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Mild Solution for the Time Fractional Hall-Magneto-Hydrodynamics Stochastic Equations

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Abstract In this paper, we establish the existence and uniqueness of mild solutions for the time fractional hall-magneto-hydrodynamics stochastic equations with a fractional derivative of Caputo. Initially, we focus on the existence and uniqueness in the deterministe case. Using the Mittag-Leffler operators $\{\mathcal{Q}_\alpha(-t^\alpha \mathbb{J}) : t \geq 0\}$ and $\{\mathcal{Q}_{\alpha,\alpha}(-t^\alpha \mathbb{J}) : t \geq 0\}$ and applying the bilinear fixed-point theorem, we will prove the frist result. Next, by Itô integral, and by similair analogy we will establish the existence and uniqueness in the stochastic case in $\mathcal{EN}^\mu_a \cap \mathrm{N}^{2\alpha}_{a,\mu}$.

Keywords Time fractional hall-magneto-hydrodynamics equations, Itô integral, derivative of Caputo, stochastic

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1. Introduction

The hall-magneto-hydrodynamics equations (HMHD) describes the evolution of a system consisting of charged particles that can be approximated as a conducting fluid. The HMHD equation is given by:

$$\begin{cases} v_t + (v \cdot \nabla)v + \mu(-\Delta)^\beta v + \nabla\pi = (b \cdot \nabla)b & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ b_t + (v \cdot \nabla)b + \nu(-\Delta)^\gamma b + \nabla \times ((\nabla \times b) \times b) = (b \cdot \nabla)v & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ \nabla \cdot v = 0, \quad \nabla \cdot b = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ v|_{t=0} = v_0, \, b|_{t=0} = b_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $v = (v_1, v_2, v_2)$ represents the velocity field of the flow, $b = (b_1, b_2, b_3)$ denotes the magnetic field, π denotes the pressure function, $\mu > 0$ denotes the viscosity

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coefficient, and ν represents the diffusivity coefficient, v_0 and b_0 are respectively the initial velocity and the initial magnetic field with free divergence (i.e $\nabla \cdot v_0 = 0$ and $\nabla \cdot b_0 = 0$). The operator $(-\Delta)^\beta$ is the Fourier multiplier of symbol $|\mathcal{K}|^{2\beta}$ given by

$$\mathcal{F}\big((-\Delta)^\beta v\big) = |\mathcal{K}|^{2\beta}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. To simplify and without loss of generality, we consider only the case where $\mu = \nu = 1$.

The fractional calculus has a long history, going back to the early days of differential calculus. Many mathematicians like Abel, Liouville, Euler, Riemann, Leibniz, l'Hôpital, and Fourier have discussed and studied it. In the last forty years, several rechearchers have studied it deeply and made amazing discoveries. At first, it was seen as something abstract and not useful in the real world. But recently, scientists have found that it can be applied to many different domains of science. This new usefulness comes from the unique way fractional calculus deals with nonlocal characteristics.

Much work has been done on fractional calculus, please refer to the complete study [9] and associated references. For instance, regarding its applications in physics, more specifically in electromaginetism, see [7, 21] and for viscoelasticity see [1, 2, 5, 18, 26]. Yimin Xiao in [29] gives an important application to stochastic processes induced by fractional Brownian motion. For further examples, see the extensive survey [9, 27] and the references therein.

In this respect, and by the same reasoning as Shinbrot [20], we can show some lemmas about the regularity of the fractional derivative of HMHD equations.

So it is not surprising to start studying this topic by proposing hall-magneto-hydrodynamics equations with a time fractional differential operator in time:

$$\begin{cases} {}^c\mathrm{D}_t^\alpha v + (v\cdot\nabla)v + \mu(-\Delta)^\beta v + \nabla\pi = (b\cdot\nabla)b & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ {}^c\mathrm{D}_t^\alpha b + (v\cdot\nabla)b + \nu(-\Delta)^\gamma b + \nabla\times((\nabla\times b)\times b) = (b\cdot\nabla)v & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ v|_{t=0} = v_0,\, b|_{t=0} = b_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where ${}^c\mathrm{D}_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ defined by

$${}^cD_t^\alpha\omega(t) := \frac{d}{dt}\left\{W_t^{1-\alpha}[\omega(t)-\omega(0)]\right\} = \frac{d}{dt}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}[\omega(\rho)-\omega(0)]ds\right\},$$

where $W_t^\alpha\omega(t)$ is the Riemann-Liouville fractional integral of order α of a function $\omega \in L^1(0,T;X)$ given by

$$W_t^\alpha\omega(t) := f_\alpha * \omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds, \quad t\in[0,T],$$

and $\Gamma(\alpha)$ is the Euler's Gamma function for any positive value of α while g_α is defined as follows:

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)}t^{\alpha-1}, & t>0, \\ 0, & t\le 0. \end{cases}$$

Note that, replacing the classical time differential operator with the time fractional differential operator gives a new description for the MHD equations.

In this paper, we will use the integral equation for the equations (1.2) which contain a set of operators called Mittag-Leffler operators for proving the mild solution.

Therefore, we have a strong interest in the following equations:

$$\begin{cases} {}^c\mathrm{D}_t^\alpha v + (v\cdot\nabla)v + \mu(-\Delta)^\beta v + \nabla\pi - (b\cdot\nabla)b = G_1\left(t, v_t\right)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ {}^c\mathrm{D}_t^\alpha b + (v\cdot\nabla)b + \nu(-\Delta)^\gamma b + \nabla\times((\nabla\times b)\times b) - (b\cdot\nabla)v & \\ = G_2\left(t, b_t\right)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ v|_{t=0} = v_0,\, b|_{t=0} = b_0 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Here $G_i(i=1,2)$ is a random external force and $W(t)$ is a standard Brownian motion/Wiener process on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, where $\mathbb{P}$ is a probability measure on Ω, $\mathcal{F}$ is a σ-algebra, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible subsets and $W(t) = W(\omega, t), \omega\in\Omega$ is a standard Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$.

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. Section 3 is concerned with the first result of our paper, which concerns the existence and uniqueness of mild solutions to the Eq. (1.2) (deterministic case). Finally, in Section 4,we give our basic assumptions, and we will state and prove the second result of this research cancerning the stochastic case.

2. Preliminaries

In the analysis of Eq. (1.2), we will use the theory of the Besov-Morrey spaces and fractional derivation theory. For convenience, we only recall some basic facts which will be used later. We refer to [4, 23, 30] for more details.

We begin by recalling some properties of functional spaces $\mathcal{M}_a^\mu$.

Definition 2.1. Let $1\leq a\leq\infty$ and $0\leq\mu<d$. The homogeneous Morrey space $\mathcal{M}_a^\mu$ is defined by

$$\mathcal{M}_a^\mu(\mathbb{R}^d) := \{\phi\in L^1_{loc}(\mathbb{R}^d), \|\phi\|_{\mathcal{M}_a^\mu} < \infty\}$$

with

$$\|\phi\|_{\mathcal{M}_a^\mu} := \sup_{x_0\in\mathbb{R}^d}\sup_{R>0} R^{-\frac{\mu}{a}}\|\phi\|_{L^a(B(x_0,R))}, \tag{2.1}$$

where $B(y,R)$ is the open ball in $\mathbb{R}^d$ centered at y and with radius $R>0$.

The space $\mathcal{M}_a^\mu$ endowed with the norm $\|\phi\|_{\mathcal{M}_a^\mu}$ is a Banach space and has the following scaling property

$$\|\phi(\beta x)\|_{\mathcal{M}_a^\mu} = \beta^{-\frac{n-\mu}{a}}\|\phi(x)\|_{\mathcal{M}_a^\mu} \qquad for \qquad \mu > 0.$$

In the case of $p = 1$, the norm $|\cdot|_{L^1}$ in equation (2.1) corresponds to the total variation of the measure ϕ on the ball $B(y,R)$, and the space $\mathcal{M}_a^\mu$ is regarded as a subset of Radon measures. When $\mu = 0$, $\mathcal{M}_a^\mu$ is equal to L^a.

Let $\phi \in C_0^\infty$ with $supp(\phi) \subseteq D_0 = \{\mathcal{K} \in \mathbb{R}^n : 2^{-1} < |\mathcal{K}| < 2\}$ such that

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\mathcal{K}) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k(\mathcal{K}) = 1, \quad \text{for all} \quad \mathcal{K} \neq 0,$$

where ϕ_k is defined by means of Fourier transform as $\hat{\phi}_k(\mathcal{K}) = \phi(2^{-k}\mathcal{K})$ for all integer k. For $f \in S'$ we define the quantity

$$\|f\|_{\mathcal{N}^s_{a,\mu,b}} := \begin{cases} \Big(\sum_{k\in\mathrm{Z}}(2^{ks}\|\phi_k * f\|_{\mathcal{M}_a^\mu})^b\Big)^{\frac{1}{b}} & 1 \le a \le \infty,\ 1 \le b < \infty,\ s \in \mathbb{R}, \\ \sup_{k\in\mathrm{Z}}(2^{ks}\|\phi_k * f\|_{\mathcal{M}_a^\mu}), & 1 \le a \le \infty,\ b = \infty,\ s \in \mathbb{R}. \end{cases}$$

It should be noted that the pair $(\mathcal{N}^s_{a,\mu,b},\ \|\cdot\|_{\mathcal{N}^s_{a,\mu,b}})$ is a Banach space.

For $1 \le a < \infty$ and $s > 0$ we have the following equivalence [8]:

$$\|f\|_{N^{-s}_{a,\mu}} = \sup_{t>0} t^{\frac{s}{2}}\|T(t)f\|_{\mathcal{M}_a^\mu}, \tag{2.2}$$

where $\{T(t)\}$ is a fractional heat semigroup.

Definition 2.2. Let $1 \le a \le \infty$ and $0 \le \mu < 3$, the function space $\mathcal{EN}_a^\mu(\mathbb{R}^3)$ is defined as follows:

$$\mathcal{EN}_a^\mu(\mathbb{R}^3) = \{h \in \mathcal{S}', \|h\|_{\mathcal{EN}_a^\mu} < \infty\},$$
$$\|h(t)\|_{\mathcal{EN}_a^\mu} = \Big(\sup_{t>0}\mathbb{E}\|h(t)\|_{\mathcal{M}_a^\mu}\Big)^{\frac{1}{2}}.$$

We are now presenting the Hölder's inequality.

Lemma 2.1. *([16]) (***Hölder's inequality***)* *Let $0 \le \mu_1,\ \mu_2,\ \mu_3 < n$ and $1 \le r_1, r_2, r_3 < \infty$, such that $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$ and $\frac{\mu_3}{r_3} = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$. Then we have*

$$\|fg\|_{\mathcal{M}^{\mu_3}_{r_3}} \le \|f\|_{\mathcal{M}^{\mu_1}_{r_1}}\|g\|_{\mathcal{M}^{\mu_2}_{r_2}}. \tag{2.3}$$

Lemma 2.2. *([16]) Let $1 \le r_1 \le r_2 \le \infty$ and $\frac{d-\mu_1}{r_1} \le \frac{d-\mu_2}{r_2}$. Then*

$$\mathcal{M}^{\mu_2}_{r_2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^{\mu_1}_{r_1}(\mathbb{R}^d).$$

By combining [25, Lemma 3] with [8, Lemma 2.3], we can derive the subsequent lemma.

Lemma 2.3. *Let $1 \le q_1 \le q_2 \le \infty$, $0 \le \mu < d$ and $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{N}\cup\{0\})^2$. If $f \in \mathcal{S}'(\mathbb{R}^d)$, then there exists a constant C depending only on d such that*

$$\|e^{-t(-\Delta)^\beta} f\ \|_{\mathcal{M}^\mu_{a_2}} \le Ct^{-\frac{1}{2\beta}(\frac{d-\mu}{q_1} - \frac{d-\mu}{q_2})}\|f\|_{\mathcal{M}^\mu_{a_1}}.$$

$$\|\partial^\gamma e^{-t(-\Delta)^\beta} f\|_{\mathcal{M}^\mu_{a_2}} \le Ct^{-\frac{|\gamma|}{2\beta} - \frac{1}{2\beta}(\frac{d-\mu}{q_1} - \frac{d-\mu}{q_2})}\|f\|_{\mathcal{M}^\mu_{a_1}}.$$

2.1. The Mainardi function

Letting $\alpha \in (0,1)$, the Mainardi function $\mathcal{U}_\alpha$ is a mapping from $\mathbb{C}$ to $\mathbb{C}$ given by

$$\mathcal{U}_\alpha(x) := \mathcal{G}_{-\alpha,1-\alpha}(-x) \ , \ x \in \mathbb{C}, \tag{2.4}$$

where $\mathcal{G}_{\theta,\eta}(x)$ is the Wright function defined by the following convergent complex series in the complex plane:

$$\mathcal{G}_{\theta,\eta}(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(\theta n + \eta)} \quad \forall \theta > -1 \, , \, \eta \in \mathbb{C}.$$

The $\mathcal{U}_\alpha$ function is introduced by Mainardi in his book [6] as a specific instance of Wright's function.

The following proposition gives two important properties of the Mainardi function.

Proposition 2.1. *For $-1 < q < \infty$, we have:*

- $\mathcal{U}_\alpha(x) \geq 0 \quad$ *for all* $\quad x \geq 0.$
- $\displaystyle\int_0^\infty x^q \mathcal{U}_\alpha(x)dx = \frac{\Gamma(q+1)}{\Gamma(\alpha q + 1)}.$

2.2. The Mittag-Leffler operators

This subsection is devoted to introducing Mittag-Leffler operators. Let $\mathcal{Y}$ be a Banach space and $-\mathbb{J}$: $D(\mathbb{J}) \subset \mathcal{Y} \to \mathcal{Y}$ be the infinitesimal generator of a semigroup $\{\mathcal{T}(t) : t \geq 0\}$. For any $\alpha \in (0,1)$, we define the Mittag-Leffler families $\{\mathcal{Q}_\alpha(-t^\alpha \mathbb{J}) : t \geq 0\}$ and $\{\mathcal{Q}_{\alpha,\alpha}(-t^\alpha \mathbb{J}) : t \geq 0\}$ as follows:

$$\mathcal{Q}_\alpha(-t^\alpha \mathbb{J}) = \int_0^\infty \mathcal{U}_\alpha(\tau)\mathcal{T}(\tau t^\alpha)d\tau, \tag{2.5}$$

and

$$\mathcal{Q}_{\alpha,\alpha}(-t^\alpha \mathbb{J}) = \int_0^\infty \alpha\tau\mathcal{U}_\alpha(\tau)\mathcal{T}(\tau t^\alpha)d\tau. \tag{2.6}$$

It is remarkable that the Mainardi functions link the classical theories and the abstract fractional. More detail can be found in [3].

Remark 2.1.
- Note that in the definition of the Mittag-Leffler families, it is important to specify that $\alpha < 1$, because if not, it would lose its coherence. Although [30, Theorem 5] provides sufficient information on these families for our objective, it is essential to realize that they do not form semigroups.
- It is important to note that for every fixed $x \in \mathcal{Y}$, the function $t \mapsto \mathcal{Q}_{\alpha,\alpha}(-t^\alpha \mathbb{J})x$ stays continuous (and analytical when $\{\mathcal{Q}(t) : t \geq 0\}$ forms an analytical semigroup) and fulfills:

$$^{c}\mathrm{D}_t^\alpha \mathcal{Q}_\alpha(-t^\alpha \mathbb{J})x = -\mathbb{J}\mathcal{Q}_\alpha(-t^\alpha \mathbb{J})x, \qquad t > 0.$$

3. Deterministic case

This section is devoted to proving the existence and uniqueness of mild solutions to the Eq. (1.2).

Put

$$\mathcal{X} := \mathcal{M}_a^{\mu} \cap \mathrm{N}_{a,\mu}^{\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}-2\beta\right)} \cap \mathrm{N}_{a,\mu}^{\frac{\alpha}{2\beta}\left(\frac{3}{\alpha}+\frac{3-\mu}{a}-2\beta\right)}.$$

Notation: Let X_1 , X_2 be Banach spaces. We denote $\|\cdot\|_{X_1\cap X_2} := \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$.

Theorem 3.1. *Let β, μ, a satisfy $0 \le \mu < 3$ and $\frac{1}{2} \le \beta < 1$. Then, there exists a positive constant $\delta = \delta(\alpha, \mu, a)$, such that for the initial velocity $(v_0, b_0) \in \mathcal{M}_a^{\mu}(\mathbb{R}^3)$ satisfying $div\, v_0 = div\, b_0 = 0$ and*

$$\|(v_0, b_0)\|_{\mathcal{M}_a^{\mu}} \le \delta.$$

The equations (1.2) have a unique mild solution (v, b) satisfying

$$\|v(t), b(t)\|_{\mathcal{X}} \le 2\|(v_0, b_0)\|_{\mathcal{M}_a^{\mu}}.$$

In order to solve the equations (1.2), we consider the following equivalent integral equation coming from Duhamel's principle

$$\begin{cases} v = \mathcal{Q}_\alpha(-t^\alpha(-\Delta)^\beta)v_0 - \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot \mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes v - b\otimes b)(\cdot, s')ds' \\ \qquad\qquad := \mathcal{A}_1^\beta(v,b), \\ b = \mathcal{Q}_\alpha(-t^\alpha(-\Delta)^\beta)b_0 - \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot \mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes b - b\otimes v)(\cdot, s')ds' \\ \qquad - \displaystyle\int_0^t (t-s')^{\alpha-1}\nabla\cdot \mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(\cdot, s')ds' := \mathcal{A}_2^\beta(v,b), \end{cases} \tag{3.1}$$

where $\mathbb{P} = Id - \nabla\Delta^{-1}div$ is the Leray-Hopf projector, which is a pseudo differential operator of order 0.

We will first estimate the Mittag-Leffler families.

Proposition 3.1. *Let $1 < a \le p$, $\alpha \in (0,1)$ and $0 < \beta < 1$ Then, there exists a positive constant C_1, C_2 such that*

$$\|\nabla\mathcal{Q}_{\alpha,\alpha}\big(\tau^\alpha(-\Delta)^\beta\big)f\|_{\mathcal{M}_a^\mu} \le C_1\tau^{-\frac{\alpha}{2\beta}(\frac{1}{\alpha}+(\frac{3-\mu}{q}-\frac{3-\mu}{a}))}\|f\|_{\mathcal{M}_q^\mu} \tag{3.2}$$

$$\|\mathcal{Q}_{\alpha}\big(\tau^\alpha(-\Delta)^\beta\big)f\|_{\mathcal{M}_a^\mu} \le C_2\tau^{-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}\|f\|_{\mathcal{M}_q^\mu}. \tag{3.3}$$

Proof: Let us show the first estimation (3.2). By using the Lemma 2.3 we obtain

$$\begin{aligned} \|\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big(\tau^\alpha(-\Delta)^\beta\big)f\|_{\mathcal{M}_a^\mu} &\le \int_0^\infty \|\nabla\alpha s\mathcal{U}_\alpha(s)S(s\tau^\alpha)(-\Delta)^\beta\|_{\mathcal{M}_a^\mu}ds \\ &\le C\tau^{-\frac{1}{2\beta}-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}\Big(\int_0^\infty s\mathcal{U}_\alpha(s)s^{-\frac{1}{2\beta}-\frac{1}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}ds\Big)\|f\|_{\mathcal{M}_q^\mu} \\ &\le C\tau^{-\frac{\alpha}{2\beta}(\frac{1}{\alpha}+(\frac{3-\mu}{q}-\frac{3-\mu}{a}))}\Big(\int_0^\infty \mathcal{U}_\alpha(s)s^{\frac{1}{2\beta}\left(2\beta-1-(\frac{3-\mu}{q}-\frac{3-\mu}{a})\right)}ds\Big)\|f\|_{\mathcal{M}_q^\mu}. \end{aligned}$$

Since $\frac{1}{2\beta}\big(2\beta-1-(\frac{3-\mu}{q}-\frac{3-\mu}{a})\big)>-1$, it follows from Proposition 2.1 that

$$\begin{aligned}&\|\mathcal{Q}_{\alpha,\alpha}\big(\tau^{\alpha}(-\Delta)^{\beta}\big)f\|_{\mathcal{M}^{\mu}_{a}}\\ \le& C\tau^{-\frac{\alpha}{2\beta}(\frac{1}{\alpha}+(\frac{3-\mu}{q}-\frac{3-\mu}{a}))}\frac{\Gamma\Big(1+2\beta-1-(\frac{3-\mu}{q}-\frac{3-\mu}{a})\Big)}{\Gamma\Big[1+\frac{\alpha}{2\beta}\big(2\beta-1-(\frac{3-\mu}{q}-\frac{3-\mu}{a})\big)\Big]}\|f\|_{\mathcal{M}^{\mu}_{q}}\\ \le& C_1\tau^{-\frac{\alpha}{2\beta}(\frac{1}{\alpha}+(\frac{3-\mu}{q}-\frac{3-\mu}{a}))}\|f\|_{\mathcal{M}^{\mu}_{a}}.\end{aligned}$$

We are now in the position to prove the estimation (3.3)

$$\begin{aligned}\|\mathcal{Q}_{\alpha}\big(\tau^{\alpha}(-\Delta)^{\beta}\big)f\|_{\mathcal{M}^{\mu}_{a}}\le&\int_0^{\infty}\|\mathcal{U}_{\alpha}(\tau)S(s\tau^{\alpha})\|_{\mathcal{M}^{\mu}_{a}}ds\\ \le& C\tau^{-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}\Big(\int_0^{\infty}\mathcal{U}_{\alpha}(s)s^{-\frac{1}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}ds\Big)\|f\|_{\mathcal{M}^{\mu}_{q}}.\end{aligned}$$

Since $-\frac{1}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})>-1$, it follows from Proposition 2.1 that

$$\begin{aligned}\|\mathcal{Q}_{\alpha}\big(\tau^{\alpha}(-\Delta)^{\beta}\big)f\|_{\mathcal{M}^{\mu}_{a}}\le& C\tau^{-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}\frac{\Gamma\Big(1+-\frac{1}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})\Big)}{\Gamma\Big[1-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})\Big]}\|f\|_{\mathcal{M}^{\mu}_{q}}\\ \le& C_2\tau^{-\frac{\alpha}{2\beta}(\frac{3-\mu}{q}-\frac{3-\mu}{a})}\|f\|_{\mathcal{M}^{\mu}_{a}}.\end{aligned}$$

Before proceeding to demonstrate Theorem 3.1, let's initially establish the bilinear estimate of equation (3.1) as presented in the following Lemma 3.1.

Lemma 3.1. *Let β,μ,a satisfy $a>1,\quad 0\le\mu<3\quad$ and $\quad 0<\beta<1$. Then, there exists a positive constant C_3 such that*

$$\|\mathbb{B}(v,b)\|_{\mathrm{N}^{\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}-2\beta\right)}_{a,\mu}}\le C_3\|v\|_{\mathcal{M}^{\mu}_{a}}\|b\|_{\mathcal{M}^{\mu}_{a}}. \tag{3.4}$$

where

$$\mathbb{B}(v,b):=\int_0^t(t-s')^{\alpha-1}\mathbb{P}\nabla\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big](v\otimes b)(s',x)ds'.$$

Proof: Since $\mathbb{P}$ is bounded in $\mathcal{M}^{\mu}_{q}(\mathbb{R}^3)$ [25], it follows from Proposition 3.1 and Hölder's inequality, we have

$$\begin{aligned}\|\mathbb{B}(v,b)\|_{\mathcal{M}^{\mu}_{a}}\le&\int_0^t(t-s')^{\alpha-1}\|\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big]\left(v\otimes b\right)\right)(s',x)\|_{\mathcal{M}^{\mu}_{a}}ds'\\ \le& C\int_0^t(t-s')^{\alpha-1-\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}\right)}\|(v\otimes b)(s',x)\|_{\mathcal{M}^{\mu}_{\frac{a}{2}}}ds'\\ \le& C\|v\|_{\mathcal{M}^{\mu}_{a}}\|b\|_{\mathcal{M}^{\mu}_{a}}\int_0^t(t-s')^{-\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}-2\beta\right)-1}ds'.\end{aligned}$$

Multiplying $t^{\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}-2\beta\right)}$ on both sides of the above two inequalities, we get

$$\|\mathbb{B}(v,b)\|_{\mathrm{N}^{\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}-2\beta\right)}_{a,\mu}}\le C_3\|v\|_{\mathcal{M}^{\mu}_{a}}\|b\|_{\mathcal{M}^{\mu}_{a}}.$$

Remark 3.1. By following a similar argument used in Lemma 3.1 we can show

$$\|\mathcal{Q}_\alpha\big(\tau^\alpha(-\Delta)^\beta\big)a\|_{\mathcal{M}^\mu_a} \leq C_4\|a_0\|_{\mathcal{M}^\mu_a}. \tag{3.5}$$

Lemma 3.2. *Let $\beta, \mu,$ a satisfy $a > 1, \quad 0 \leq \mu < 3 \quad$ and $\quad 0 < \beta < 1$. Then, there exists a positive constant C_5 such that*

$$\Big\|\int_0^t (t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(s',x)ds'\Big\|_{\mathrm{N}^s_{a,\mu}} \leq C_5\|b\|^2_{\mathcal{M}^\mu_a}, \tag{3.6}$$

where $s := \frac{\alpha}{2\beta}\big(\frac{3}{\alpha} + \frac{3-\mu}{a} - 2\beta\big)$.

Proof: Using the Proposition 3.1 and Hölder's inequality, we have

$$\begin{aligned}
&\Big\|\int_0^t (t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(s',x)ds'\Big\|_{\mathcal{M}^\mu_a}\\
&\qquad\leq \int_0^t \Big\|(t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(s',x)\Big\|_{\mathcal{M}^\mu_a} ds'\\
&\qquad\leq C\int_0^t (t-s')^{\alpha-1-\frac{2}{2\beta}-\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}\right)}\|(b\otimes b)(s',x)\|_{\mathcal{M}^\mu_{\frac{a}{2}}}ds'\\
&\qquad\leq C\|b\|_{\mathcal{M}^\mu_a}\|b\|_{\mathcal{M}^\mu_a}\int_0^t (t-s')^{-\frac{\alpha}{2\beta}\left(\frac{3}{\alpha}+\frac{3-\mu}{a}-2\beta\right)-1}ds'.
\end{aligned}$$

Multiplying $t^{\frac{\alpha}{2\beta}\left(\frac{3}{\alpha}+\frac{3-\mu}{a}-2\beta\right)}$ on both sides of the above two inequalities, we get

$$\Big\|\int_0^t (t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(s',x)ds'\Big\|_{\mathrm{N}^s_{a,\mu}} \leq C_5\|b\|^2_{\mathcal{M}^\mu_a}.$$

Proof of Theorem 3.1. It is simple to verify that the indices β, μ and a provided in Theorem 3.1 satisfy the assumptions of Lemma 3.1 and Lemma 2.3. Suppose $(v_0, b_0) \in \mathcal{M}^\mu_a(\mathbb{R}^3)$ with divergence free. Then, Let us introduce the map Φ and the complete metric space $(\mathbf{Y}, d)$, defined as follows:

$$\begin{aligned}
\mathbf{Y} &:= \big\{(v,b)\in\mathcal{X}(\mathbb{R}^3)^3, \|(v,b)\|_{\mathcal{X}} \leq 2\|(v_0,b_0)\|_{\mathcal{M}^\mu_a}\big\},\\
d(\vartheta_1,\vartheta_2) &:= \|\vartheta_1-\vartheta_2\|_{\mathcal{X}},\\
\Theta(v,b) &:= (\mathcal{A}^\alpha_1(v,b), \mathcal{A}^\alpha_2(v,b)).
\end{aligned}$$

Applying the inequalities (3.5) and (3.6), then for all $(v,b)\in\mathbf{Y}$, we have

$$\|\mathcal{A}^\beta_1(v,b)\|_{\mathcal{X}} \leq C_4\|v_0\|_{\mathcal{M}^\mu_a} + C_3(\|v\|^2_{\mathcal{X}} + \|b\|^2_{\mathcal{X}}).$$

Similary, we obtain

$$\|\mathcal{A}^\beta_2(v,b)\|_{\mathcal{X}} \leq C_4\|b_0\|_{\mathcal{M}^\mu_a} + 2C_3(\|v\|_{\mathcal{X}} + \|b\|_{\mathcal{X}}) + C_5\|b\|^2_{\mathcal{X}}.$$

Then, we can see

$$\begin{aligned}
\|\Theta(v,b)\|_{\mathcal{X}} &\leq C_4\|(v_0,b_0)\|_{\mathcal{M}^\mu_a} + C_3\|(v,b)\|^2_{\mathcal{X}} + C_5\|(v,b)\|^2_{\mathcal{X}}\\
&\leq C_4\|(v_0,b_0)\|_{\mathcal{M}^\mu_a} + C_6\|(v_0,b_0)\|_{\mathcal{M}^\mu_a}.
\end{aligned}$$

Then, ther exists a constant $\epsilon_1 = \max(C_4, C_6)$ such that

$$\|\Theta(v,b)\|_{\mathcal{X}} \leq \epsilon_1 \|(v_0,b_0)\|_{\mathcal{M}^{\mu}_a}. \tag{3.7}$$

On the other hand, for any $(v_1,b_1),(v_2,b_2) \in \mathrm{Y}$, ther exists a constant ϵ_2 such that

$$\begin{aligned}
&\|\mathcal{A}^{\alpha}_1(v_1,b_1) - \mathcal{A}^{\alpha}_1(v_2,b_2)\|_{\mathcal{X}}\\
\leq&\|\mathbb{B}(v_1,v_1) - \mathbb{B}(v_2,v_2)\|_{\mathcal{X}} + \|\mathbb{B}(b_1,b_1) - \mathbb{B}(b_2,b_2)\|_{\mathcal{X}} + C_5\|b_1-b_2\|^2_{\mathcal{X}}\\
\leq&\|\mathbb{B}(v_1,v_1-v_2) - \mathbb{B}(v_1-v_2,v_2)\|_{\mathcal{X}} + \|\mathbb{B}(b_1-b_2,b_2) - \mathbb{B}(b_1,b_1-b_2)\|_{\mathcal{X}} + \|b_1-b_2\|^2_{\mathcal{X}}\\
\leq&\epsilon_2\Big\{(\|v_1\|_{\mathcal{X}} + \|v_2\|_{\mathcal{X}})\|v_1-v_2\|_{\mathcal{X}} + (\|b_1\|_{\mathcal{X}} + \|b_2\|_{\mathcal{X}} + \|b_1-b_2\|_{\mathcal{X}})\|b_1-b_2\|_{\mathcal{X}}\Big\}\\
\leq&\epsilon_2\Big\{(\|(v_1,b_1)\|_{\mathcal{X}} + \|(v_2,b_2)\|_{\mathcal{X}})(\|v_1-v_2\|_{\mathcal{X}} + \|b_1-b_2\|_{\mathcal{X}})\Big\}\\
\leq&2\epsilon_2\|(v_0,b_0)\|_{\mathcal{M}^{\mu}_a}\Big\{\|v_1-v_2\|_{\mathcal{X}} + \|b_1-b_2\|_{\mathcal{X}}\Big\}.
\end{aligned}$$

Therefore,

$$\|\Theta(v_1,b_1) - \Theta(v_2,b_2)\|_{\mathcal{X}} \leq 2\epsilon_2\|(v_0,b_0)\|_{\mathcal{M}^{\mu}_a}\Big\{\|v_1-v_2\|_{\mathcal{X}} + \|b_1-b_2\|_{\mathcal{X}}\Big\}. \tag{3.8}$$

Now, let us assume that initial velocity $(v_0,b_0) \in \mathrm{N}^s_{a,\mu}(\mathbb{R}^3)$ satisfies

$$\|(v_0,b_0)\|_{\mathcal{M}^{\mu}_a} \leq \min\{\frac{1}{4\epsilon_2,}, \frac{1}{4\epsilon_1}\},$$

we get from (3.7) and (3.8) that

$$\begin{aligned}
&\|\Theta(v,b)\|_{\mathcal{X}} \leq 2\|(v_0,b_0)\|_{\mathcal{M}^{\mu}_a},\\
&\|\Theta(v_1,b_1) - \Theta(v_2,b_2)\|_{\mathcal{X}} \leq \frac{1}{2}\Big\{\|v_1-v_2\|_{\mathcal{X}} + \|b_1-b_2\|_{\mathcal{X}}\Big\}.
\end{aligned}$$

Thus, applying the contraction mapping principle, we can conclude that there exists a unique solution $(v,b) \in \mathbf{Y}$ that satisfies (3.1) for all $t > 0$. This completes the proof of Theorem 3.1.

4. Stochastic case

In this section, we show our main result concerning the mild solution for time-fractional hall-magneto-hydrodynamics stochastic equations.

We begin by stating the assumptions imposed on the external forcing terms in our equations (1.3). Let $G : [0,+\infty) \times \mathcal{M}^{\mu}_p(\mathbb{R}^2) \to \mathcal{M}^{\mu}_p(\mathbb{R}^2)$ and assume that :

(H_1) For each $\psi \in \mathcal{M}^{\mu}_a(\mathbb{R}^2)$, the mappings $t \in [0,+\infty) \to G(t,\psi)$ are measurable.

(H_2) $G(\cdot,0) = 0$ (for simplicity).

(H_3) There exist positive constants A_G, such that, for all $t \in [0,\infty)$ and $\psi,\phi \in \mathcal{M}^{\mu}_a(\mathbb{R}^2)$

$$\|G(t,\psi) - G(t,\phi)\|^2_{\mathcal{M}^{\mu}_a(\mathbb{R}^2)} \leq A_G\|\psi-\phi\|^2_{\mathcal{M}^{\mu}_a(\mathbb{R}^2)}.$$

We can now establish a mild solution to equations (1.3).

Theorem 4.1. *Under the conditions of Theorem 3.1, let* (H_1)*-*(H_3) *hold and* $\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\Big)$ *be a filtered probability basis. Then there exists a positive constant* η *that if* (v_0, b_0) *is* $\mathcal{F}_0$ *measurable and satisfies* $\|(v_0,b_0)\|_{\mathcal{EN}^\mu_a} \leq \eta$*, the equation* (1.3) *admits a unique global mild solution* (v,b)*, such that* $\|(v,b)\|_{\mathcal{EN}^\mu_a \cap \mathrm{N}^{2\alpha}_{a,\mu}} \leq \|(v_0,b_0)\|_{\mathcal{EN}^\mu_a}$.

To solve the equations (1.3), we consider the following equivalent integral equation coming from Duhamel's principle

$$\begin{cases} v = \mathcal{Q}_\alpha(-t^\alpha(-\Delta)^\beta)v_0 - \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes v - b\otimes b)(\cdot,s')ds' \\ \qquad\qquad + \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]G_1(s,b_s)dW(s) := \mathcal{D}^\beta_1(v,b). \\ b = \mathcal{Q}_\alpha(-t^\alpha(-\Delta)^\beta)b_0 - \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes b - b\otimes v)(\cdot,s')ds' \\ \qquad\qquad - \displaystyle\int_0^t (t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]((\nabla\times b)\times b)(\cdot,s')ds' \\ \qquad\qquad + \displaystyle\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]G_2(s,b_s)dW(s) := \mathcal{D}^\beta_2(v,b). \end{cases} \tag{4.1}$$

Proof of Theorem 4.1. We begin by estimating the solution $\mathcal{D}^\beta_1(v,b)$ which is defined in (4.1). For $t>0$ we have

$$\begin{aligned} &\mathbb{E}\|\mathcal{D}^\beta_1(v,b)\|^2_{\mathcal{M}^\mu_a} \\ \leq\ & \mathbb{E}\|\mathcal{Q}_\alpha(-t^\alpha(-\Delta)^\beta)v_0\|^2_{\mathcal{M}^\mu_a} \\ &+\mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes v - b\otimes b)(\cdot,s')ds'\|^2_{\mathcal{M}^\mu_a} \\ &+\mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big]G_1(s,b_s)dW(s)\|^2_{\mathcal{M}^\mu_a} \\ \leq\ & \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3. \end{aligned} \tag{4.2}$$

For estimate $\mathcal{K}_1$, by using Proposition 3.1 it is simple to see that

$$\begin{aligned} \mathcal{K}_1 &\leq C_7\mathbb{E}\|v_0\|^2_{\mathcal{M}^\mu_a} \\ &\leq C_7 \sup\nolimits_{t\in\mathbb{R}^+}\mathbb{E}\|v_0\|^2_{\mathcal{M}^\mu_a} \\ &\leq C_7\|v_0\|_{\mathcal{EN}^\mu_a}. \end{aligned} \tag{4.3}$$

For $\mathcal{K}_2$, by Proposition 3.1, (H_1)-(H_3) and **Fubini's** theorem, we obtain

$$\begin{aligned} \mathcal{K}_2 &\leq \mathbb{E}\Big(\int_0^t (t-s)^{2\alpha-2}\|\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^\alpha(-\Delta)^\beta\big](v\otimes v-b\otimes b)(s',x)\|_{\mathcal{M}^\mu_a}ds'\Big) \\ &\leq C_8\mathbb{E}\Big(\int_0^t (t-s)^{2\alpha-2-\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}\right)}\|(v\otimes v-b\otimes b)(s',x)\|_{\mathcal{M}^\mu_{\frac{a}{2}}}\,ds'\Big) \\ &\leq C'_8 t^{2\alpha(1-\frac{1}{2\alpha}-\frac{1}{4\beta}(\frac{1}{\alpha}+\frac{3-\mu}{a}))}\int_0^t\Big(\mathbb{E}\|(v\otimes v-b\otimes b)(s',x)\|_{\mathcal{M}^\mu_{\frac{a}{2}}}\,ds\Big) \\ &\leq C'_8 t^{2\alpha(1-\frac{1}{2\alpha}-\frac{1}{4\beta}(\frac{1}{\alpha}+\frac{3-\mu}{a}))}\|v\|^2_{\mathcal{EN}^\mu_a}\|b\|^2_{\mathcal{EN}^\mu_a}. \end{aligned} \tag{4.4}$$

For $\mathcal{K}_3$, by Proposition 3.1, Itô's isometry and (H_1)-(H_3), we have

$$\begin{aligned}
\mathcal{K}_3 &= \mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big]G_1(s,b_s)dW(s)\|^2_{\mathcal{M}^{\mu}_a} \\
&\leq C_9\mathbb{E}\int_0^t (t-s)^{2\alpha-2}\|\nabla\mathcal{Q}_{\alpha,\alpha}\big[(t-s)^{\alpha}(-\Delta)^{\beta}\big]G_1\left(s,\theta_s\right)\|^2_{\mathcal{M}^{\mu}_P}ds \\
&\leq C_9A_{G_1}\int_0^t (t-s)^{2\alpha-2}\|\theta_s\|^2_{\mathcal{M}^{\mu}_P}ds \\
&\leq C_9A_{G_1}\sup_{t\in\mathbb{R}^+} t^{2\alpha}\|\theta_s\|^2_{\mathcal{M}^{\mu}_P} \\
&\leq C_9A_{G_1}\|\theta_s\|^2_{\mathrm{N}^{2\alpha}_{a,\mu}}.
\end{aligned} \tag{4.5}$$

In the ensuing part, our aim is to evaluate the solution $\mathcal{D}^{\beta}_2(v,b)$ as described in equation (4.1). For $t>0$ we have

$$\begin{aligned}
&\mathbb{E}\|\mathcal{D}^{\beta}_2(v,b)\|^2_{\mathcal{M}^{\mu}_a} \\
\leq\ &\mathbb{E}\|\mathcal{Q}_{\alpha}(-t^{\alpha}(-\Delta)^{\beta})b_0\|^2_{\mathcal{M}^{\mu}_a} \\
&+\mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big](v\otimes b-b\otimes v)(\cdot,s')ds'\|^2_{\mathcal{M}^{\mu}_a} \\
&\mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big]((\nabla\times b)\times b)(\cdot,s')ds'\|^2_{\mathcal{M}^{\mu}_a} \\
&+\mathbb{E}\|\int_0^t (t-s')^{\alpha-1}\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big]G_2(s,b_s)dW(s)\|^2_{\mathcal{M}^{\mu}_a} \\
\leq\ &\mathcal{K}'_1+\mathcal{K}'_2+\mathcal{K}'_3+\mathcal{K}'_4.
\end{aligned} \tag{4.6}$$

The same approach used to illustrate $\mathcal{K}_1, \mathcal{K}_2and\mathcal{K}_3$ is applicable here.

$$\mathcal{K}'_1 \leq C_{10}\|b_0\|_{\mathcal{EN}^{\mu}_a}.$$

$$\mathcal{K}'_4 \leq C_{11}A_{G_2}\|\theta_s\|^2_{\mathrm{N}^{2\alpha}_{a,\mu}}.$$

$$\mathcal{K}'_2 \leq C_{12}t^{2\alpha(1-\frac{1}{2\alpha}-\frac{1}{4\beta}(\frac{1}{\alpha}+\frac{3-\mu}{a}))}\|v\|^2_{\mathcal{EN}^{\mu}_a}\|b\|^2_{\mathcal{EN}^{\mu}_a}.$$

It remains to estimate $\mathcal{K}'_3$. By Proposition 3.1, (H_1)-(H_3) and **Fubini's** theorem, we obtain

$$\begin{aligned}
\mathcal{K}_3 &\leq \mathbb{E}\Big(\int_0^t (t-s)^{2\alpha-2}\|\mathbb{P}\nabla\cdot\mathcal{Q}_{\alpha,\alpha}\big[(t-s')^{\alpha}(-\Delta)^{\beta}\big]((\nabla\times b)\times b)(s',x)\|_{\mathcal{M}^{\mu}_a}ds'\Big) \\
&\leq C_{13}\mathbb{E}\Big(\int_0^t (t-s)^{2\alpha-2-\frac{2}{2\beta}-\frac{\alpha}{2\beta}\left(\frac{1}{\alpha}+\frac{3-\mu}{a}\right)}\|(b\otimes b)(s',x)\|_{\mathcal{M}^{\mu}_{\frac{a}{2}}}ds'\Big) \\
&\leq C'_{13}t^{2\alpha(1-\frac{1}{2\alpha}-\frac{1}{4\beta}(\frac{3}{\alpha}+\frac{3-\mu}{a}))}\int_0^t\Big(\mathbb{E}\|(b\otimes b)(s',x)\|_{\mathcal{M}^{\mu}_{\frac{a}{2}}}ds\Big) \\
&\leq C'_{13}t^{2\alpha(1-\frac{1}{2\alpha}-\frac{1}{4\beta}(\frac{3}{\alpha}+\frac{3-\mu}{a}))}\|b\|^2_{\mathcal{EN}^{\mu}_a}.
\end{aligned}$$

Following the systematic steps detailed in the proof of Theorem 3.1,we successfully conclude the proof for Theorem 4.1.

Conflict of interest

The authors declare no conflict of interest.

Data Availability

We do not involve any data in our work.

References

[1] A. Heibig and L. I. Palade, *Well posedness of a linearized fractional derivative fluid model*, Journal of Mathematical Analysis and Applications, 2011, 380(1), 188–203.

[2] M. Caputo, *Vibrations of an infinite viscoelastic layer with a dissipative memory*, The Journal of the Acoustical Society of America, 1974, 56(3), 897–904.

[3] D. Chae and J. Lee, *Global well-posedness in the super-critical dissipative quasi-geostrophic equations*, Communications in Mathematical Physics, 2003, 233, 297–311.

[4] H. Khaider, A. Azanzal, and A. Raji, *Well-posedness of solutions for the 2D stochastic quasi-geostrophic equation in critical Fourier-Besov-Morrey spaces*, Electronic Journal of Differential Equations, 2024(01-12), 74–10.

[5] D. Yuan, W. Jiang, Y. Hou, J. Xiao, X. Ling, and C. Xing, *Fractional derivative viscoelastic response of high-viscosity modified asphalt*, Construction and Building Materials, 2022, 350, 128915.

[6] F. Mainardi, *On the initial value problem for the fractional diffusion-wave equation*, Series on Advances in Mathematics for Applied Sciences, 1994, 23, 246–251.

[7] J. Gulgowski and T. P. Stefański, *On applications of fractional derivatives in electromagnetic theory,* *Proceedings of the 2020 23rd International Microwave and Radar Conference (MIKON), 2020, 13–17. DOI: 10.1109/MIKON.

[8] J. Sun, Z. Fu, Y. Yin, and M. Yang, *Global existence and Gevrey regularity to the Navier-Stokes-Nernst-Planck-Poisson system in critical Besov-Morrey spaces*, Journal of Discrete and Continuous Dynamical Systems-B, 2020, 26(6), 3409–3425.

[9] J. T. Machado, V. Kiryakova, and F. Mainardi, *Recent history of fractional calculus,* Communications in Nonlinear Science and Numerical Simulation, 2011, 16(3), 1140–1153.

[10] H. Bahouri, J. Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Springer Science and Business Media, Berlin, 2011.

[11] H. Khaider, A. Azanzal, and A. Raji, *Global well-posedness and analyticity for the fractional stochastic Hall-magnetohydrodynamics system in the Besov-Morrey spaces*, Arabian Journal of Mathematics, 2024, 1–12.

[12] H. Tang, *On the pathwise solutions to the Camassa-Holm equation with multiplicative noise,* SIAM Journal on Mathematical Analysis , 2018, 50, 1322–1366.

[13] I. Khan, M. Saqib, and F. Ali, *Application of time-fractional derivatives with non-singular kernel to the generalized convective flow of Casson fluid in a microchannel with constant walls temperature*, The European Physical Journal Special Topics, 2017, 226, 3791–3802.

[14] M. Krasnoschok and N. Vasylyeva, *On a nonclassical fractional boundary-value problem for the Laplace operator*, Journal of Differential Equations, 2014, 257(6), 1814–1839.

[15] S. Kumar and N. Sukavanam, *Approximate controllability of fractional order semilinear systems with bounded delay*, Journal of Differential Equations, 2012, 252(11), 6163–6174.

[16] L. C. F. Ferreira and L. S. M. Lima, *Self-similar solutions for active scalar equations in Fourier-Besov-Morrey spaces*, Monatshefte für Mathematik, 2014, 175, 491–509.

[17] H. Khaider, F. Ouidirne, A. Azanzal, and C. Allalou, *Well-posedness for the stochastic incompressible Hall-magnetohydrodynamic system*, Gulf Journal of Mathematics, 2024, 17(1), 191–204.

[18] M. Di Paola, A. Pirrotta, and A. Valenza, *Visco-elastic behavior through fractional calculus: an easier method for best fitting experimental results*, Mechanics of Materials, 2011, 43(12), 799–806.

[19] M. Sen, *Introduction to fractional-order operators and their engineering applications*, University of Notre Dame, Netherlands, 2014.

[20] M. Shinbrot, *Fractional derivatives of solutions of the Navier-Stokes equations*, Archive for Rational Mechanics and Analysis, 1971, 40(2), 139–154.

[21] N. Engheta, *On Fractional Calculus and Fractional Multipoles in Electromagnetism*, 1996.

[22] P. Chen and Y. Li, *Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions*, Zeitschrift für Angewandte Mathematik und Physik, 2014, 65, 711–728.

[23] P. M. D. Carvalho Neto, *Fractional Differential Equations: a Novel Study of Local and Global Solutions in Banach Spaces (Doctoral dissertation, Universidade de Sao Paulo)*, 2013.

[24] S. D. Eidelman and A. N. Kochubei, *Cauchy problem for fractional diffusion equations*, Journal of Differential Equations, 2004, 199(2), 211–255.

[25] X. Sun, J. Liu, and J. Zhang,Global Well-Posedness for the Fractional Navier-Stokes-Coriolis Equations in Function Spaces Characterized by Semigroups. Preprint. https://doi. org/10.20944/preprints202111, 408, v1.2021.

[26] K. Hassan, A. Achraf, A. Chakir, M. Said, *Well-posedness and analyticity for quasigeostrophic equation in the Besov-Morrey spaces characterized by semigroup*, Filomat, 38 (2024), 6237–6244.

[27] Y. Chen, J. Dong, and H. Ni, *varepsilon-Strong simulation of fractional Brownian motion and related stochastic differential equations*, Mathematics of Operations Research, 2021, 46(2), 559–594.

[28] Y. Li, Y. Wang, and W. Deng, *Galerkin finite element approximations for stochastic space-time fractional wave equations*, SIAM Journal on Numerical Analysis, 2017, 55(6), 3173–3202.

[29] Y. Xiao, *Packing measure of the sample paths of fractional Brownian motion*, Transactions of the American Mathematical Society, 1996, 348(8), 3193–3213.

[30] H. Khaider, A. Azanzal, R. Abderrahmane, and M. Said, *Mild solution for the time fractional magneto-hydrodynamics system*, Analysis and Mathematical Physics, 2024, 14(2), 14.