q_{p,q} \left(M^{q-1}_{\mathcal{Q}} ||h_2||_{L^\infty(\mathcal{Q})} + ||h_3||_{L^\infty(\mathcal{Q})} \right) ||w||^{q-1}_{\mathbb{H}} \\
& + \ M^s_{p,s} M^s_{\mathcal{Q}} ||h_4||_{L^\infty(\mathcal{Q})} ||w||^{s-1}_{\mathbb{H}} + \ M_{\mathcal{Q}} ||w||^{p-1}_{\mathbb{H}} + \|w\|^{t-1}_{\mathbb{H}} + M^q_{t,q} \|w\|^{q-1}_{\mathbb{H}} \Big] \|v\|_{\mathbb{H}}.
\end{aligned}
$$

Then for each $w \in \mathbb{H}$, $\mathcal{F}(w, .)$ is linear and continuous . Thus, there exists a linear and continuous operator on $\mathbb{H}$ denoted by $\mathcal{A}$ such that

$$
\langle \mathcal{A}w, v \rangle = \mathcal{F}(w,v), \quad \text{for all } w, v \in \mathbb{H}.
$$

Moreover, we have

$$
\begin{aligned}
& \|\mathcal{A}w\|_* \\
\le & ||\tau_1||_{L^{p'}(\mathcal{Q},v_1)} + ||h_1||_{L^\infty(\mathcal{Q})} ||w||^{p-1}_H + M_{\mathcal{Q}} M_{p,s} ||\tau_3||_{L^{s'}(\mathcal{Q},v_3)} \\
& + M_{p,q} ||\tau_2||_{L^{q'}(\mathcal{Q},v_2)} + M^q_{p,q} \left(M^{q-1}_{\mathcal{Q}} ||h_2||_{L^\infty(\mathcal{Q})} + ||h_3||_{L^\infty(\mathcal{Q})} \right) ||w||^{q-1}_{\mathbb{H}} \\
& + M^s_{p,s} M^s_{\mathcal{Q}} ||h_4||_{L^\infty(\mathcal{Q})} ||w||^{s-1}_{\mathbb{H}} + M^p_{\mathcal{Q}} M^p_{p,p'} ||w||^{p-1}_{\mathbb{H}} + \|w\|^{t-1}_{\mathbb{H}} + M^q_{t,q} \|w\|^{q-1}_{\mathbb{H}},
\end{aligned}
$$

where

$$
\|\mathcal{A}w\|_* := \sup \Big\{ |\langle \mathcal{A}w, v \rangle| = |\mathcal{F}(w,v)| : v \in \mathbb{H}, \|v\|_{\mathbb{H}} = 1 \Big\}
$$

is the norm in $\mathbb{H}^*$. This gives us the operator

$$
\begin{aligned}
\mathcal{A} : \mathbb{H} &\longrightarrow \mathbb{H}^* \\
w &\longmapsto \mathcal{A}w.
\end{aligned}
$$

It is therefore possible that the equation of Problem (1.1) is equivalent to the equation of the operator

$$
\mathcal{A}w = \mathcal{G}, \quad w \in \mathbb{H}.
$$

4.2.2. Monotonicity and Coercivity of the operator $\mathcal{A}$

$\star$ Now, we show that $\mathcal{A}$ is strictly monotone.
Let $v_1, v_2 \in \mathbb{H}$ with $v_1 \neq v_2$. We have

$$\begin{aligned}
&\Big\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2 \Big\rangle \\
&= \mathcal{F}(v_1, v_1 - v_2) - \mathcal{F}(v_2, v_1 - v_2) \\
&= \int_{\mathcal{Q}} \Big(|\Delta v_1|^{t-2} \Delta v_1 - |\Delta v_2|^{p-2} \Delta v_2 \Big) \Delta (v_1 - v_2)\, \phi_1 dz \\
&\quad + \int_{\mathcal{Q}} \Big(|\Delta v_1|^{q-2} \Delta v_1 - |\Delta v_2|^{q-2} \Delta v_2 \Big) \Delta (v_1 - v_2)\, \phi_2 dz \\
&\quad + \int_{\mathcal{Q}} \Big\langle \mathcal{K}(z, \nabla v_1), \nabla (v_1 - v_2) \Big\rangle v_1 dz - \int_{\mathcal{Q}} \Big\langle \mathcal{K}(z, \nabla v_2), \nabla (v_1 - v_2) \Big\rangle v_1 dz \\
&\quad + \int_{\mathcal{Q}} \Big\langle \mathcal{L}(z, v_1, \nabla v_1), \nabla (v_1 - v_2) \Big\rangle v_2 dz \\
&\quad - \int_{\mathcal{Q}} \Big\langle \mathcal{L}(z, v_2, \nabla v_2), \nabla (v_1 - v_2) \Big\rangle v_2 dz + \int_{\mathcal{Q}} b(z, v_1)(v_1 - v_2) v_3 dz \\
&\quad - \int_{\mathcal{Q}} b(z, v_2)(v_1 - v_2) v_3 dz \\
&\quad + \int_{\mathcal{Q}} |v_1|^{p-2} v_1 (v_1 - v_2) v_4 dz - \int_{\mathcal{Q}} |v_2|^{p-2} v_2 (v_1 - v_2) v_4 dz \\
&= \int_{\mathcal{Q}} \Big(|\Delta v_1|^{t-2} \Delta v_1 - |\Delta v_2|^{p-2} \Delta v_2 \Big) \Delta (v_1 - v_2)\, \phi_1 dz \\
&\quad + \int_{\mathcal{Q}} \Big(|\Delta v_1|^{q-2} \Delta v_1 - |\Delta v_2|^{q-2} \Delta v_2 \Big) \Delta (v_1 - v_2)\, \phi_2 dz \\
&\quad + \int_{\mathcal{Q}} \Big\langle \mathcal{K}(z, \nabla v_1) - \mathcal{K}(z, \nabla v_2), \nabla (v_1 - v_2) \Big\rangle v_1 dz \\
&\quad + \int_{\mathcal{Q}} \Big\langle \mathcal{L}(z, v_1, \nabla v_1) - \mathcal{L}(z, v_2, \nabla v_2), \nabla (v_1 - v_2) \Big\rangle v_2 dz \\
&\quad + \int_{\mathcal{Q}} \Big(b(z, v_1) - b(z, v_2) \Big) \Big(v_1 - v_2 \Big) v_3 dz \\
&\quad + \int_{\mathcal{Q}} \Big(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2 \Big) \Big(v_1 - v_2 \Big) v_4 dz.
\end{aligned}$$

Thanks to **(III)** and Proposition 2.3 (ii), we obtain

$$\begin{aligned}
\Big\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2 \Big\rangle &\geq \beta_q \int_{\mathcal{Q}} (|\Delta v_1| + |\Delta v_2|)^{q-2} |\Delta v_1 - \Delta v_2|^2\, \phi_2 dz \\
&\quad + \alpha \textstyle\int_{\mathcal{Q}} |\nabla (v_1 - v_2)|^p\ v_1\ dz \\
&\quad + \beta_p \textstyle\int_{\mathcal{Q}} \Big(|v_1||v_2| \Big)^{p-2} |v_1 - v_2|^2\ v_4\ dz \\
&\geq \alpha \int_{\mathcal{Q}} |\nabla (v_1 - v_2)|^p v_1 dz + \int_{\mathcal{Q}} |\Delta v_1 - \Delta v_2|^q\, \phi_2 dz \\
&\geq \alpha \|\nabla (v_1 - v_2)\|^p_{L^p(\mathcal{Q}, v_1)}.
\end{aligned}$$

Therefore, $\mathcal{A}$ is strictly monotone .

$\star$ In this note, we prove that the operator $\mathcal{A}$ is coercive.

Letting $w \in \mathbb{H}$, we have

$$\begin{aligned}\langle \mathcal{A}w, w\rangle &= \mathcal{F}(w,w)\\ &= \mathcal{F}_1(w,w)+\mathcal{F}_2(w,w)+\mathcal{F}_3(w,w)+\mathcal{F}_4(w,w)+\mathcal{F}_5(w,w)+\mathcal{F}_6(w,w)\\ &= \int_{\mathcal{Q}} |\Delta w|^p \phi_1 dz + \int_{\mathcal{Q}} |\Delta w|^q \phi_2 dz\\ &\quad + \int_{\mathcal{Q}} \langle \mathcal{K}(z,\nabla w), \nabla w\rangle v_1 dz + \int_{\mathcal{Q}} \langle \mathcal{L}(z,w,\nabla w), \nabla w\rangle v_2 dz\\ &\quad + \int_{\mathcal{Q}} b(z,w)w\, v_3 dz + \int_{\mathcal{Q}} |w|^p v_4 dz.\end{aligned}$$

Moreover, from **(IV)** and Theorem 2.2(with $\nu = 1$), we obtain

$$\begin{aligned}\langle \mathcal{A}w, w\rangle &\geq \int_{\mathcal{Q}} |\Delta w|^p \phi_1 dz + \beta_1 \int_{\mathcal{Q}} |\nabla w|^p v_1 dz + \beta_2 \int_{\mathcal{Q}} |\nabla w|^q v_2 dz\\ &\quad + \beta_3 \int_{\mathcal{Q}} |w|^q v_2 dz + \int_{\mathcal{Q}} |w|^p v_4 dz\\ &\geq \int_{\mathcal{Q}} |\Delta w|^p \phi_1 dz + min(\beta_1, 1)\left[\int_{\mathcal{Q}} |\nabla w|^p v_1 dz + \int_{\mathcal{Q}} |w|^p v_4 dz\right]\\ &\quad + min(\beta_2,\beta_3)\left[\int_{\mathcal{Q}} |\nabla w|^q v_2 dz + \int_{\mathcal{Q}} |w|^q v_2 dz\right]\\ &\geq min(\beta_1, 1)\|w\|_{\mathbb{H}}^p.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathcal{A}w, w\rangle}{\|w\|_{\mathbb{H}}} \geq min(\beta_1, 1)\|w\|_{\mathbb{H}}^{p-1}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathcal{A}w, w\rangle}{\|w\|_{\mathbb{H}}} \longrightarrow +\infty \text{ as } \|w\|_{\mathbb{H}} \longrightarrow +\infty,$$

that is, $\mathcal{A}$ is coercive.

4.2.3. Continuity of the operator $\mathcal{A}$

Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. Then $\nabla w_n \longrightarrow \nabla w$ in $(L^p(\mathcal{Q}, v_1))^i$. Hence, thanks to Theorem 2.1, there exists a sub sequence (w_{n_m}) and $\psi \in L^p(\mathcal{Q}, v_1)$ such that

$$\begin{aligned}&\nabla w_{n_m}(z) \longrightarrow \nabla w(z), \quad \text{a.e. in } \mathcal{Q},\\ &|\nabla w_{n_m}(z)| \leq \psi(z), \quad \text{a.e. in } \mathcal{Q}.\end{aligned} \tag{4.2}$$

We will show that $\mathcal{A}w_n \longrightarrow \mathcal{A}w$ in $\mathbb{H}^*$.

The following notes are required to demonstrate this convergence.

Note 1:

For $k = 1, ..., n$, we define the operator

$$\begin{aligned}&B_k : H \longrightarrow L^{p'}(\mathcal{Q}, v_1)\\ &(B_k w)(z) = \mathcal{K}_k(z, \nabla w(z)).\end{aligned}$$

We need to show that $B_k w_n \longrightarrow B_k w$ in $L^{p'}(\mathcal{Q}, v_1)$.

In Banach spaces, we will use the convergence principle and the Lebesgue theorem.

- Let $w \in \mathbb{H}$. Using **(II)** and Theorem 2.2(with $\nu = 1$), we obtain

$$\begin{aligned}\|B_k w\|^{p'}_{L^{p'}(\mathcal{Q},v_1)} &= \int_{\mathcal{Q}} |B_k w(z)|^{p'} v_1 dz = \int_{\mathcal{Q}} |\mathcal{K}_k(z, \nabla w)|^{p'} v_1 dz \\ &\leq \int_{\mathcal{Q}} \left(\tau_1 + h_1 |\nabla w|^{p-1}\right)^{p'} v_1 dz \\ &\leq M_p \int_{\mathcal{Q}} \left(\tau_1^{p'} + h_1^{p'} |\nabla w|^p\right) v_1 dz \\ &\leq M_p \left[\|\tau_1\|^{p'}_{L^{p'}(\mathcal{Q},v_1)} + \|h_1\|^{p'}_{L^\infty(\mathcal{Q})} \|\nabla w\|^p_{L^p(\mathcal{Q},v_1)}\right] \\ &\leq M_p \left[\|\tau_1\|^{p'}_{L^{p'}(\mathcal{Q},v_1)} + \|h_1\|^{p'}_{L^\infty(\mathcal{Q})} \|w\|^p_{\mathbb{H}}\right],\end{aligned}$$

where the constant M_p depends only on p.

- Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. By **(II)** and (4.2), we obtain

$$\begin{aligned}\|B_k w_{n_m} - B_k w\|^{p'}_{L^{p'}(\mathcal{Q},v_1)} &= \int_{\mathcal{Q}} |B_k w_{n_m}(z) - B_k w(z)|^{p'} v_1 dz \\ &\leq \int_{\mathcal{Q}} \Big(|\mathcal{K}_k(z, \nabla w_{n_m})| + |\mathcal{K}_k(z, \nabla w)|\Big)^{p'} v_1 dz \\ &\leq M_p \int_{\mathcal{Q}} \Big(|\mathcal{K}_k(z, \nabla w_{n_m})|^{p'} + |\mathcal{K}_k(z, \nabla w)|^{p'}\Big) v_1 dz \\ &\leq M_p \int_{\mathcal{Q}} [\left(\tau_1 + h_1 |\nabla w_{n_m}|^{p-1}\right)^{p'} \\ &+ \left(\tau_1 + h_1 |\nabla w|^{p-1}\right)^{p'}] v_1 dz \\ &\leq M_p \int_{\mathcal{Q}} \left[\left(\tau_1 + h_1 \psi^{p-1}\right)^{p'} + \left(\tau_1 + h_1 \psi^{p-1}\right)^{p'}\right] v_1 dz \\ &\leq 2 M_p M_p' \int_{\mathcal{Q}} \left(\tau_1^{p'} + h_1^{p'} \psi^p\right) v_1 dz \\ &\leq 2 M_p M_p' \left[\|\tau_1\|^{p'}_{L^{p'}(\mathcal{Q},v_1)} + \|h_1\|^{p'}_{L^\infty(\mathcal{Q})} \|\psi\|^p_{L^p(\mathcal{Q},v_1)}\right].\end{aligned}$$

Hence, thanks to (**I**), we get, as $n \longrightarrow \infty$

$$B_k w_{n_m}(z) = \mathcal{K}_k(z, \nabla w_{n_m}(z)) \longrightarrow \mathcal{K}_k(z, \nabla w(z)) = B_k w(z), \quad \text{a.e. } z \in \mathcal{Q}.$$

Therefore, by Lebesgue's theorem, we obtain

$$\|B_k w_{n_m} - B_k w\|_{L^{p'}(\mathcal{Q},v_1)} \longrightarrow 0,$$

that is,

$$B_k w_{n_m} \longrightarrow B_k w \quad \text{in} \quad L^{p'}(\mathcal{Q}, v_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$B_k w_n \longrightarrow B_k w \quad \text{in} \quad L^{p'}(\mathcal{Q}, v_1). \tag{4.3}$$

Note 2:
For $k = 1, ..., n$, we define the operator

$$O_k : H \longrightarrow L^{q'}(\mathcal{Q}, v_2),$$
$$(O_k w)(z) = \mathcal{L}_k(z, w(z), \nabla w(z)).$$

We will prove that $O_k w_n \longrightarrow O_k w$ in $L^{q'}(\mathcal{Q}, v_2)$.

- Let $w \in \mathbb{H}$. Using **(II)**, Remark 3.1 (i) and Theorem 2.2(with $\nu = 1$), we obtain

$$\begin{aligned}
&\|O_k w\|^{q'}_{L^{q'}(\mathcal{Q},v_2)} \\
&= \int_{\mathcal{Q}} |\mathcal{L}_k(z, w, \nabla w)|^{q'} v_2\, dz \\
&\leq \int_{\mathcal{Q}} \left(\tau_2 + h_2|w|^{q-1} + h_3|\nabla w|^{q-1}\right)^{q'} v_2\, dz \\
&\leq M_q \int_{\mathcal{Q}} \left[\tau_2^{q'} + h_2^{q'}|w|^q + h_3^{q'}|\nabla w|^q\right] v_2\, dz \\
&\leq M_q \left[\|\tau_2\|^{q'}_{L^{q'}(\mathcal{Q},v_2)} + \|h_2\|^{q'}_{L^\infty(\mathcal{Q})}\|w\|^q_{L^q(\mathcal{Q},v_2)} + \|h_3\|^{q'}_{L^\infty(\mathcal{Q})}\|\nabla w\|^q_{L^q(\mathcal{Q},v_2)}\right] \\
&\leq M_q \left[\|\tau_2\|^{q'}_{L^{q'}(\mathcal{Q},v_2)} + \|h_2\|^{q'}_{L^\infty(\mathcal{Q})} C^q_{p,q}\|w\|^q_{L^p(\mathcal{Q},v_1)} + \|h_3\|^{q'}_{L^\infty(\mathcal{Q})} C^q_{p,q}\|\nabla w\|^q_{L^p(\mathcal{Q},v_1)}\right] \\
&\leq M_q \left[\|\tau_2\|^{q'}_{L^{q'}(\mathcal{Q},v_2)} + C^q_{p,q}\left(M^q_{\mathcal{Q}}\|h_2\|^{q'}_{L^\infty(\mathcal{Q})} + \|h_3\|^{q'}_{L^\infty(\mathcal{Q})}\right)\|w\|^q_{\mathbb{H}}\right],
\end{aligned}$$

  where the constant M_q depends only on q.
- Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. According to **(II)**, Remark 3.1 (i) and the same arguments used in Note 1 **(ii)**, we obtain analogously,

$$O_k w_n \longrightarrow O_k w \quad \text{in} \quad L^{q'}(\mathcal{Q}, v_2). \tag{4.4}$$

Note 3:
We define the operator

$$N : \mathbb{H} \longrightarrow L^{s'}(\mathcal{Q}, v_3),$$
$$(Nw)(z) = b(z, w(z)).$$

In this note, we will show that $Nw_n \longrightarrow Nw$ in $L^{s'}(\mathcal{Q}, v_3)$.

- Let $w \in H$. Using **(II)** and Remark 3.1 (2i), we obtain

$$\begin{aligned}
\|Nw\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} &= \int_{\mathcal{Q}} |b(z,w)|^{s'} v_3 dz \\
&\leq \int_{\mathcal{Q}} \left(\tau_3 + h_4|w|^{s-1}\right)^{s'} v_3 dz \\
&\leq M_s \int_{\mathcal{Q}} \left(\tau_3^{s'} + h_4^{s'}|w|^s\right) v_3 dz \\
&\leq M_s \left[\|\tau_3\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} + \|h_4\|^{p'}_{L^\infty(\mathcal{Q})}\|w\|^s_{L^s(\mathcal{Q},v_3)}\right] \\
&\leq M_s \left[\|\tau_3\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} + M^s_{p,s}\|h_4\|^{p'}_{L^\infty(\mathcal{Q})}\|w\|^s_{L^p(\mathcal{Q},v_1)}\right] \\
&\leq M_s \left[\|\tau_3\|_{L^{s'}(\mathcal{Q},v_1)} + M^s_{p,s} M^s_{\mathcal{Q}}\|h_4\|^{s'}_{L^\infty(\mathcal{Q})}\|w\|^s_{\mathbb{H}}\right],
\end{aligned}$$

  where the constant M_s depends only on s.

- Let $w_n \longrightarrow w$ in H as $n \longrightarrow \infty$. By **(II)** and Remark 3.1 (2i), we get

$$\begin{aligned}
\|Nw_{n_m} - Nw\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} &= \int_{\mathcal{Q}} \Big|Nw_{n_m}(z) - Nw(z)\Big|^{p'} v_3 dz \\
&\leq \int_{\mathcal{Q}} \Big(|b(z,w_{n_m})| + |b(z,w)|\Big)^{s'} v_3 dz \\
&\leq M_s \int_{\mathcal{Q}} \Big(|b(z,w_{n_m})|^{s'} + |b(z,w)|^{s'}\Big) v_3 dz \\
&\leq M_s \int_{\mathcal{Q}} \Big[\Big(\tau_3 + h_4|w_{n_m}|^{s-1}\Big)^{s'} + \Big(\tau_3 + h_4|w|^{s-1}\Big)^{s'}\Big] v_3 dz \\
&\leq M_s \int_{\mathcal{Q}} \Big[\Big(\tau_3 + h_4|\psi|^{s-1}\Big)^{s'} + \Big(\tau_3 + h_4\psi^{s-1}\Big)^{s'}\Big] v_3 dz \\
&\leq 2M_s M_s' \Big[\|\tau_3\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} + \|h_4\|^{s'}_{L^\infty(\mathcal{Q})} \|\psi\|^{s}_{L^{s}(\mathcal{Q},v_3)}\Big] \\
&\leq 2M_s M_s' \Big[\|\tau_3\|^{s'}_{L^{s'}(\mathcal{Q},v_3)} + M^s_{p,s}\|h_4\|^{s'}_{L^\infty(\mathcal{Q})} \|\psi\|^{s}_{L^{p}(\mathcal{Q},v_1)}\Big].
\end{aligned}$$

Next using condition $\mathcal{A}\infty$, we deduce, as $n \longrightarrow \infty$

$$Nw_{n_m}(z) = b(z, w_{n_m}(z)) \longrightarrow b(z, w(z)) = Nw(z), \quad \text{a.e. } z \in \mathcal{Q}.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Nw_{n_m} - Nw\|_{L^{s'}(\mathcal{Q},v_3)} \longrightarrow 0,$$

that is,

$$Nw_{n_m} \longrightarrow Nw \quad \text{in} \quad L^{s'}(\mathcal{Q}, v_3).$$

We conclude, from the convergence principle in Banach spaces, that

$$Nw_n \longrightarrow Nw \quad \text{in} \quad L^{s'}(\mathcal{Q}, v_3). \tag{4.5}$$

Note 4:
We define the operator

$$\begin{aligned}
&J : H \longrightarrow L^{p'}(\mathcal{Q}, v_4), \\
&(Jw)(z) = |w(z)|^{p-2} w(z).
\end{aligned}$$

In this note, we will demonstrate that $Jw_n \longrightarrow Jw$ in $L^{p'}(\mathcal{Q}, v_4)$.

- Let $w \in \mathbb{H}$. We have

$$\begin{aligned}
\|Jw\|^{p'}_{L^{p'}(\mathcal{Q},v_4)} &= \int_{\mathcal{Q}} |Jw|^{p'} v_4 dz \\
&= \int_{\mathcal{Q}} |w|^{(p-1)p'} v_4 dz \\
&= \int_{\mathcal{Q}} |w|^{p} v_4 dz \\
&\leq \ M^p_{\mathcal{Q}} M^p_{p,p'} \|w\|^p_{\mathbb{H}}.
\end{aligned}$$

- Let $w_n \longrightarrow w$ in $\mathbb{H}$ as $n \longrightarrow \infty$. Then $w_n \longrightarrow w$ in $L^p(\mathcal{Q}, v_4)$. Hence, thanks to Theorem 2.1, there exists a subsequence (w_{n_m}) and $\varphi \in L^p(\mathcal{Q}, v_4)$ such that

$$w_{n_m}(z) \longrightarrow w(z), \quad \text{a.e. in } \mathcal{Q},$$

$$|w_{n_m}(z)| \leq \varphi(z), \qquad \text{a.e. in } \mathcal{Q}.$$

Next, we get

$$\begin{aligned}
\|Jw_{n_m} - Jw\|^{p'}_{L^{p'}(\mathcal{Q},v_4)} &= \int_{\mathcal{Q}} \Big| Jw_{n_m}(z) - Jw(z)\Big|^{p'} v_4 dz \\
&\leq \int_{\mathcal{Q}} \Big(|Jw_{n_m}(z)| + |Jw(z)|\Big)^{p'} v_4 dz \\
&\leq M_p \int_{\mathcal{Q}} \Big(|Jw_{n_m}(z)|^{p'} + |Jw(z)|^{p'}\Big) v_4 dz \\
&\leq M_p \int_{\mathcal{Q}} \Big(||w_{n_m}|^{p-2} w_{n_m}|^{p'} + ||w|^{p-2} w|^{p'}\Big) v_4 dz \\
&\leq M_p \int_{\mathcal{Q}} \Big(|w_{n_m}|^{(p-1)p'} + |w|^{(p-1)p'}\Big) v_4 dz \\
&\leq M_p \int_{\mathcal{Q}} \Big(|w_{n_m}|^{p} + |w|^{p}\Big) v_4 dz \\
&\leq M_p \int_{\mathcal{Q}} \Big(|\varphi|^{p} + |\varphi|^{p}\Big) v_4 dz \\
&\leq 2M_p \int_{\mathcal{Q}} |\varphi|^{p} v_4 dz \\
&\leq 2M_p \|\varphi\|^{p}_{L^p(\mathcal{Q},v_4)}.
\end{aligned}$$

Therefore, by Lebesgue's theorem, we obtain

$$\|Jw_{n_m} - Jw\|_{L^{p'}(\mathcal{Q},v_4)} \longrightarrow 0,$$

that is,

$$Jw_{n_m} \longrightarrow Jw \quad \text{in} \quad L^{p'}(\mathcal{Q}, v_4).$$

We conclude, in view of the convergence principle in Banach spaces, that

$$Jw_n \longrightarrow Jw \quad \text{in} \quad L^{p'}(\mathcal{Q}, v_4). \tag{4.6}$$

Note 5:
We define the operator $K : \mathbb{H} \to L^{t'}(\mathcal{Q}, \phi_1)$ by

$$(Kw)(z) = |\Delta w(z)|^{t-2} \Delta w(z).$$

We now show that, the operator K is bounded and continuous.

• We have

$$\begin{aligned}
\|Kw\|^{t'}_{L^{t'}(\mathcal{Q},\phi_1)} &= \int_{\mathcal{Q}} |F_5 w(z)|^{t'} \phi_1 dz \\
&= \int_{\mathcal{Q}} \Big| |\Delta w|^{t-2} \Delta w\Big|^{t'} \phi_1 dz \\
&= \int_{\mathcal{Q}} |\Delta w|^{t} \phi_1 dz \\
&= \|\Delta w\|^{t}_{L^t(\mathcal{Q},\phi_1)} \\
&\leq \|w\|^{t}_{\mathbb{H}}.
\end{aligned}$$

Therefore, we obtain

$$\|Kw\|_{L^{t'}(\mathcal{Q},\phi_1)} \leq \|w\|_{\mathbb{H}}^{t-1}$$

and hence the boundedness.

• Let $w_m \to w$ in $\mathbb{H}$ as $m \to 0$. We need to show that $Kw_m \to Kw$ in $L^{t'}(\mathcal{Q},\phi_1)$. If $w_m \to w$ in H then $\Delta w_m \to \Delta w$ in $L^t(\mathcal{Q},\phi_1)$. Using Theorem 2 , there exists a subsequence $\{w_{m_k}\}$ and a function $\Phi \in L^t(\mathcal{Q},\phi_1)$ such that

$$\begin{aligned} &\Delta w_{m_k}(z) \to \Delta w(z) \text{ a.e. in } \mathcal{Q}, \\ &|\Delta w_{m_k}(z)| \leq \Phi(z) \text{ a.e. in } \mathcal{Q}. \end{aligned} \tag{4.7}$$

Now, since $t > 2$, using (4.7), $a = t/t' = t-1$ and $a' = (t-1)/(t-2)$, there exists a constant $\alpha_t > 0$ (by Proposition 2.3(i)) such that

$$\begin{aligned}
\|Kw_{m_k} - Kw\|_{L^{t'}(\mathcal{Q},\phi_1)}^{t'} &= \int_{\mathcal{Q}} |Kw_{m_k} - Kw|^{t'} \phi_1 dz \\
&= \int_{\mathcal{Q}} \left| |\Delta w_{m_k}|^{t-2}\Delta w_{m_k} - |\Delta w|^{t-2}\Delta w \right|^{t'} \phi_1 dz \\
&\leq \int_{\mathcal{Q}} \left[\alpha_t |\Delta w_{m_k} - \Delta w| \left(|\Delta w_{m_k}| + |\Delta w|\right)^{t-2} \right]^{t'} \phi_1 dz \\
&\leq \alpha_t^{t'} \int_{\mathcal{Q}} |\Delta w_{m_k} - \Delta w|^{t'} (2\Phi)^{(t-2)t'} \phi_1 dz \\
&= 2^{(t-2)t'} \alpha_t^{t'} \int_{\mathcal{Q}} |\Delta w_{m_k} - \Delta w|^{t'} \Phi^{(t-2)t'} \phi_1 dz \\
&\leq 2^{(t-2)t'} \alpha_t^{t'} \left(\int_{\mathcal{Q}} |\Delta w_{m_k} - \Delta w|^{t'a} \phi_1 dz \right)^{1/a} \left(\int_{\mathcal{Q}} \Phi^{(t-2)t'a'} \phi_1 dz \right)^{1/aa'} \\
&= 2^{(t-2)t'} \alpha_t^{t'} \left(\int_{\mathcal{Q}} |\Delta w_{m_k} - \Delta w|^{t} \phi_1 dz \right)^{t'/t} \left(\int_{\mathcal{Q}} \Phi^{t} \phi_1 dz \right)^{(t-2)/(t-1)} \\
&= 2^{(t-2)t'} \alpha_t^{t'} \|\Delta w_{m_k} - \Delta w\|_{L^t(\mathcal{Q},\phi_1)}^{t'} \|\Phi\|_{L^t(\mathcal{Q},\phi_1)}^{t'(t-2)} \\
&\leq 2^{(t-2)t'} \alpha_t^{t'} \|w_{m_k} - w\|_X^{t'} \|\Phi\|_{L^t(\mathcal{Q},\phi_1)}^{t'(t-2)}.
\end{aligned}$$

Therefore (since $2 < t < \infty$), we obtain $\|Kw_{m_k} - Kw\|_{L^{t'}(\mathcal{Q},\phi_1)} \to 0$, that is,

$$Kw_{m_k} \to Kw \text{ in } L^{t'}(\mathcal{Q},\phi_1)$$

By the convergence principle in Banach spaces (see Proposition 10.13 in [22]), we have

$$Kw_m \to Kw \text{ in }, L^{t'}(\mathcal{Q},\phi_1). \tag{4.8}$$

Note 6:
Define the operator $G : \mathbb{H} \to L^{q'}(\mathcal{Q},\phi_2), (Gw)(z) = |\Delta w(z)|^{q-2}\Delta w(z)$. We also have that the operator G is continuous and bounded. In fact:

• If $q > 2$, we have by Remark 3.1 (i), that

$$\begin{aligned}\|Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}^{q'} = & \int_{\mathcal{Q}}\left||\Delta w|^{q-2}\Delta w\right|^{q'}\phi_2 dz = \int_{\mathcal{Q}}|\Delta w|^{q}\phi_2 dz \\ & = \|\Delta w\|_{L^q(\mathcal{Q},\phi_2)}^{q} \\ & \leq M_{t,q}^{q}\|\Delta w\|_{L^t(\mathcal{Q},\phi_1)}^{q} \\ & \leq M_{t,q}^{q}\|w\|_{\mathbb{H}}^{q}.\end{aligned}$$

Hence,

$$\|Gw\|_{L^{q'}(\mathcal{Q},\phi_2)} \leq M_{t,q}^{q-1}\|w\|_{\mathbb{H}}^{q-1}.$$

• Now using (4.7), Remark 3.1 (i), $b = q/q' = q-1$ and $b' = (q-1)/(q-2)$ (if $q > 2$), there exists a constant $\alpha_q > 0$ (by Lemma 4(a)) such that

$$\begin{aligned}\|Gw_{m_k} - Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}^{q'} &= \int_{\mathcal{Q}}|Gw_{m_k} - Gw|^{q'}\phi_2 dz \\ &= \int_{\mathcal{Q}}\left||\Delta w_{m_k}|^{q-2}\Delta w_{m_k} - |\Delta w|^{q-2}\Delta w\right|^{q'}\phi_2 dz \\ &\leq \int_{\mathcal{Q}}\left[\alpha_q\,|\Delta w_{m_k} - \Delta w|\,(|\Delta w_{m_k}| + |\Delta w|)^{(q-2)}\right]^{q'}\phi_2 dz \\ &\leq \alpha_q^{q'}\int_{\mathcal{Q}}|\Delta w_{m_k} - \Delta w|^{q'}(2\Phi)^{(q-2)q'}\phi_2 dz \\ &\leq 2^{(q-2)q'}\alpha_q^{q'}\left(\int_{\mathcal{Q}}|\Delta w_{m_k} - \Delta w|^{q'b}\phi_2 dz\right)^{1/b}\left(\int_{\mathcal{Q}}\Phi^{(q-2)q'b'}\phi_2 dz\right)^{1/b'} \\ &= \alpha_q^{q'}2^{(q-2)q'}\left(\int_{\mathcal{Q}}|\Delta w_{m_k} - \Delta w|^{q}\phi_2 dz\right)^{q'/q}\left(\int_{\mathcal{Q}}\Phi^{q}\phi_2 dz\right)^{(q-2)/(q-1)} \\ &= \alpha_q^{q'}2^{(q-2)q'}\,\|\Delta w_{m_k} - \Delta w\|_{L^q(\mathcal{Q},\phi_2)}^{q'}\,\|\Phi\|_{L^q(\mathcal{Q},\phi_2)}^{q'(q-2)} \\ &\leq \alpha_q^{q'}2^{(q-2)q'}M_{t,q}^{q'}\,\|\Delta w_{m_k} - \Delta w\|_{L^t(\mathcal{Q},\phi_1)}^{q'}\,M_{t,q}^{q'(q-2)}\,\|\Phi\|_{L^t(\mathcal{Q},\phi_1)}^{q'(q-2)} \\ &\leq \alpha_q^{q'}2^{(q-2)q'}M_{t,q}^{q}\,\|w_{m_k} - w\|_{\mathbb{H}}^{q'}\,\|\Phi\|_{L^t(\mathcal{Q},\phi_1)}^{q'(q-2)}.\end{aligned}$$

Hence

$$\|Gw_{m_k} - Gw\|_{L^{q'}(\mathcal{Q},\phi_2)} \leq 2^{q-2}\alpha_q M_{t,q}^{q-1}\,\|\Phi\|_{L^t(\mathcal{Q},\phi_1)}^{q-2}\,\|w_{m_k} - w\|_{\mathbb{H}}.$$

In the case $q = 2$ we have $(Gw)(z) = \Delta w(z)$. Hence,

$$\begin{aligned}&\|Gw\|_{L^2(\mathcal{Q},\phi_2)} = \|\Delta w\|_{L^2(\mathcal{Q},\phi_2)} \leq M_{t,q}\|\Delta w\|_{L^t(\mathcal{Q},\phi_1)} \leq M_{t,q}\|w\|_{\mathbb{H}}, \\ &\|Gw_{m_k} - Gw\|_{L^2(\mathcal{Q},\phi_2)} \leq M_{t,q}\,\|\Delta w_{m_k} - \Delta w\|_{L^t(\mathcal{Q},\phi_1)} \leq M_{t,q}\,\|w_{m_k} - w\|_{\mathbb{H}}.\end{aligned}$$

Therefore (for $2 \leq q < \infty$), we obtain $\|Gw_{m_k} - Gw\|_{L^{q'}(\mathcal{Q},\phi_2)} \to 0$, that is, $Gw_{m_k} \to Gw$ in $L^{q'}(\mathcal{Q},\phi_2)$. By the convergence principle in Banach spaces (see Proposition 10.13 in [22]), we have

$$Gw_m \to Gw \text{ in } L^{q'}(\mathcal{Q},\phi_2). \tag{4.9}$$

Finally, letting $v \in H$ and using Hölder inequality, we obtain

$$\begin{aligned}
|\mathcal{F}_1(w_n, v) - \mathcal{F}_1(w, v)| &= |\int_{\mathcal{Q}} \langle \mathcal{K}(z, \nabla w_n) - \mathcal{K}(z, \nabla w), \nabla v \rangle v_1 dz| \\
&\leq \sum_{k=1}^{n} \int_{\mathcal{Q}} |\mathcal{K}_k(z, \nabla w_n) - \mathcal{K}_k(z, \nabla w)||D_k v| v_1 dz \\
&= \sum_{k=1}^{n} \int_{\mathcal{Q}} |B_k w_n - B_k w||D_k v| v_1 dz \\
&\leq \sum_{k=1}^{n} \|B_k w_n - B_k w\|_{L^{p'}(\mathcal{Q}, v_1)} \|D_k v\|_{L^p(\mathcal{Q}, v_1)} \\
&\leq \left(\sum_{k=1}^{n} \|B_k w_n - B_k w\|_{L^{p'}(\mathcal{Q}, v_1)} \right) \|v\|_H,
\end{aligned}$$

and by Remark 3.1 (2i), we get

$$\begin{aligned}
|\mathcal{F}_2(w_n, v) - \mathcal{F}_2(w, v)| &= |\int_{\mathcal{Q}} \langle \mathcal{L}(z, w_n, \nabla w_n) - \mathcal{L}(z, w, \nabla w), \nabla v \rangle \mathcal{Q}_2 dz| \\
&\leq \sum_{k=1}^{n} \int_{\mathcal{Q}} |\mathcal{L}_k(z, w_n, \nabla w_n) - \mathcal{L}_k(z, w, \nabla w)||D_k v| v_2 dz \\
&= \sum_{k=1}^{n} \int_{\mathcal{Q}} |O_k w_n - O_k w||D_k v| v_2 dz \\
&\leq \left(\sum_{k=1}^{n} \|O_k w_n - O_k w\|_{L^{q'}(\mathcal{Q}, v_2)} \right) \|\nabla v\|_{L^q(\mathcal{Q}, v_2)} \\
&\leq M_{p,q} \left(\sum_{k=1}^{n} \|O_k w_n - O_k w\|_{L^{q'}(\mathcal{Q}, v_2)} \right) \|\nabla v\|_{L^p(\mathcal{Q}, v_1)} \\
&\leq M_{p,q} \left(\sum_{k=1}^{n} \|O_k w_n - O_k w\|_{L^{q'}(\mathcal{Q}, v_2)} \right) \|v\|_{\mathbb{H}}.
\end{aligned}$$

By Remark 3.1 (3i), we obtain

$$\begin{aligned}
|\mathcal{F}_3(w_n, v) - \mathcal{F}_3(w, v)| &\leq \int_{\mathcal{Q}} |b(z, w_n) - b(z, w)||v| v_3 dz \\
&= \int_{\mathcal{Q}} |N w_n - N w||v| v_3 dz \\
&\leq \|N w_n - N w\|_{L^{s'}(\mathcal{Q}, v_3)} \|v\|_{L^s(\mathcal{Q}, v_3)} \\
&\leq M_{p,s} \|N w_n - N w\|_{L^{s'}(\mathcal{Q}, v_3)} \|v\|_{L^p(\mathcal{Q}, v_1)} \\
&\leq M_{p,s} M_{\mathcal{Q}} \|N w_n - N w\|_{L^{s'}(\mathcal{Q}, v_3)} \|v\|_{\mathbb{H}}.
\end{aligned}$$

And by Note 4, we get

$$\begin{aligned}
|\mathcal{F}_4(w_n, v) - \mathcal{F}_4(w, v)| &\leq \int_{\mathcal{Q}} \left| |w_i|^{p-2} w_i - |w|^{p-2} w \right| |v| v_4 dz \\
&= \int_{\mathcal{Q}} |J w_n - J w||v| v_4 dz \\
&\leq M_{\mathcal{Q}} M_{p,p'} \|J w_n - J w\|_{L^{p'}(\mathcal{Q}, v_4)} \|v\|_{\mathbb{H}}.
\end{aligned}$$

We also have

$$\begin{aligned}|\mathcal{F}_5\left(w_m,\varphi\right)-\mathcal{F}_5(w,v)| &\leq \int_{\mathcal{Q}}\Big| \, |\Delta w_m|^{t-2}\Delta w_m-|\Delta w|^{t-2}\Delta w||\Delta v \mid \phi_1 dz \\ &= \int_{\mathcal{Q}} |Kw_m-Kw|\,|\Delta v|\phi_1 dz \\ &\leq \|Kw_m-Kw\|_{L^{t'}(\mathcal{Q},\phi_1)}\,\|\Delta v\|_{L^{t}(\mathcal{Q},\phi_1)} \\ &\leq \|Kw_m-Kw\|_{L^{t'}(\mathcal{Q},\phi_1)}\,\|v\|_{\mathbb{H}},\end{aligned}$$

and by Remark 3.1 (i), we get

$$\begin{aligned}|\mathcal{F}_6(w_m,v)-\mathcal{F}_6(w,v)| &\leq \int_{\mathcal{Q}}\Big||\Delta w_m|^{q-2}\Delta w_m-|\Delta w|^{q-2}\Delta w\Big||\Delta v|\phi_2 dz \\ &= \int_{\mathcal{Q}} |Gw_m-Gw\|\Delta v|\,\phi_2 dz \\ &\leq \|Gw_m-Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}\,\|\Delta v\|_{L^{q}(\mathcal{Q},\phi_2)} \\ &\leq M_{t,q}\,\|Gw_m-Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}\,\|\Delta v\|_{L^{t}(\mathcal{Q},\phi_1)} \\ &\leq M_{t,q}\|Gw_m-Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}\|v\|_{\mathbb{H}}.\end{aligned}$$

Hence, for all $v \in H$, we have

$$\begin{aligned}&|\mathcal{F}(w_n,v)-\mathcal{F}(w,v)| \\ &\leq \sum_{j=1}^{6}\Big|\mathcal{F}_j(w_n,v)-\mathcal{F}_j(w,v)\Big| \\ &\leq \Big[\sum_{k=1}^{n}\Big(\|B_k w_n-B_k w\|_{L^{p'}(\mathcal{Q},v_1)}+M_{p,q}\|O_k w_n-O_k w\|_{L^{q'}(\mathcal{Q},v_2)}\Big) \\ &\quad + M_{p,s}M_{\mathcal{Q}}\|Hw_n-Hw\|_{L^{s'}(\mathcal{Q},v_3)}+M_{\mathcal{Q}}M_{p,p'}\|Jw_n-Jw\|_{L^{p'}(\mathcal{Q},v_4)} \\ &\quad + \|Kw_m-Kw\|_{L^{t'}(\mathcal{Q},\phi_1)}+M\,\|Gw_m-Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}\Big]\|v\|_{\mathbb{H}}.\end{aligned}$$

Then, we get

$$\begin{aligned}\|\mathcal{A}w_n-\mathcal{A}w\|_* \leq &\sum_{k=1}^{n}\Big(\|B_k w_n-B_k w\|_{L^{p'}(\mathcal{Q},v_1)}+M_{p,q}\|O_k w_n-O_k w\|_{L^{q'}(\mathcal{Q},v_2)}\Big) \\ &+ M_{p,s}M_{\mathcal{Q}}\|Hw_n-Hw\|_{L^{s'}(\mathcal{Q},v_3)}+M_{\mathcal{Q}}M_{p,p'}\|Jw_n-Jw\|_{L^{p'}(\mathcal{Q},v_4)} \\ &+ \|Kw_m-Kw\|_{L^{t'}(\mathcal{Q},\phi_1)}+M\,\|Gw_m-Gw\|_{L^{q'}(\mathcal{Q},\phi_2)}\,.\end{aligned}$$

Combining (4.3), (4.4), (4.5), (4.6), (4.8) and (4.9), we deduce that

$$\|\mathcal{A}w_n-\mathcal{A}w\|_* \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

that is, $\mathcal{A}w_n \longrightarrow \mathcal{A}w$ in $\mathbb{H}^*$. Hence, $\mathcal{A}$ is continuous and this implies that $\mathcal{A}$ is hemicontinuous.

Therefore, by Theorem 2.3, the operator equation $\mathcal{A}w = \mathcal{G}$ has exactly one solution $w \in \mathbb{H}$ and it is the unique solution for problem (1.1).

Finally, the proof of Theorem 4.1 is completed.

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Conflict of interest

The authors declare that they have no conflict of interest.

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