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An Inertial Tseng's Extragradient Method for Approximating Solution of Split Problems in Banach Spaces

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Abstract In this paper, we introduce a new inertial type algorithm with a self-adaptive step size for approximating a common element of the set of solutions of split common null point and pseudomonotone variational inequality problem as well as the set of common fixed point of a finite family of quasi nonexpansive mappings in uniformly smooth and 2-uniformly convex real Banach space. The proposed algorithm is constructed in such a way that its convergence analysis does not require a prior estimate of the operator norm. We also give numerical examples to illustrate the performance of our algorithm. Our results generalize and improve many existing results in the literature.

Keywords Variational inequality problem, inertial Tseng's extragradient method, fixed point, Banach spaces

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1. Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and let $F : C \to E^*$ be a mapping. The problem of finding a point $x^* \in C$ such that

$$\langle y - x^*, Fx^* \rangle \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

is called a *variational inequality problem*. The set of solutions of variational inequality problem (1.1) is denoted by $VI(C, F)$. The study of variational inequality problem originates from solving minimization problems involving infinite-dimensional functions and calculus of variation (see, for example, [33] and reference therein). The concept of variational inequality problem was initially introduced by Hartman and Stampacchia [18] as a generalization of boundary value problems in 1966. Such problems are applicable in a wide range of applied sciences and mathematics. Later in 1967 Lions and Stampacchia [28] studied the existence and uniqueness of the solution. Since then, the theory of variational inequality problem has received

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much attention due to its wide applications in various areas of pure and applied sciences, such as optimal control, image recovery, resource allocations, networking, transportation, signal processing, game theory, operation research and so on (see, for example, [3,23,39] and references therein). The constraints can clearly be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of nonlinear operators has become an interesting area of research for many researchers working in the area of nonlinear operator theory (see, for example, [30,31] and the references contained in them). In view of this, many researchers in their quest to find solutions of variational inequality problems have proposed and analyzed various iterative approximation methods (see for example, [13, 20])in which most of them are based on projection methods. The simplest and earliest form of projection method is due to Goldstein [17], which is a natural extension of the gradient projected technique considered for solving optimization problems. A number of results on iterative methods proposed for approximating solutions of variational inequality problems are studied such that the operator F was often considered to be either strongly monotone or inverse strongly monotone (see, for instance [17, 26] and references therein) for convergence to be guaranteed. In order to relax the strong monotonicity condition imposed on the operator F, Korpelevich [25] proposed the following extragradient method in a finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda F(y_n)) \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\lambda \in (0, \frac{1}{L})$, F is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.2) converges weakly to a solution of problem (1.1). However, the extragradient method requires the computation at each step of the iteration process two projections onto an arbitrary closed and convex subset C of H. This might affect the efficiency of the extragradient method if the feasible set is not simple enough which might also increase the computational cost.

In order to overcome this barrier, several modifications of the extragradient method were proposed (see, for example [12,19,44] and references therein) for solving variational inequality problem (1.1). In particular, Tseng [44] proposed the following Tseng's extragradient method

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = y_n - \lambda(F(y_n) - F(x_n)) \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, F is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to a solution of problem (1.1) in a real Hilbert space. Another modification of the extragradient method was proposed by Censor et al. [12] as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ T_n = \{z \in H : \langle z - y_n, x_n - \lambda F(x_n) - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)), \quad \forall \ n \geq 0. \end{cases} \tag{1.4}$$

They modified the extragradient method (1.2) by replacing the second projection onto a closed and convex subset C with a projection onto the half space T_n. Algorithm (1.4) is therefore called subgradient extragradient method. Observe that, the set T_n is a half space, making algorithm (1.4) simpler to implement than algorithm (1.2). They proved that the sequence $\{x_n\}$ generated by algorithm (1.4) converges weakly to a solution of problem (1.1) in a real Hilbert space under some mild assumptions. Observe that all the methods mentioned above require a prior knowledge of the Lipschitz constant of the operator F as an input parameter which is very difficult to estimate when solving some practical problems.

Let C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively, where H_1 and H_2 are two real Hilbert spaces, and $T : H_1 \longrightarrow H_2$ be a bounded linear operator. The split feasibility problem (SFP) is defined as follows:

$$find \quad x^* \in C \quad such \ that \quad Tx^* \in Q. \tag{1.5}$$

The set of solutions of problem (1.5) is denoted by SFP(C,Q,T)$=\{x^* \in C : Tx^* \in Q\}$. The concept of SFP was first introduced in [10], in the setting of finite dimensional space for modeling inverse problems arising from medical image reconstruction and phase retrieval. Since its inception in 1994, the SFP has received much attention due to its applications in various areas such as signal processing, image restoration, data compression with particular progress in intensity modulated radiation therapy and so on, (see, for example [7, 8, 11]). In order to solve problem (1.5), Byrne [8] proposed the following iterative algorithm

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C(x_n - \gamma T^*(I - P_Q)Tx_n), \end{cases} \tag{1.6}$$

where $\gamma \in (0, \frac{2}{||T||^2})$, P_C and P_Q are the metric projections onto C and Q respectively and T^* is the adjoint operator of T. They proved that the sequence $\{x_n\}$ generated by algorithm (1.6) converges weakly to a solution of the SFP (1.5). Later, Byrne et al. [6] introduced the concept of split common null point problem (SCNPP) in the setting of real Hilbert spaces, which is defined as follows: let $A_i : H_1 \longrightarrow 2^{H_1},\ 1 \leq i \leq m$ and $B_j : H_2 \longrightarrow 2^{H_2},\ 1 \leq j \leq n$ be set valued mappings respectively, and $T_j : H_1 \longrightarrow H_2,\ 1 \leq j \leq n$ be a bounded linear operator. Then the SCNPP [6] is defined as follows:

Find $x^* \in H_1$ such that

$$x^* \in (\bigcap_{i=1}^{m}(A_i^{-1}0) \ \bigcap \ (\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)), \tag{1.7}$$

where $(A_i^{-1}0)$ and $(B_j^{-1}0)$ are the null point sets of A_i and B_j respectively and the null points set of A_i is defined by $A_i^{-1}0 = \{x^* \in H_1 : 0 \in A_i x^*\}$. In solving problem

(1.7), Byrne et al. [6] proposed the following algorithm

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = J_\lambda^{A_1}\big(x_n - \gamma T^*(I - J_\lambda^{B_1})Tx_n\big), \end{cases} \tag{1.8}$$

where $\lambda > 0, J_\lambda$ is the resolvent and $\gamma \in (0, \frac{2}{||T||^2})$. They proved that the sequence $\{x_n\}$ generated by algorithm (1.8) converges weakly to a solution of problem (1.7). The split common null point problem generalizes the split feasibility problem [10], and split variational inequality problem (see for example [6,9]). On the other hand, Takahashi [42] extends these results on the concept of SCNPP (1.7) to uniformly convex and smooth Banach spaces as follows: Let E and F be uniformly convex and smooth Banach spaces respectively, and let J_E and J_F be the duality mappings on E and F respectively. Let A and B be maximal monotone mappings of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$ respectively. Let Q_μ be the metric resolvent of B for $\mu \geq 0$. Let $T : E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$, and let T^* be the adjoint operator of T. Suppose that $(A^{-1}0) \ \bigcap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$, and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - \mu_n J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n), \\ C_n = \{z \in A^{-1}0 : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in A^{-1}0 : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \qquad n \geq 0, \end{cases} \tag{1.9}$$

where $\{\mu_n\} \subset (0, \infty)$ satisfies that for some $a, b \in \mathbb{R}$, $0 < a \leq \mu_n \leq b < \frac{1}{||T||^2}, \ \ n \geq 0$.

They proved that the sequence $\{x_n\}$ generated by algorithm (1.9) converges strongly to a point $z_0 \in (A^{-1}0) \ \bigcap T^{-1}(B^{-1}0)$, where $z_0 = P_{(A^{-1}0) \ \bigcap T^{-1}(B^{-1}0)} x_1$. However, the strongly convergent algorithms mentioned above share a common feature, that is, their stepsize depends on a prior estimate of the norm of the bounded linear operator which, in general is very difficult to estimate. Thus, the following questions arises naturally:

1. Can we provide a new self-adaptive iterative scheme for solving SCNPP (1.7) in a real Banach space more general than Hilbert space such that its convergence analysis does not require a prior estimate of the operator norm?
2. Can we also approximate such a solution as mentioned above which happens to be a common fixed point of a finite family of quasi-ϕ-nonexpansive mappings?

In order to answer these questions and related issues, the construction of self-adaptive stepsize iterative algorithms has aroused the interest of many researchers. Lopez et al. [29] suggested the use of a self-adaptive stepsize sequence $\{\gamma_n\}$ in place of γ in algorithm (1.6) which does not depend on the norm of the bounded linear operator T. The stepsize is given as follows:

$$\gamma_n := \frac{\rho_n ||(I - P_Q)Tx_n||^2}{||T^*(I - P_Q)Tx_n||^2}, \ \ T^*(I - P_Q)Tx_n \neq 0, \tag{1.10}$$

where $\rho_n \in (0, 4)$.

They proved that the sequence $\{x_n\}$ generated by algorithm (1.6) converges weakly to a solution of the SFP (1.5). The authors in [29] noted that for T with large data sets it may be difficult to compute the operator norm and this may have effect on the iteration process. However, in 1964 Polyak [37] introduced the technique of inertial extrapolation process as a means of speeding up the rate of convergence of iterative methods. Many researchers have proposed and analyzed a large number of inertial type iterative schemes (see, for example [33, 39] and references therein).

Recently, new methods have been proposed to improve the efficiency and convergence properties of algorithms for solving variational inequality problems. For example, Yao, Adamu, and Shehu [47] introduced forward-reflected-backward splitting algorithms with momentum, demonstrating weak, linear, and strong convergence results. Their approach enhances the convergence rates and stability of iterative methods. Additionally, Jolaoso, Shehu, and Yao [21] proposed a strongly convergent inertial proximal point algorithm without the need for an on-line rule. This method provides strong convergence guarantees and is particularly effective in dealing with non-monotone operators, further broadening the applicability of variational inequality problem-solving techniques. These recent advancements underscore the ongoing efforts to refine and optimize methods for solving variational inequality problems, highlighting the dynamic and evolving nature of research in this area.

Motivated by the above works, in this paper, we introduce a new inertial Tseng's extragradient algorithm with self-adaptive step-size technique for approximating common element in the set of solutions of split common null point and pseudomonotone variational inequality problem and the set of common fixed point of a finite family of quasi nonexpansive mappings in uniformly smooth and 2 - uniformly convex Banach space.

Again, we prove a strong convergence theorem of our algorithm to a solution of the stated problem without prior knowledge of Lipschitz constant of the operator under some mild assumptions. we give some numerical examples in order to illustrate the performance of our algorithm and compare it with some existing ones in the literature. Our results generalize and extend many existing results in the literature.

2. Preliminaries

A Banach space E is called smooth if the limit

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t}$$

exists for all $x, y \in S_E$ and for any $\lambda \in (0,1)$, if $||\lambda x+(1-\lambda)y|| < 1$ for all $x, y \in S_E$ with $x \neq y$, then E is said to be strictly convex. Furthermore, E is said to be uniformly convex if for any $\epsilon \in (0,2]$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| = 1, ||y|| = 1$ and $||x-y|| \geq \epsilon$, then $\frac{||x+y||}{2} \leq 1-\delta$, for all $x, y \in S_E$, and $S_E(x) = \{x \in E : ||x|| = 1\}$ is the unit sphere of E.

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\{\frac{||x+\tau y||-||x-\tau y||}{2} - 1 : x, y \in S_E\}.$$

E is called uniformly smooth if the $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$; q - uniformly smooth if there exists a positive constant C_q such that $\rho_E(\tau) \leq C_q(\tau)^q$ for any $\tau > 0$.

Observe that every q - uniformly smooth Banach space is uniformly smooth. Also, every uniformly convex Banach space is strictly convex and reflexive. Typical examples of such spaces, (see, for example Chidume [15], pp. 34, 54) are L_p, l_p and W_p^m which are q - uniformly smooth for $1 \leq q < 2$; 2 - uniformly smooth and uniformly convex (see, for instance [45]). The normalized duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||.||x^*||, ||x^*|| = ||x||\}$$

for all $x \in E$.

Remark 2.1. Observe that the normalized duality mapping J has the following basic properties (see, for more details [16, 38]):

(T1) If E is smooth Banach space, then J is single - valued mapping from E into E^*;

(T2) If E is strictly convex Banach space, then J is one to one;

(T3) If E is uniformly smooth Banach space, then J is uniformly norm to norm continuous on each bounded subset of E;

(T4) If E is reflexive Banach space, then J is surjective;

(T5) If E is reflexive, smooth and strictly convex Banach space with dual E^* and $J^* : E^* \longrightarrow E$ is the normalized duality mapping in E^*, then $J^* = J^{-1}$;

(T6) If E is reflexive, smooth and strictly convex Banach space, then the normalized duality mapping J is single - valued, one to one and onto.

Let E be a reflexive, smooth and strictly convex Banach space and C be a nonempty, closed and convex subset of E (see, for more details [2]).

A mapping $\phi : E \times E \longrightarrow [0, \infty)$ denotes the Lyapunov functional defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2, \quad \forall \ x, y \in E. \tag{2.1}$$

Observe that in a Hilbert space H, $\phi(x, y) = ||x - y||^2, \quad \forall \ x, y \in H$.

Obviously, the functional ϕ satisfies the following properties (see, for more details [2]).

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \quad \forall \ x, y \in E;$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall \ x, y, z \in E; \tag{2.2}$$

$$\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \quad \forall \ x, y, z \in E; \tag{2.3}$$

$$\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq ||x||||Jx - Jy|| + ||y - x||||y||, \forall x, y \in E; \tag{2.4}$$

$$\phi(z, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \leq \alpha\phi(z, x) + (1-\alpha)\phi(z, y), \quad \forall \ x, y \in E, \quad and \quad \alpha \in (0, 1). \tag{2.5}$$

Define a functional $V : E \times E^* \longrightarrow [0, \infty)$ (see for example [2]) by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||y||^2, \ \forall \, x \in E, \quad and \quad x^* \in E^*. \tag{2.6}$$

The following relation is easily verified,

$$V(x, x^*) = \phi(x, J^{-1}(x^*)), \ \forall \, x \in E, \quad and \quad x^* \in E^*. \tag{2.7}$$

Observe that the mapping g defined by fixing $x \in E$, and $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into $\mathbb{R}$.

Lemma 2.1. *[2] Let E be a strictly convex, reflexive and smooth Banach space, and let V be as defined in* (2.6)*. Then*

$$V(x,x^*)+2\langle J^{-1}x^*-x,y^*\rangle \leq V(x,x^*+y^*)\ \ \forall\, x\in E,\ \ and\ \ x^*,y^*\in E^*. \tag{2.8}$$

Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty, closed and convex subset of E.

It is shown that, see Alber [2] for each $x \in E$, there exists a unique element $k \in C$ (written as $\Pi_C x$) such that

$$\phi(k,x)=\inf_{y\in C}\phi(y,x).$$

The mapping $\Pi_C : E \longrightarrow C$ defined by $\Pi_C x = k$, is called generalized projection (see, for example [2]).

Note that if E is a Hilbert space, then Π_C is a metric projection onto C.

Lemma 2.2. *(see for more details [1,2,22]) Let E be a smooth, reflexive and strictly convex Banach space and C be a nonempty, closed and convex subset of E. Then the following inequalities hold:*

$$\phi(x,\Pi_C y)+\phi(\Pi_C y,y)\leq \phi(x,y),\quad \forall\ x\in C\ \ and\ \ y\in E; \tag{2.9}$$

$$If\quad x\in E\quad and\quad z\in C,\quad then\quad z=\Pi_C x \Longleftrightarrow \langle z-y,Jx-Jz\rangle\geq 0,\quad \forall\ y\in C; \tag{2.10}$$

Lemma 2.3. *[14] Let E be a uniformly smooth Banach space, $r > 0$ a positive number, and $B_r(0)$ a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any sequence of positive real numbers $\{\lambda_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty}\lambda_i = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$, the following inequality hold,*

$$||\sum_{n=1}^{\infty}\lambda_n x_n||^2 \leq \sum_{n=1}^{\infty}\lambda_n||x_n||^2-\lambda_i\lambda_j g(||x_i-x_j||). \tag{2.11}$$

Lemma 2.4. *[35] Let E be a uniformly convex and smooth Banach space and $\{\mu_n\}$ and $\{\lambda_n\}$ be two sequences in E. If $\lim_{n\to\infty}\phi(\mu_n,\lambda_n)=0$ and either $\{\mu_n\}$ or $\{\lambda_n\}$ is bounded, then $\lim_{n\to\infty}||\mu_n-\lambda_n||=0$.*

Lemma 2.5. *[4] Let E be a 2-uniformly convex Banach space. Then, there exists $\tau > 0$ such that*

$$\frac{1}{\tau}||x-y||^2\leq\phi(x,y),\quad \forall\ x,y\in E. \tag{2.12}$$

Lemma 2.6. *[45] Let E be a 2-uniformly smooth Banach space with the smoothness constants $\kappa > 0$ and for all $x, y \in E$. Then the following inequality holds:*

$$||x+y||^2\leq ||x||^2+2\langle y,Jx\rangle+2\kappa^2||y||^2. \tag{2.13}$$

Definition 2.1. Let $T : C \to C$ be a mapping.

1. A point $x \in C$ is called a fixed point of T if $Tx = x$, where $F(T) := \{x \in C : Tx = x\}$ is the set of the fixed point of T.

2. A point $x \in C$ is said to be an asymptotic fixed point of T, if there exists a sequence $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.
3. T is said to be quasi - ϕ - nonexpansive if $F(T) \neq \emptyset$, and
$$\phi(p, Tx) \leq \phi(p, x), \quad \forall\ x \in C \ \ and\ \ p \in F(T). \tag{2.14}$$
4. T is called demiclosed at zero if for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x \in C$ and
$$||x_n - Tx_n|| \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \quad then \quad Tx = x.$$
5. A multi - valued mapping $M : E \to 2^{E^*}$ is called monotone if $\forall\ x, y \in E$, with
$$u^* \in Mx \quad and \quad v^* \in My, \quad then \quad \langle x - y, u^* - v^*\rangle \geq 0 \quad holds.$$
6. A monotone mapping $M : E \to 2^{E^*}$ is said to be maximal if M is monotone and the graph of M, $\mathrm{G}(M) := \{(x, u) \in E \times E^* : u \in Mx\}$, is not properly contained in the graph of any other monotone mapping defined on E.

Clearly, when M is a maximal monotone operator and $\lambda > 0$, then the resolvent of M is defined as:
$$\Im_\lambda x = (J + \lambda M)^{-1} Jx, \ \ \forall\ \ x \in E.$$

The following lemma is due to Browder [5].

Lemma 2.7. *[5] Let E be a uniformly convex and smooth Banach space, and let J be the normalized duality mapping of E into E^*. Let M be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $\lambda > 0$,*
$$R(J + \lambda M) = E^*,$$
where $R(J + \lambda M)$ is the range of $J + \lambda M$.

Let E be a uniformly convex Banach space with a Gateaux differentiable norm, and M be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $\lambda > 0$, we consider the following inclusion (see, for more details [5, 42])
$$0 \in J(x_\lambda - x) + \lambda M x_\lambda.$$
This inclusion has a unique solution x_λ. We define Q_λ^M by $x_\lambda = Q_\lambda^M x$. Such $Q_\lambda^M = (I + \lambda J^{-1} M)^{-1}$, $\lambda > 0$ is called the metric resolvent of M. The set of null points of M is defined by $M^{-1}0 = \{z \in E : 0 \in Mz\}$. We know that $M^{-1}0$ is closed and convex; and $F(J_\lambda^M) = M^{-1}0$.

Note that in Hilbert space, the metric resolvent Q_λ of M is called the resolvent of M.

Lemma 2.8. *[22] Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty, closed and convex subset of E. Let $\lambda > 0$ and $M \subset E \times E^*$ be a monotone mapping such that $D(M) \subset J^{-1}R(J + \lambda M)$. Then, the resolvent of M which is defined by $\Im_\lambda x = (J + \lambda M)^{-1} Jx$ for all $x \in C$ is a firmly nonexpansive type mapping.*

Lemma 2.9. *[24] Let E be a reflexive, strictly convex and smooth Banach space and $M : E \longrightarrow 2^{E^*}$ be a maximal monotone mapping such that $M^{-1}0 \neq \emptyset$ and $\Im_\lambda = (J + \lambda M)^{-1}J$ for all $\lambda > 0$, then*

$$\phi(x^*, \Im_\lambda y) + \phi(\Im_\lambda y, y) \leq \phi(x^*, y), \quad \forall\ x^* \in F(\Im_\lambda), \ \textit{and}\ \ y \in E. \tag{2.15}$$

Definition 2.2. Let $F : C \to E^*$ be a mapping. Then F is said to be

1. monotone if the following inequality holds
$$\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall\, x, y \in C;$$
2. pseudomonotone if
$$\langle x - y, F(x)\rangle \geq 0 \Rightarrow \langle x - y, F(y)\rangle \geq 0, \quad \forall\, x, y \in C;$$
3. Lipschitz continuous if there exists a constant $L > 0$ such that
$$||Fx - Fy|| \leq L||x - y||, \quad \forall\, x, y \in C;$$
4. weakly sequentially continuous if for any $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Lemma 2.10. *[34] Consider the variational inequality problem VIP. Suppose that the mapping $h : [0, 1] \longrightarrow E^*$ defined by $h(t) = F(tx + (1 - t)y)$ and $t \in [0, 1]$ is continuous for all $x, y \in C$ (i.e, h is hemicontinuous). Then $M(C, F) \subset VI(C, F)$. Moreover, if F is pseudomonotone, then $VI(C, F)$ is closed, convex and $VI(C, F) = M(C, F)$. Note that for some existing results for Minty variational inequality problem $(MVIP)$, see [27, 41] for more details.*

Lemma 2.11. *[46] If $\{b_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*

$$b_{n+1} \leq (1 - \psi_n)b_n + \psi_n\sigma_n + \gamma_n, \ \ n \geq 0,$$

where (i) $\{\psi_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \psi_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then, $b_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.12. *[32] Let $\{b_n\}$ be a sequence of real numbers such that there exists a subsequence $\{b_{n_i}\}$ of $\{b_n\}$ such that $b_{n_i} < b_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$ and the following properties are satisfied for all $k \in \mathbb{N}$:*

$$b_{m_k} \leq b_{m_k+1} \quad \textit{and} \quad b_k \leq b_{m_k+1},$$
$$In\ \ fact, \quad m_k = \max\{j \leq k : b_j < b_{j+1}\}.$$

Lemma 2.13. *[42] Let E and F be strictly convex, reflexive, and smooth Banach spaces respectively, and let J_E and J_F be the normalized duality mappings on E and F respectively. Let A and B be maximal monotone mappings of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$ respectively. Let J_λ and Q_μ be the metric resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$, and let T^* be the adjoint operator of T. Suppose that $(A^{-1}0) \ \bigcap\ T^{-1}(B^{-1}0) \neq \emptyset$. Let $\lambda, \mu, r > 0$, and $z \in E$. Then, the following are equivalent:*

1. $z = J_\lambda(J_{E^*}^{-1}(J_E(z) - rT^*J_F(Tz - Q_\mu Tz)))$;
2. $z \in (A^{-1}0) \ \bigcap\ T^{-1}(B^{-1}0)$;

3. Main results

In this section, we first establish two important lemmas and then prove a strong convergence theorem for finding a common element of the set of solutions of split common null point and pseudomonotone variational inequality problem and common fixed point of a finite family of quasi nonexpansive mappings in uniformly smooth and 2 - uniformly convex real Banach space. Furthermore, to obtain a strong convergence of our algorithm, we make the following assumptions:

Assumption A

(A1) Let E_1 and E_2 be uniformly smooth and 2 - uniformly convex real Banach spaces and C and D be nonempty, closed and convex subsets of E_1 and E_2 respectively. Let $\{A_i\}_{i=1}^N, \{B_i\}_{i=1}^N$ be finite families of maximal monotone mappings of E_1 into $2^{E_1^*}$ and E_2 into $2^{E_2^*}$ such that $A_i^{-1}0 \neq \emptyset$ and $B_i^{-1}0 \neq \emptyset$ for each $i \in \{1, 2, ..., N\}$. Let $Q_{\lambda_i,n}^{B_i}$ and $\Psi_{\mu_i,n}^{A_i}$ be metric and generalized resolvents of B_i for $\{\lambda_{i,n}\} > 0$ and A_i for $\{\mu_{i,n}\} > 0$ respectively. Let $L : E_1 \longrightarrow E_2$ be a bounded linear operator with its adjoint $L^* : E_2^* \longrightarrow E_1^*$ such that $L \neq 0$.

(A2) The operator $F : E_1 \longrightarrow E_1^*$ is pseudomonotone, L - Lipschitz continuous and weakly sequentially continuous on E_1.

(A3) For $i \in \{1, 2, ..., M\}, \{T_i\}$ is a finite family of quasi nonexpansive mappings of E into itself.

(A4) The solution set $\Gamma = VI(C, F) \bigcap \Omega \neq \emptyset$, where $\Omega = \{\bar{x} \in (\bigcap_{i=1}^M F(T_i) \bigcap (\bigcap_{i=1}^N (A_i^{-1}0)) \quad such \quad that \quad L\bar{x} \in (\bigcap_{i=1}^N (B_i^{-1}0))\}$.

Condition B we assume that the control sequences satisfy:

(B1) $\{\beta_{n,i}\} \subset (0, 1)$, $\sum_{i=0}^{\infty} \beta_{n,i} = 1$ and $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ for all $i = 1, 2, ..., M$;

(B2) $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In order to prove the strong convergence result of Algorithm 3, we first prove the following lemma which plays an important role in the proof of the main result.

Lemma 3.1. *Suppose that* $\{u_n\}$, $\{t_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{\lambda_n\}$ *are sequences generated by Algorithm* 3 *and assumptions* $(A1)$-$(A4)$ *and conditions* $(B1)$-$(B2)$ *hold: Then*

1. *If* $t_n = y_n$ *for some* $n \geq 1$, *then* $t_n \in VI(C, F)$.
2. *The sequence* $\{\lambda_n\}$ *generated by* (3.2) *is a nonincreasing sequence and* $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

Proof. (1) Suppose that $t_n = y_n$ for some $n \geq 1$. Then from Algorithm 3, we have

$$t_n = \Pi_C J_{E_1}^{-1}(J_{E_1} t_n - \lambda_n F(t_n)).$$

Thus, $t_n \in C$. Using the definition of $\{y_n\}$ in Algorithm 3 and the property of generalized projection Π_C onto C in equation (2.10) of Lemma 2.2, we have

$$\langle t_n - y, J_{E_1} t_n - \lambda_n F(t_n) - J_{E_1} t_n \rangle \geq 0, \quad \forall\, y \in C.$$

Initialization: Take $\lambda_0 > 0, \gamma > 0, \mu \in (0, \frac{1}{\kappa\sqrt{2\tau}}), \theta > 0$. Select initial data $x_0, x_1, u \in E_1$, and set $n = 1$. Choose a positive sequence $\{\rho_n\}$ such that $\lim_{n\to\infty} \frac{\rho_n}{\alpha_n} = 0$.

Step 1: Given x_{n-1}, x_n and θ_n for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \hat{\theta}_n]$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\{\frac{\rho_n}{||x_n - x_{n-1}||}, \theta\}, & if \quad x_n \neq x_{n-1}, \\ \theta, & otherwise. \end{cases} \tag{3.1}$$

Step 2: Compute

$$u_n = J_{E_1}^{-1}(J_{E_1}x_n + \theta_n(J_{E_1}x_{n-1} - J_{E_1}x_n)).$$

Step 3: Compute

$$t_n = \Psi_{\mu_{i,n}}^{A_i}(J_{E_1}^{-1}(J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n)).$$

Step 4: Compute

$$y_n = \Pi_C J_{E_1}^{-1}(J_{E_1}t_n - \lambda_n F(t_n))$$

If $y_n = t_n$, then set $z_n = t_n$ and go to step 6. Else go to step 5.

Step 5: Compute

$$z_n = J_{E_1}^{-1}(J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)).$$

Step 6: Compute

$$w_n = J_{E_1}^{-1}(\beta_{n,0}J_{E_1}z_n + \sum_{i=1}^{M} \beta_{n,i}J_{E_1}(T_i z_n)).$$

Step 7: Compute

$$x_{n+1} = J_{E_1}^{-1}(\alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n)$$

where λ_{n+1} and γ_n are updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu||y_n - t_n||}{||F(y_n) - F(t_n)||}, \lambda_n\}, & if \quad F(y_n) - F(t_n) \neq 0, \\ \lambda_n, & otherwise. \end{cases} \tag{3.2}$$

For $\epsilon > 0$ small enough, $\epsilon = \frac{1}{3\kappa^2||L^*||^2}$, the step-size γ_n is chosen as follows:

$$0 < \epsilon \leq \gamma_n \leq \frac{||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2}{\kappa^2||L^*J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2} - \epsilon, \qquad if \quad Lu_n \neq Q_{\lambda_{i,n}}^{B_i}Lu_n, \tag{3.3}$$

otherwise, $\gamma_n = \gamma(\gamma \geq 0)$. Set $n := n + 1$ and return to step 1.

Thus,

$$\langle t_n - y, -\lambda_n F(t_n)\rangle = \lambda_n \langle y - t_n, F(t_n)\rangle \geq 0, \quad \forall\, y \in C.$$

Since $\lambda_n \geq 0$, we obtain that $\langle y - t_n, F(t_n)\rangle \geq 0$. Hence, $t_n \in VI(C, F)$.
(2) It follows from (3.2) that $\lambda_{n+1} \leq \lambda_n$, for all $n \in \mathbb{N}$. Furthermore, since A is a Lipschitz continuous mapping with a positive constant L, in a case where $F(t_n) - F(y_n) \neq 0$, we obtain

$$\frac{\mu||t_n - y_n||}{||F(t_n) - F(y_n)||} \geq \frac{\mu||t_n - y_n||}{L||t_n - y_n||} = \frac{\mu}{L}.$$

Since $\{\lambda_n\}$ is a nonincreasing sequence which bounded below by $\min\{\frac{\mu}{L}, \lambda_1\}$, we conclude that

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}.$$

□

Remark 3.1. From Definition 3.1, we have that

$$\lim_{n\to\infty} \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) = 0.$$

Proof. We have that $\theta_n||x_n - x_{n-1}|| \leq \rho_n$ for each $n \geq 1$, which together with $\lim_{n\to\infty} \frac{\rho_n}{\alpha_n} = 0$ implies

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq \lim_{n\to\infty} \frac{\rho_n}{\alpha_n} = 0. \tag{3.4}$$

Hence,

$$\begin{aligned}
\phi(x^*, x_{n-1}) - \phi(x^*, x_n) &= ||x^*||^2 - 2\langle x^*, J_{E_1}x_{n-1}\rangle + ||x_{n-1}||^2 \\
&\quad -(||x^*||^2 - 2\langle x^*, J_{E_1}x_n\rangle + ||x_n||^2) \\
&= ||x_{n-1}||^2 - ||x_n||^2 + 2\langle x^*, J_{E_1}x_n - J_{E_1}x_{n-1}\rangle \\
&\leq ||x_{n-1} - x_n||(||x_n|| + ||x_{n-1}||) \\
&\quad +2||x^*||||J_{E_1}x_{n-1} - J_{E_1}x_n||.
\end{aligned} \tag{3.5}$$

Since E_1 is uniformly smooth, then J_{E_1} is norm to norm uniformly continuous on a bounded subset of E_1, and we obtain from (3.4) that

$$\lim_{n\to\infty} \alpha_n.\frac{\theta_n}{\alpha_n}||J_{E_1}x_n - J_{E_1}x_{n-1}|| = 0. \tag{3.6}$$

Thus,

$$\lim_{n\to\infty} \alpha_n(\frac{\theta_n}{\alpha_n}||x_{n-1} - x_n||(||x_n|| + ||x_{n-1}||) + 2\frac{\theta_n}{\alpha_n}||x^*||||J_{E_1}x_{n-1} - J_{E_1}x_n||) = 0. \tag{3.7}$$

$$\lim_{n\to\infty} \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) = 0. \tag{3.8}$$

□

We know that, the following lemma, which was carefully proved in [33], plays an important role in the proof of our main result.

Lemma 3.2. *([33],Lemma 9). Suppose that assumptions (A1)-(A4) and conditions* $(B1)$*-*$(B2)$ *hold, and let* $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by Algorithm* 3 *and* $\{u_{n_k}\}$ *be a subsequence of* $\{u_n\}$ *which converges weakly to* $\bar{x} \in E$ *and* $\lim_{k\to\infty} ||u_{n_k} - y_{n_k}|| = 0$. *Then* $\bar{x} \in VI(C, F)$.

Lemma 3.3. *Let* $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ *be sequences defined iteratively by Algorithm* 3, *and let* $x^* \in \Gamma$ *which satisfies the following inequality*

$$\phi(x^*, z_n) \leq \phi(x^*, t_n) - (1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n). \tag{3.9}$$

Proof. Let $x^* \in \Gamma$. Then from Algorithm 3, we have

$$\begin{aligned}
\phi(x^*, z_n) &= \phi(x^*, J_{E_1}^{-1}(J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)) \\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}^{-1}(J_{E_1}y_n - \lambda_n(Fy_n - Ft_n))\rangle \\
&\quad + ||J_{E_1}^{-1}(J_{E_1}y_n - \lambda_n(Fy_n - Ft_n))||^2 \\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)\rangle \\
&\quad + ||J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)||^2 \\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}y_n\rangle + 2\lambda_n\langle x^*, Fy_n - Ft_n\rangle \\
&\quad + ||J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)||^2.
\end{aligned} \tag{3.10}$$

Using Lemma 2.6 and since E_1^* is 2-uniformly smooth, we have from (3.10) that

$$\begin{aligned}
||J_{E_1}y_n - \lambda_n(Fy_n - Ft_n)||^2 &\leq ||J_{E_1}y_n||^2 - 2\lambda_n\langle y_n, Fy_n - Ft_n\rangle \\
&\quad + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2.
\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) and applying equation (2.2), we obtain

$$\begin{aligned}
\phi(x^*, z_n) &\leq ||x^*||^2 - 2\langle x^*, J_{E_1}y_n\rangle + 2\lambda_n\langle x^*, Fy_n - Ft_n\rangle + ||J_{E_1}y_n||^2 \\
&\quad - 2\lambda_n\langle y_n, Fy_n - Ft_n\rangle + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2 \\
&= \phi(x^*, y_n) + 2\lambda_n\langle x^*, Fy_n - Ft_n\rangle - 2\lambda_n\langle y_n, Fy_n - Ft_n\rangle \\
&\quad + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2 \\
&= \phi(x^*, y_n) + 2\lambda_n\langle x^* - y_n, Fy_n - Ft_n\rangle + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2 \\
&= \phi(x^*, t_n) + \phi(t_n, y_n) + 2\langle x^* - t_n, J_{E_1}t_n - J_{E_1}y_n\rangle \\
&\quad + 2\lambda_n\langle x^* - y_n, Fy_n - Ft_n\rangle + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2.
\end{aligned} \tag{3.12}$$

Applying equation (2.3), we have

$$\phi(t_n, y_n) = -\phi(y_n, t_n) + 2\langle y_n - t_n, J_{E_1}y_n - J_{E_1}t_n\rangle. \tag{3.13}$$

Substituting equation (3.13) into (3.12), we obtain

$$\begin{aligned}
\phi(x^*, z_n) &\leq \phi(x^*, t_n) - \phi(y_n, t_n) + 2\langle t_n - y_n, J_{E_1}t_n - J_{E_1}y_n\rangle \\
&\quad + 2\langle x^* - t_n, J_{E_1}t_n - J_{E_1}y_n\rangle + 2\lambda_n\langle x^* - y_n, F(y_n) - F(t_n)\rangle \\
&\quad + 2\kappa^2\lambda_n^2||Fy_n - Ft_n||^2 \\
&= \phi(x^*, t_n) - \phi(y_n, t_n) + 2\langle t_n - y_n, J_{E_1}t_n - J_{E_1}y_n\rangle \\
&\quad + 2\langle x^* - y_n, J_{E_1}t_n - J_{E_1}y_n\rangle - 2\langle t_n - y_n, J_{E_1}t_n - J_{E_1}y_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&+2\lambda_n\langle x^*-y_n, F(y_n)-F(t_n)\rangle + 2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)+2\langle x^*-y_n, J_{E_1}t_n-J_{E_1}y_n\rangle\\
&+2\lambda_n\langle x^*-y_n, F(y_n)-F(t_n)\rangle + 2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2. \qquad (3.14)
\end{aligned}
$$

Using the definition of $\{y_n\}$ and Lemma 2.2, we have

$$\langle x^*-y_n, J_{E_1}t_n-\lambda_nF(t_n)-J_{E_1}y_n\rangle \leq 0$$

$$\langle x^*-y_n, J_{E_1}t_n-J_{E_1}y_n\rangle - \lambda_n\langle x^*-y_n, F(t_n)\rangle \leq 0$$

$$\langle x^*-y_n, J_{E_1}t_n-J_{E_1}y_n\rangle \leq \lambda_n\langle x^*-y_n, F(t_n)\rangle. \qquad (3.15)$$

Substituting (3.15) into (3.14),we have

$$
\begin{aligned}
\phi(x^*,z_n) &\leq \phi(x^*,t_n)-\phi(y_n,t_n)+2\lambda_n\langle x^*-y_n, F(t_n)\rangle\\
&\quad+2\lambda_n\langle x^*-y_n, F(y_n)-F(t_n)\rangle + 2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)+2\lambda_n\langle x^*-y_n, F(t_n)+F(y_n)-F(t_n)\rangle\\
&\quad+2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)+2\lambda_n\langle x^*-y_n, F(y_n)\rangle\\
&\quad+2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)-2\lambda_n\langle y_n-x^*, F(y_n)\rangle\\
&\quad+2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2. \qquad (3.16)
\end{aligned}
$$

Observe that $x^*\in VI(C,F)$ and $\langle y_n-x^*, F(x^*)\rangle\geq 0$. Thus, $\langle y_n-x^*, F(x^*)\rangle\geq 0$ implies $\langle y_n-x^*, F(y_n)\rangle\geq 0$, since F is pseudomonotone.

Furthermore, we have from (3.16) that

$$
\begin{aligned}
\phi(x^*,z_n) &\leq \phi(x^*,t_n)-\phi(y_n,t_n)+2\kappa^2\lambda_n^2||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)+2\kappa^2\lambda_n^2\frac{\lambda_{n+1}^2}{\lambda_{n+1}^2}||Fy_n-Ft_n||^2\\
&\leq \phi(x^*,t_n)-\phi(y_n,t_n)+\frac{2\kappa^2\mu^2\lambda_n^2}{\lambda_{n+1}^2}\frac{||y_n-t_n||^2}{||Fy_n-Ft_n||^2}||Fy_n-Ft_n||^2\\
&=\phi(x^*,t_n)-\phi(y_n,t_n)+\frac{2\kappa^2\mu^2\lambda_n^2}{\lambda_{n+1}^2}||y_n-t_n||^2. \qquad (3.17)
\end{aligned}
$$

Using Lemma 2.5, we obtain from (3.17)

$$
\begin{aligned}
\phi(x^*,z_n) &\leq \phi(x^*,t_n)-\phi(y_n,t_n)+\frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\phi(y_n,t_n)\\
&=\phi(x^*,t_n)-(1-\frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n,t_n)\\
&=\phi(x^*,t_n). \qquad (3.18)
\end{aligned}
$$

We obtain from Lemma 2.6, Lemma 2.13 (1) and (3.3) the following

$$\phi(x^*,t_n)=\phi(x^*,\Psi_{\mu_{i,n}}^{A_i}(J_{E_1}^{-1}(J_{E_1}u_n-\gamma_nL^*J_{E_2}(I-Q_{\lambda_{i,n}}^{B_i})Lu_n)))$$

$$
\begin{aligned}
&\le \phi(x^*, J_{E_1}^{-1}(J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n))\\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}(J_{E_1}^{-1}(J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n))\rangle\\
&\quad + ||J_{E_1}^{-1}(J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n)||^2\\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad + ||J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&\le ||x^*||^2 - 2\langle x^*, J_{E_1}u_n\rangle + 2\gamma_n\langle x^*, L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle + ||u_n||^2\\
&\quad - 2\gamma_n\langle u_n, L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle + \kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}u_n\rangle + 2\gamma_n\langle Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle + ||u_n||^2\\
&\quad - 2\gamma_n\langle Lu_n, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) + 2\gamma_n\langle Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad - 2\gamma_n\langle Lu_n, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) - 2\gamma_n\langle Lu_n - Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) - 2\gamma_n\langle Lu_n - Q_{\lambda_{i,n}}^{B_i}Lu_n + Q_{\lambda_{i,n}}^{B_i}Lu_n - Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) - 2\gamma_n\langle Lu_n - Q_{\lambda_{i,n}}^{B_i}Lu_n, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad - 2\gamma_n\langle Q_{\lambda_{i,n}}^{B_i}Lu_n - Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&\quad + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) - 2\gamma_n||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2 + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&\quad - 2\gamma_n\langle Q_{\lambda_{i,n}}^{B_i}Lu_n - Lx^*, J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n\rangle\\
&= \phi(x^*, u_n) - 2\gamma_n||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2 + 2\kappa^2\gamma_n^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2\\
&= \phi(x^*, u_n) - 2\gamma_n[||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2 - \kappa^2\gamma_n||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2]\\
&\le \phi(x^*, u_n) - \gamma_n[||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2 - \kappa^2\gamma_n||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2]\\
&= \phi(x^*, u_n) - \gamma_n[||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2 - \frac{||J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2}{\kappa^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2}\\
&\quad \times\kappa^2||L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n||^2]. \qquad (3.19)
\end{aligned}
$$

$$
\phi(x^*, t_n) \le \phi(x^*, u_n). \qquad (3.20)
$$

Using the definition of $\{u_n\}$ in Algorithm 3, we obtain

$$
\begin{aligned}
\phi(x^*, u_n) &= \phi(x^*, J_{E_1}^{-1}((1-\theta_n)J_{E_1}x_n + \theta_n J_{E_1}x_{n-1}))\\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}(J_{E_1}^{-1}((1-\theta_n)J_{E_1}x_n + \theta_n J_{E_1}x_{n-1}))\rangle\\
&\quad + ||J_{E_1}^{-1}((1-\theta_n)J_{E_1}x_n + \theta_n J_{E_1}x_{n-1})||^2\\
&\le ||x^*||^2 - 2(1-\theta_n)\langle x^*, J_{E_1}x_n\rangle - 2\theta_n\langle x^*, J_{E_1}x_{n-1}\rangle\\
&\quad + (1-\theta_n)||J_{E_1}x_n||^2 + \theta_n||J_{E_1}x_{n-1}||^2
\end{aligned}
$$

$$\leq (1-\theta_n)\phi(x^*, x_n) + \theta_n \phi(x^*, x_{n-1}). \tag{3.21}$$

Let $x^* \in \Gamma$. Since T_i is quasi nonexpansive, we have

$$\begin{aligned}
\phi(x^*, w_n) &= \phi(x^*, (J_{E_1}^{-1}(\beta_{n,0} J_{E_1} z_n + \sum_{i=1}^{M} \beta_{n,i} J_{E_1} T_i z_n))) \\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}(J_{E_1}^{-1}(\beta_{n,0} J_{E_1} z_n + \sum_{i=1}^{M} \beta_{n,i} J_{E_1} T_i z_n))\rangle \\
&\quad + ||J_{E_1}^{-1}(\beta_{n,0} J_{E_1} z_n + \sum_{i=1}^{M} \beta_{n,i} J_{E_1} T_i z_n)||^2 \\
&\leq ||x^*||^2 - 2\beta_{n,0}\langle x^*, J_{E_1} z_n\rangle - 2\sum_{i=1}^{M} \beta_{n,i}\langle x^*, J_{E_1} T_i z_n\rangle + \beta_{n,0}||J_{E_1} z_n||^2 \\
&\quad + \sum_{i=1}^{M} \beta_{n,i}||J_{E_1} T_i z_n||^2 \\
&= \beta_{n,0}||x^*||^2 - 2\beta_{n,0}\langle x^*, J_{E_1} z_n\rangle + \beta_{n,0}||z_n||^2 + \sum_{i=1}^{M} \beta_{n,i}||x^*||^2 \\
&\quad - 2\sum_{i=1}^{M} \beta_{n,i}\langle x^*, J_{E_1} T_i z_n\rangle + \sum_{i=1}^{M} \beta_{n,i}||T_i z_n||^2 \\
&= \beta_{n,0}\phi(x^*, z_n) + \sum_{i=1}^{M} \beta_{n,i}\phi(x^*, T_i z_n) \\
&\leq \beta_{n,o}\phi(x^*, z_n) + \sum_{i=1}^{M} \beta_{n,i}\phi(x^*, z_n) \\
&= \beta_{n,0}\phi(x^*, z_n) + (1-\beta_{n,0})\phi(x^*, z_n) \\
&= \phi(x^*, z_n).
\end{aligned} \tag{3.22}$$

This implies that

$$\phi(x^*, w_n) \leq \phi(x^*, z_n). \tag{3.23}$$

Using the definition of $\{x_{n+1}\}$ in Algorithm 3, (3.23) and (3.18), we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J_{E_1}^{-1}(\alpha_n J_{E_1} u + (1-\alpha_n) J_{E_1}(w_n))) \\
&= ||x^*||^2 - 2\langle x^*, J_{E_1}(J_{E_1}^{-1}\alpha_n J_{E_1} u + (1-\alpha_n) J_{E_1}(w_n))\rangle \\
&\quad + ||J_{E_1}^{-1}(\alpha_n J_{E_1} u + (1-\alpha_n) J_{E_1}(w_n))||^2 \\
&= ||x^*||^2 - 2\langle x^*, \alpha_n J_{E_1} u + (1-\alpha_n) J_{E_1}(w_n)\rangle \\
&\quad + ||\alpha_n J_{E_1} u + (1-\alpha_n) J_{E_1}(w_n)||^2 \\
&\leq ||x^*||^2 - 2\alpha_n\langle x^*, J_{E_1} u\rangle - 2(1-\alpha_n)\langle x^*, J_{E_1}(w_n)\rangle \\
&\quad + \alpha_n||J_{E_1} u||^2 + (1-\alpha_n)||J_{E_1}(w_n)||^2 \\
&= \alpha_n||x^*||^2 - 2\alpha_n\langle x^*, J_{E_1} u\rangle - 2(1-\alpha_n)\langle x^*, J_{E_1}(w_n)\rangle \\
&\quad + \alpha_n||u||^2 + (1-\alpha_n)||w_n||^2 + (1-\alpha_n)||x^*||^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n)\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, z_n)\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, t_n)\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, u_n). \qquad (3.24)
\end{aligned}
$$

Substituting (3.21) into (3.24), we have

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &\le \alpha_n\phi(x^*, u) + (1-\alpha_n)[(1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})]\\
&\le \max\{\phi(x^*, u), \max\{\phi(x^*, x_n), \phi(x^*, x_{n-1})\}\}\\
&\vdots\\
&\le \max\{\phi(x^*, u), \max\{\phi(x^*, x_1), \phi(x^*, x_0)\}\}. \qquad (3.25)
\end{aligned}
$$

Hence, $\{\phi(x^*, x_n)\}$ is bounded. Since $\frac{1}{\tau}||x_n - x^*||^2 \le \phi(x^*, x_n)$, we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded. □

Theorem 3.1. *Suppose that assumptions (A1)-(A4) hold, and the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be the sequence generated by Algorithm 3. Then $\{x_n\}$ converges strongly to a solution $\bar{x} = \Pi_\Gamma u$.*

Proof. Let $x^* \in \Gamma$. We estimate $\phi(x^*, x_{n+1})$ using inequalities (3.18) and (3.21), and we obtain

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J_{E_1}^{-1}(\alpha_n J_{E_1}u + (1-\alpha_n)J_{E_1}(w_n)))\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n)\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, z_n)\\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)[\phi(x^*, t_n) - (1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n)]\\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, t_n) - (1-\alpha_n)(1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n)\\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, u_n) - (1-\alpha_n)(1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n)\\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)[(1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})]\\
&\quad -(1-\alpha_n)(1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n)\\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)[\phi(x^*, x_n) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n))]\\
&\quad -(1-\alpha_n)(1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})\phi(y_n, t_n). \qquad (3.26)
\end{aligned}
$$

The remaining part of the proof will be divided into two cases.

Case I. Suppose that the sequence $\{\phi(x^*, x_n)\}_{n=1}^{\infty}$ is nonincreasing sequence of real numbers. Since the sequence $\{\phi(x^*, x_n)\}_{n=1}^{\infty}$ is bounded then it converges for all $n \ge n_0$. That is

$$\lim_{n\to\infty}(\phi(x^*, x_n) - \phi(x^*, x_{n+1})) = 0. \qquad (3.27)$$

This implies from (3.26) that

$$(1-\alpha_n)[(1 - \frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})]\phi(y_n, t_n) \le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n)$$

$$\begin{aligned} &+(1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))]-\phi(x^*,x_{n+1})\\ &=\alpha_n(\phi(x^*,u)-\phi(x^*,x_n))+\phi(x^*,x_n)-\phi(x^*,x_{n+1})\\ &+(1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))]. \end{aligned} \tag{3.28}$$

Using (3.27), equation (3.8) of Remark 3.1 and the fact that $(1-\frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})>0, (1-\alpha_n)>0$ together with condition $(B2)$, we have from (3.28) that

$$\begin{aligned} &(1-\alpha_n)[(1-\frac{2\tau\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2})]\phi(y_n,t_n)\leq\alpha_n(\phi(x^*,u)-\phi(x^*,x_n))\\ &+\phi(x^*,x_n)-\phi(x^*,x_{n+1})+(1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))]\longrightarrow 0 \end{aligned}$$

$as\ n\longrightarrow\infty$. Hence,

$$\lim_{n\to\infty}\phi(y_n,t_n)=0. \tag{3.29}$$

Thus, from Lemma 2.4, we have that

$$\lim_{n\to\infty}||y_n-t_n||=0. \tag{3.30}$$

Using the definition of $\{z_n\}$ in Algorithm 3, (3.30) and the fact that A is Lipschitz continuous, we have

$$J_{E_1}z_n=J_{E_1}y_n-\lambda_n(Fy_n-Ft_n),$$

$$\begin{aligned} ||J_{E_1}y_n-J_{E_1}z_n|| &= ||\lambda_n(Fy_n-Ft_n)||\\ &\leq\lambda_n||Fy_n-Ft_n||\\ &\leq\frac{\lambda_n\mu}{\lambda_{n+1}}||y_n-t_n||\longrightarrow 0,\ as\ n\longrightarrow\infty. \end{aligned} \tag{3.31}$$

This implies from (3.31) that

$$\lim_{n\to\infty}||J_{E_1}y_n-J_{E_1}z_n||=0. \tag{3.32}$$

Since E_1^* is uniformly smooth, then $J_{E_1}^{-1}$ is uniformly norm to norm continuous on bounded subsets of E_1^*. Hence, we have from (3.32) that

$$\lim_{n\to\infty}||J_{E_1}y_n-J_{E_1}z_n||=\lim_{n\to\infty}||J_{E_1}^{-1}(J_{E_1}y_n)-J_{E_1}^{-1}(J_{E_1}z_n)||. \tag{3.33}$$

Hence,

$$\lim_{n\to\infty}||y_n-z_n||=0. \tag{3.34}$$

From Lemma 2.3 and the definitions of $\{x_{n+1}\},\{u_n\},\{w_n\}$ in Algorithm 3, we obtain

$$\begin{aligned} \phi(x^*,w_n) &= \phi(x^*,(J_{E_1}^{-1}(\beta_{n,0}J_{E_1}z_n+\sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n)))\\ &= ||x^*||^2-2\langle x^*,J_{E_1}(J_{E_1}^{-1}(\beta_{n,0}J_{E_1}z_n+\sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n))\rangle \end{aligned}$$

$$
\begin{aligned}
&+||J_{E_1}^{-1}(\beta_{n,0}J_{E_1}z_n+\sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n)||^2\\
&=||x^*||^2-2\langle x^*,\beta_{n,0}J_{E_1}z_n+\sum_{i=1}^{M}\beta_{n,i}T_iz_n\rangle\\
&\quad+||\beta_{n,0}J_{E_1}z_n+\sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n||^2\\
&\leq ||x^*||^2-2\beta_{n,0}\langle x^*,J_{E_1}z_n\rangle-\sum_{i=1}^{M}2\beta_{n,0}\langle x^*,J_{E_1}T_iz_n\rangle+\beta_{n,0}||J_{E_1}z_n||^2\\
&\quad+\sum_{i=1}^{M}\beta_{n,i}||J_{E_1}T_iz_n||^2-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)\\
&=\beta_{n,0}||x^*||^2-2\beta_{n,0}\langle x^*,J_{E_1}z_n\rangle+\beta_{n,0}||z_n||^2+\sum_{i=1}^{M}\beta_{n,i}||x^*||^2\\
&\quad-\sum_{i=1}^{M}2\beta_{n,i}\langle x^*,J_{E_1}T_iz_n\rangle+\sum_{i=1}^{M}\beta_{n,i}||T_iz_n||^2-\beta_{n,0}\beta_{n,0}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)\\
&=\beta_{n,0}\phi(x^*,z_n)+\sum_{i=1}^{M}\beta_{n,0}\phi(x^*,T_iz_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)\\
&\leq \beta_{n,0}\phi(x^*,z_n)+\sum_{i=1}^{M}\beta_{n,i}\phi(x^*,z_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)\\
&=\beta_{n,0}\phi(x^*,z_n)+(1-\beta_{n,0})\phi(x^*,z_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)\\
&=\phi(x^*,z_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||).
\end{aligned}
\tag{3.35}
$$

From (3.35), we have

$$
\begin{aligned}
\phi(x^*,x_{n+1})&=\phi(x^*,J_{E_1}^{-1}(\alpha_nJ_{E_1}u+(1-\alpha_n)J_{E_1}(w_n)))\\
&\leq \alpha_n\phi(x^*,u)+(1-\alpha_n)\phi(x^*,w_n)\\
&\leq \alpha_n\phi(x^*,u)+(1-\alpha_n)[\phi(x^*,z_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&\leq \alpha_n\phi(x^*,u)+(1-\alpha_n)[\phi(x^*,t_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&\leq \alpha_n\phi(x^*,u)+(1-\alpha_n)[\phi(x^*,u_n)-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&=\alpha_n\phi(x^*,u)+(1-\alpha_n)[(1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})\\
&\quad-\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&=\alpha_n\phi(x^*,u)+(1-\alpha_n)[(1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})]\\
&\quad-(1-\alpha_n)[\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&=\alpha_n\phi(x^*,u)+(1-\alpha_n)\phi(x^*,x_n)\\
&\quad+(1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))]\\
&\quad-(1-\alpha_n)[\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)]\\
&=\alpha_n(\phi(x^*,u)-\phi(x^*,x_n))+\phi(x^*,x_n)\\
&\quad+(1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))]\\
&\quad-(1-\alpha_n)[\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n-J_{E_1}T_iz_n||)].
\end{aligned}
\tag{3.36}
$$

From (3.27), equation (3.8) of Remark 3.1 together with condition ($B2$), we have from (3.36) that

$$(1-\alpha_n)\beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n - J_{E_1}T_iz_n||) \le \alpha_n(\phi(x^*,u)-\phi(x^*,x_n)) + \phi(x^*,x_n)$$
$$-\phi(x^*,x_{n+1}) + (1-\alpha_n)[\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))] \longrightarrow 0, \quad as \quad n \longrightarrow \infty.$$

Hence,

$$\lim_{n\to\infty} \beta_{n,0}\beta_{n,i}g(||J_{E_1}z_n - J_{E_1}T_iz_n||) = 0. \tag{3.37}$$

Thus, using the property of g in Lemma 2.3 and since $\liminf_{n\to\infty}\beta_{n,0}\beta_{n,i} > 0$, we have from (3.37) that

$$\lim_{n\to\infty} ||J_{E_1}z_n - J_{E_1}T_iz_n|| = 0. \tag{3.38}$$

Since E_1^* is uniformly smooth, then $J_{E_1}^{-1}$ is uniformly norm to norm continuous on bounded subsets of E_1^*. Hence, we have from (3.38) that

$$\begin{aligned}\lim_{n\to\infty} ||J_{E_1}z_n - J_{E_1}T_iz_n|| &= \lim_{n\to\infty} ||J_{E_1}^{-1}(J_{E_1}z_n) - J_{E_1}^{-1}(J_{E_1}T_iz_n)|| \\ &= \lim_{n\to\infty} ||z_n - T_iz_n|| = 0. \end{aligned} \tag{3.39}$$

Using the definition of $\{w_n\}$ in Algorithm 3 and (3.39), we have

$$\begin{aligned}||J_{E_1}w_n - J_{E_1}z_n|| &= ||(\beta_{n,0}J_{E_1}z_n + \sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n) - J_{E_1}z_n|| \\ &= ||\beta_{n,0}(J_{E_1}z_n - J_{E_1}z_n) + \sum_{i=1}^{M}\beta_{n,i}(J_{E_1}T_iz_n - J_{E_1}z_n)|| \\ &= ||\sum_{i=1}^{M}\beta_{n,i}J_{E_1}T_iz_n - J_{E_1}z_n|| \\ &\le \sum_{i=1}^{M}\beta_{n,i}||J_{E_1}T_iz_n - J_{E_1}z_n||.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||J_{E_1}w_n - J_{E_1}z_n|| = 0. \tag{3.40}$$

Since E_1^* is uniformly smooth, we have that

$$\lim_{n\to\infty} ||J_{E_1}^{-1}(J_{E_1}w_n) - J_{E_1}^{-1}(J_{E_1}z_n)|| = \lim_{s\to\infty} ||w_n - z_n|| = 0. \tag{3.41}$$

From the definition of $\{x_{n+1}\}$ in Algorithm 3, we have

$$x_{n+1} = J_{E_1}^{-1}(\alpha_nJ_{E_1}u + (1-\alpha_n)J_{E_1}w_n),$$

$$J_{E_1}x_{n+1} - J_{E_1}w_n = (\alpha_nJ_{E_1}u + (1-\alpha_n)J_{E_1}w_n) - J_{E_1}w_n,$$

$$||J_{E_1}x_{n+1} - J_{E_1}w_n|| = ||\alpha_nJ_{E_1}u + (1-\alpha_n)J_{E_1}w_n - (\alpha_nJ_{E_1}w_n + (1-\alpha_n)J_{E_1}w_n)||$$

$$= ||\alpha_n J_{E_1} u - \alpha_n J_{E_1} w_n + (1-\alpha_n) J_{E_1} w_n - (1-\alpha_n) J_{E_1} w_n)||$$
$$= \alpha_n ||J_{E_1} u - J_{E_1} w_n||.$$

Now, using condition $(B2)$, we obtain

$$\lim_{n\to\infty} ||J_{E_1} x_{n+1} - J_{E_1} w_n|| = 0. \tag{3.42}$$

Since E_1^* is uniformly smooth, we have from (3.42) that

$$\lim_{n\to\infty} ||x_{n+1} - w_n|| = 0. \tag{3.43}$$

From the definition of $\{u_n\}$ in Algorithm 3 and equation (3.6) of Remark 3.1, we obtain

$$u_n = J_{E_1}^{-1}(J_{E_1} x_n + \theta_n (J_{E_1} x_{n-1} - J_{E_1} x_n),$$

$$\begin{aligned} ||J_{E_1} u_n - J_{E_1} x_n|| &= ||\theta_n (J_{E_1} x_{n-1} - J_{E_1} x_n)|| \\ &= \alpha_n . \frac{\theta_n}{\alpha_n} ||J_{E_1} x_{n-1} - J_{E_1} x_n|| \longrightarrow 0, \quad as \;\; n \longrightarrow \infty. \end{aligned} \tag{3.44}$$

Hence,

$$\lim_{n\to\infty} ||J_{E_1} u_n - J_{E_1} x_n|| = 0. \tag{3.45}$$

Since E_1^* is uniformly smooth, then $J_{E_1}^{-1}$ is uniformly norm to norm continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} ||J_{E_1} u_n - J_{E_1} x_n|| = \lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.46}$$

From (3.19), we obtain

$$\begin{aligned} &\phi(x^*, t_n) \\ &\leq \phi(x^*, u_n) - \gamma_n [||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}}) L u_n||^2 - \kappa^2 \gamma_n ||L^* J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}}) L u_n||^2]. \end{aligned} \tag{3.47}$$

$$\begin{aligned} &\gamma_n [||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}}) L u_n||^2 - \kappa^2 \gamma_n ||L^* J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}}) L u_n||^2] \leq \phi(x^*, u_n) \\ &\quad - \phi(x^*, t_n) \\ &= \phi(x^*, u_n) - \phi(x^*, x_{n+1}) + \phi(x^*, x_{n+1}) - \phi(x^*, t_n) \\ &= [(1-\theta_n)\phi(x^*, x_n) + \theta_n \phi(x^*, x_{n-1})] - \phi(x^*, x_{n+1}) + (\alpha_n \phi(x^*, u) \\ &\quad + (1-\alpha_n)\phi(x^*, w_n)) - \phi(x^*, t_n) \\ &= \phi(x^*, x_n) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) - \phi(x^*, x_{n+1}) + \alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] \\ &\quad + \phi(x^*, w_n) - \phi(x^*, t_n) \\ &\leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) + \alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] \\ &\quad + \phi(x^*, z_n) - \phi(x^*, t_n) \\ &\leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) + \alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] \\ &\quad + \phi(x^*, t_n) - \phi(x^*, t_n) \\ &= \alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) \end{aligned}$$

$$+\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \longrightarrow 0, \;\; as \;\; n \longrightarrow \infty.$$

Hence, we have

$$\lim_{n\to\infty}[||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 - \kappa^2\gamma_n||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2] = 0. \tag{3.48}$$

Next, we have from the definition of γ_n in (3.3) that, there exists a very small number $\epsilon > 0$ such that

$$0 < \gamma_n \le \frac{||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n)||^2}{\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n)||^2} - \epsilon. \tag{3.49}$$

This implies that

$$\gamma_n\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \le ||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\ -\epsilon\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2. \tag{3.50}$$

Thus, we have from (3.50) that

$$\epsilon\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \le ||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\ -\gamma_n\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2. \tag{3.51}$$

Hence, from (3.48), we have

$$\lim_{n\to\infty}\epsilon\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 = 0.$$

Thus, we have

$$\lim_{n\to\infty}||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 = 0. \tag{3.52}$$

Now, we have from (3.19) and (3.52) that

$$\begin{aligned}
0 &\le \gamma_n||J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&\le \phi(x^*, u_n) - \phi(x^*, t_n) + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= \phi(x^*, u_n) - \phi(x^*, x_{n+1}) + \phi(x^*, x_{n+1}) - \phi(x^*, t_n) + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= [(1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})] - \phi(x^*, x_{n+1}) + (\alpha_n\phi(x^*, u) \\
&\quad +(1-\alpha_n)\phi(x^*, w_n)) - \phi(x^*, t_n) + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= \phi(x^*, x_n) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) - \phi(x^*, x_{n+1}) - \phi(x^*, t_n) \\
&\quad +\alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] + \phi(x^*, w_n) + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&\le \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) - \phi(x^*, t_n) \\
&\quad +\alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] + \phi(x^*, z_n) + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&\le \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n)) \\
&\quad +\alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] + \gamma_n^2\kappa^2||L^*J_{E_2}(I - Q^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&\quad +\phi(x^*, t_n) - \phi(x^*, t_n) \\
&= \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n))
\end{aligned}$$

$$+\alpha_n[\phi(x^*,u)-\phi(x^*,w_n)]+\gamma_n^2\kappa^2||L^*J_{E_2}(I-Q^{B_i}_{\lambda_{i,n}})Lu_n||^2\longrightarrow 0, \quad (3.53)$$
$$as \;\; n\longrightarrow\infty.$$

Hence, we obtain

$$\lim_{n\to\infty}||J_{E_2}(I-Q^{B_i}_{\lambda_{i,n}})Lu_n||^2=0. \quad (3.54)$$

Since E_2 is uniformly smooth, then we have from (3.54) that

$$\lim_{n\to\infty}||Lu_n-Q^{B_i}_{\lambda_{i,n}}Lu_n||^2=0. \quad (3.55)$$

Let $v_n=J^{-1}_{E_1}(J_{E_1}u_n-\gamma_nL^*J_{E_2}(I-Q^{B_i}_{\lambda_{i,n}})Lu_n)$. Following the same approach as in (3.19), we obtain

$$\phi(x^*,v_n)\leq\phi(x^*,u_n). \quad (3.56)$$

From the definition of v_n, we have

$$J_{E_1}v_n=J_{E_1}u_n-\gamma_nL^*J_{E_2}(I-Q^{B_i}_{\lambda_{i,n}})Lu_n. \quad (3.57)$$

Then from (3.57) we get

$$0\leq||J_{E_1}u_n-J_{E_1}v_n||\leq\gamma_n||L||||J_{E_2}(I-Q^{B_i}_{\lambda_{i,n}})Lu_n||\longrightarrow 0, \quad as \quad n\longrightarrow\infty.$$

This implies

$$\lim_{n\to\infty}||J_{E_1}u_n-J_{E_1}v_n||=0. \quad (3.58)$$

Considering the fact that E_1^* is uniformly smooth, then we have

$$\lim_{n\to\infty}||u_n-v_n||=0. \quad (3.59)$$

Also, from the definition of $\{v_n\}$, (3.56), Lemma 2.9, (3.27), equation (3.8) of Remark 3.1 together with condition $(B2)$, we obtain

$$\begin{aligned}
\phi(v_n,t_n)&=\phi(v_n,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&\leq\phi(x^*,v_n)-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&\leq\phi(x^*,u_n)-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&=\phi(x^*,u_n)-\phi(x^*,x_{n+1})+\phi(x^*,x_{n+1})-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&=[(1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1})]-\phi(x^*,x_{n+1})-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&\quad+(\alpha_n\phi(x^*,u)+(1-\alpha_n)\phi(x^*,w_n))\\
&=\phi(x^*,x_n)+\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))-\phi(x^*,x_{n+1})-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&\quad+\alpha_n[\phi(x^*,u)-\phi(x^*,w_n)]+\phi(x^*,w_n)\\
&\leq\phi(x^*,x_n)+\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))-\phi(x^*,x_{n+1})\\
&\quad+\alpha_n[\phi(x^*,u)-\phi(x^*,w_n)]+\phi(x^*,z_n)-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)\\
&\leq\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\theta_n(\phi(x^*,x_{n-1})-\phi(x^*,x_n))\\
&\quad+\alpha_n[\phi(x^*,u)-\phi(x^*,w_n)]+\phi(x^*,t_n)-\phi(x^*,\Psi^{A_i}_{\mu_{i,n}}v_n)
\end{aligned}$$

$$= \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n))$$
$$+\alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] + \phi(x^*, t_n) - \phi(x^*, t_n)$$
$$= \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + \theta_n(\phi(x^*, x_{n-1}) - \phi(x^*, x_n))$$
$$+\alpha_n[\phi(x^*, u) - \phi(x^*, w_n)] \longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \tag{3.60}$$

Hence,

$$\lim_{n\to\infty} \phi(v_n, t_n) = 0. \tag{3.61}$$

Thus, from Lemma 2.4, we have

$$\lim_{n\to\infty} ||v_n - t_n|| = 0. \tag{3.62}$$

Again, we have from (3.59) and (3.62) that

$$\begin{aligned} ||t_n - u_n|| &= ||t_n - v_n + v_n - u_n|| \\ &\leq ||t_n - v_n|| + ||v_n - u_n|| \longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \end{aligned} \tag{3.63}$$

Hence,

$$\lim_{n\to\infty} ||t_n - u_n|| = 0. \tag{3.64}$$

From (3.64) and (3.46), we obtain

$$\begin{aligned} ||t_n - x_n|| &= ||t_n - u_n + u_n - x_n|| \\ &\leq ||t_n - u_n|| + ||u_n - x_n|| \longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||t_n - x_n|| = 0. \tag{3.65}$$

Also we have from (3.30) and (3.64) that

$$\begin{aligned} ||y_n - u_n|| &= ||y_n - t_n + t_n - u_n|| \\ &\leq ||y_n - t_n|| + ||t_n - u_n|| \longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \end{aligned} \tag{3.66}$$

Thus,

$$\lim_{n\to\infty} ||y_n - u_n|| = 0. \tag{3.67}$$

Furthermore, we have from (3.46) and (3.67) that

$$\begin{aligned} ||y_n - x_n|| &= ||y_n - u_n + u_n - x_n|| \\ &\leq ||y_n - u_n|| + ||u_n - x_n|| \longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \end{aligned} \tag{3.68}$$

Thus,

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.69}$$

From (3.69), (3.43), (3.41) and (3.34), we obtain

$$||x_{n+1} - x_n|| = ||x_{n+1} - w_n + w_n - z_n + z_n - y_n + y_n - x_n||$$

$$\leq ||x_{n+1} - w_n|| + ||w_n - z_n|| + ||z_n - y_n|| + ||y_n - x_n|| \longrightarrow 0,$$

as $n \longrightarrow \infty$. Hence,

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.70}$$

Furthermore, from (3.70) and (3.43), we have

$$\begin{aligned} ||x_n - w_n|| &= ||x_n - x_{n+1} + x_{n+1} - w_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - w_n|| \longrightarrow 0, \ as\ n \longrightarrow \infty. \end{aligned} \tag{3.71}$$

This implies from (3.71) that

$$\lim_{n\to\infty} ||x_n - w_n|| = 0. \tag{3.72}$$

Thus, from (3.34) and (3.69), we obtain

$$\begin{aligned} ||z_n - x_n|| &= ||z_n - y_n + y_n - x_n|| \\ &\leq ||z_n - y_n|| + ||y_n - x_n|| \longrightarrow 0, \ as\ n \longrightarrow \infty. \end{aligned} \tag{3.73}$$

Hence, from (3.73), we have

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{3.74}$$

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $u^* \in E_1$. Also, we know that L is linear and bounded, then we have that $\{Lu_{n_k}\}$ converges weakly to Lu^*. Also, from (3.55) we have that $\{Q^{B_i}_{\lambda_{i,n_k}} Lu_{n_k}\}$ converges weakly to Lu^*. Since $Q^{B_i}_{\lambda_{i,n}}$ is the metric resolvent of B_i for $\lambda_{i,n} \geq 0$, we have

$$Q^{B_i}_{\lambda_{i,n}} Lu_n = (I + \lambda_{i,n} J^{-1}_{E_2} B_i)^{-1} Lu_n,$$

$$\frac{J_{E_2}(Lu_n - Q^{B_i}_{\lambda_{i,n}} Lu_n)}{\lambda_{i,n}} \in B_i Q^{B_i}_{\lambda_{i,n}} Lu_n, \quad \forall\ n \in \mathbb{N} \quad and \quad i \in \mathbb{N}.$$

From the monotonicity of B_i, we obtain

$$0 \leq \langle u - Q^{B_i}_{\lambda_{i,n_k}} Lu_{n_k}, v^* - \frac{J_{E_2}(Lu_{n_k} - Q^{B_i}_{\lambda_{i,n_k}} Lu_{n_k})}{\lambda_{i,n_k}} \rangle, \quad for\ \ all\ \ (u, v^*) \in B_i.$$

Taking the limit as $k \longrightarrow \infty$ we obtain from (3.54) that

$$||J_{E_2}(Lu_{n_k} - Q^{B_i}_{\lambda_{i,n_k}} Lu_{n_k})|| = ||Lu_{n_k} - Q^{B_i}_{\lambda_{i,n_k}} Lu_{n_k}|| \longrightarrow 0, \quad as\ \ k \longrightarrow \infty$$

and $\lambda_{i,n_k} > 0$ for all $k > 1$, it follows that $0 \leq \langle u - Lu^*, v^* - 0 \rangle$ for all $(u, v^*) \in B_i$.

Thus, since B_i is maximal monotone, we have $Lu^* \in \bigcap_{i=1}^{N} (B_i^{-1} 0)$.

We also have from (3.65) that there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\{t_{n_k}\}$ converges weakly to u^*. Since $\Psi^{A_i}_{\mu_{i,n}}$ is the generalized resolvent of A_i, we have

Let $v_n = J_{E_1}^{-1}(J_{E_1}u_n - \gamma_n L^* J_{E_2}(I - Q_{\lambda_{i,n}}^{B_i})Lu_n)$. Then

$$t_n = \Psi_{\mu_{i,n}}^{A_i} v_n,$$

$$\frac{J_{E_1}v_n - J_{E_1}t_n}{\mu_{i,n}} \in A_i t_n, \quad \forall \;\; n \in \mathbb{N} \quad \; and \quad i \in \mathbb{N}.$$

From the monotonicity of A_i, it follows that

$$0 \leq \langle s - t_{n_k}, w^* - \frac{J_{E_1}v_{n_k} - J_{E_1}t_{n_k}}{\mu_{i,n_k}}\rangle, \quad for \;\; all \;\; (s, w^*) \in A_i.$$

Hence, from (3.62), $||J_{E_1}v_{n_k} - J_{E_1}t_{n_k}|| \longrightarrow 0$ as $k \longrightarrow \infty$ and $\mu_{i,n} > 0$ for all $k > 1$, we have that $0 \leq \langle s - u^*, w^* - 0\rangle$ for all $(s, w^*) \in A_i$. Since A_i is maximal monotone, we obtain $u^* \in (A_i^{-1}0)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$, which implies that $z_{n_k} \rightharpoonup u^*$ as $k \longrightarrow \infty$. Hence, by demiclosedness of $(I - T_i)$ at zero for each $i \in \mathbb{N}$ together with $\underset{k\to\infty}{lim}||T_i z_{n_k} - z_{n_k}|| = 0$, it follows that $u^* \in \bigcap_{i=1}^{M} F(T_i)$.

Next, we show that $\{x_n\}$ converges strongly to a point $\bar{x} = \Pi_\Gamma u$. By Lemma 3.2, it follows that $\underset{k\to\infty}{lim}||u_{n_k} - y_{n_k}|| = 0$, then $u^* \in VI(C, F)$. Since $\{x_n\}$ is bounded, then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle x_{n+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle &= \lim_{k\to\infty}\langle x_{n_k+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \\ &= \langle u^* - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle. \qquad (3.75)\end{aligned}$$

Thus, from equation (2.10) of Lemma 2.2 and (3.75), we have

$$\limsup_{n\to\infty}\langle x_{n+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle = \langle u^* - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \leq 0. \qquad (3.76)$$

Hence, it follows from (3.76) that

$$\limsup_{n\to\infty}\langle x_{n+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \leq 0. \qquad (3.77)$$

Furthermore, from the definition of $\phi(\bar{x}, x_{n+1})$ in Algorithm 3 and Lemma 2.1, we obtain

$$\begin{aligned}\phi(\bar{x}, x_{n+1}) &= \phi(\bar{x}, J_{E_1}^{-1}(\alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n)) \\ &= V(\bar{x}, \alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n) \\ &\leq V(\bar{x}, \alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n - \alpha_n(J_{E_1}(u) - J_{E_1}\bar{x})) \\ &\quad -2\langle J_{E_1}^{-1}(\alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n) - \bar{x}, -\alpha_n(J_{E_1}(u) - J_{E_1}\bar{x})\rangle \\ &= V(\bar{x}, \alpha_n J_{E_1}(u) + (1-\alpha_n)J_{E_1}w_n - \alpha_n(J_{E_1}(u) - J_{E_1}\bar{x})) \\ &\quad +2\alpha_n\langle x_{n+1} - \bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\ &= \phi(\bar{x}, J_{E_1}^{-1}[(1-\alpha_n)J_{E_1}w_n + \alpha_n J_{E_1}\bar{x}]) + 2\alpha_n\langle x_{n+1} - \bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\ &\leq (1-\alpha_n)\phi(\bar{x}, w_n) + \alpha_n\phi(\bar{x}, \bar{x}) + 2\alpha_n\langle x_{n+1} - \bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\ &\leq (1-\alpha_n)\phi(\bar{x}, z_n) + 2\alpha_n\langle x_{n+1} - \bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\phi(\bar{x},u_n) + 2\alpha_n\langle x_{n+1}-\bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\
&= (1-\alpha_n)[(1-\theta_n)\phi(\bar{x},x_n) + \theta_n\phi(\bar{x},x_{n-1})] \\
&\quad +2\alpha_n\langle x_{n+1}-\bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\
&= (1-\alpha_n)[\phi(\bar{x},x_n) + \theta_n(\phi(\bar{x},x_{n-1}) - \phi(\bar{x},x_n))] \\
&\quad +2\alpha_n\langle x_{n+1}-\bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle \\
&= (1-\alpha_n)\phi(\bar{x},x_n) + (1-\alpha_n)[\theta_n(\phi(\bar{x},x_{n-1}) - \phi(\bar{x},x_n))] \\
&\quad +2\alpha_n\langle x_{n+1}-\bar{x}, J_{E_1}(u) - J_{E_1}\bar{x}\rangle. \qquad (3.78)
\end{aligned}
$$

Setting $\gamma_n = [1-\alpha_n]\theta_n(\phi(\bar{x},x_{n-1})-\phi(\bar{x},x_n))$ and $\sigma_n = 2\langle x_{n+1}-\bar{x}, J_{E_1}(u)-J_{E_1}\bar{x}\rangle$ Now, applying Lemma 2.11, (3.77), (3.78) and condition $(B2)$, we obtain

$$\lim_{n\to\infty} \phi(\bar{x},x_n) = 0. \qquad (3.79)$$

Thus, from Lemma 2.4, we have

$$\lim_{n\to\infty} ||\bar{x}-x_n|| = 0. \qquad (3.80)$$

Hence, $x_n \longrightarrow \bar{x}$ where $\bar{x} = \Pi_\Gamma u$.
Case II. Suppose that the sequence $\{\phi(p,x_n)\}_{n=1}^\infty$ is not a nonincreasing sequence. Then, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\phi(p,x_{n_k}) < \phi(p,x_{n_k+1}), \quad for \;\; all \quad k\in\mathbb{N}.$$

Then, using Lemma 2.12, there exists a nondecreasing sequence $\{m_s\}\subseteq\mathbb{N}$ such that $m_s \longrightarrow \infty$ as $s\longrightarrow\infty$. Then,

$$\phi(p,x_{m_s}) \le \phi(p,x_{m_s+1}) \quad and \quad \phi(p,x_s) \le \phi(p,x_{m_s+1}).$$

Since $\{\phi(p,x_{m_s})\}$ is bounded, then $\lim_{s\to\infty}\phi(p,x_{m_s})$ exists.

Therefore, using the same approach as in Case I, we have the following

$$(i) \lim_{s\to\infty} ||x_{m_s} - w_{m_s}|| = 0,$$

$$(ii) \lim_{s\to\infty} ||u_{m_s} - y_{m_s}|| = 0,$$

$$(iii) \lim_{s\to\infty} ||z_{m_s} - y_{m_s}|| = 0,$$

$$(iv) \lim_{s\to\infty} ||x_{m_s+1} - x_{m_s}|| = 0.$$

Now, following the same steps as in the proof of Case I, we obtain

$$\limsup_{s\to\infty} \langle x_{m_s+1}-\bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \le 0. \qquad (3.81)$$

Furthermore, from (3.78) and $\phi(\bar{x},x_{m_s}) \le \phi(\bar{x},x_{m_s+1})$, we have

$$
\begin{aligned}
\phi(\bar{x},x_{m_s+1}) &\le (1-\alpha_{m_s})\phi(\bar{x},x_{m_s}) + (1-\alpha_{m_s})[\theta_{m_s}(\phi(\bar{x},x_{m_s-1}) - \phi(\bar{x},x_{m_s}))] \\
&\quad +2\alpha_{m_s}\langle x_{m_s+1}-\bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \\
&\le (1-\alpha_{m_s})\phi(\bar{x},x_{m_s+1}) + (1-\alpha_{m_s})[\theta_{m_s}(\phi(\bar{x},x_{m_s-1}) - \phi(\bar{x},x_{m_s}))] \\
&\quad +2\alpha_{m_s}\langle x_{m_s+1}-\bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle.
\end{aligned}
$$

Since $\alpha_{m_s} > 0$ for all $s \geq 0$ and $\phi(\bar{x}, x_{m_s}) \leq \phi(\bar{x}, x_{m_s+1})$, we have

$$\phi(\bar{x}, x_{m_s}) \leq \phi(\bar{x}, x_{m_s+1}) \leq 2\langle x_{m_s+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle.$$

This implies

$$\limsup_{s\to\infty} \phi(\bar{x}, x_{m_s}) \leq \limsup_{s\to\infty} 2\ \langle x_{m_s+1} - \bar{x}, J_{E_1}u - J_{E_1}\bar{x}\rangle \leq 0.$$

Thus,

$$\limsup_{s\to\infty} \phi(\bar{x}, x_{m_s}) = 0,$$

which by Lemma 2.4, we have $\lim\limits_{s\to\infty} ||\bar{x} - x_{m_s}|| = 0$.

However, we know that $\phi(\bar{x}, x_s) \leq \phi(\bar{x}, x_{m_s+1})$ for all $s \in \mathbb{N}$, hence, $\lim\limits_{s\to\infty} \phi(\bar{x}, x_s) = 0$, by Lemma 2.4, implies

$$\lim_{s\to\infty} ||\bar{x} - x_s|| = 0.$$

Hence, $x_s \longrightarrow \bar{x}$ where $\bar{x} = \Pi_\Gamma u$. □

Corollary 3.1. *Let E be uniformly smooth and 2 - uniformly convex real Banach space, $F : E \longrightarrow E^*$ be a monotone and Lipschitz continuous operator, and $\{T_i\}_{i=1}^M$ be a finite family of quasi nonexpansive mappings of E into itself. Let $\{u_n\}$, $\{t_n\}$, $\{y_n\}$, $\{w_n\}$, and $\{z_n\}$ be sequences generated by Algorithm 3 and $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$ and let $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1) - (A4)$ and condition $(B1)$ of Algorithm 3. Suppose $\Gamma = VI(C, F) \bigcap \Omega \neq \emptyset$, where $\Omega = \{\bar{x} \in (\bigcap_{i=1}^M F(T_i) \bigcap (\bigcap_{i=1}^N (A_i^{-1}0))$ such that $L\bar{x} \in (\bigcap_{i=1}^N (B_i^{-1}0))\}$. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $\bar{x} = \Pi_\Gamma u$.*

Proof. Observe that in this case the weak sequential continuity of A in assumption (A2) of Algorithm 3 has to be droped since it follows from the monotonicity of A and Lemma 3.2 (see, Lemma 9, equation (41) of [33] for more details) that

$$\frac{1}{\lambda_{n_k}}\langle z - y_{n_k}, J_{E_1}u_{n_k} - J_{E_1}y_{n_k}\rangle + \langle y_{n_k} - u_{n_k}, F(u_{n_k})\rangle \leq \langle z - u_{n_k}, F(u_{n_k})\rangle$$
$$\leq \langle z - u_{n_k}, F(z)\rangle. \quad (3.82)$$

Furthermore, passing the limit as $k \longrightarrow \infty$ in inequality (3.82) and applying the fact that $||u_{n_k} - y_{n_k}|| \longrightarrow 0$, *as* $k \longrightarrow \infty$, we obtain

$$\langle z - u^*, F(z)\rangle \geq 0, \ \ \forall \ \ z \in C.$$

Hence, it follows from Theorem (3.1) that the sequence $\{x_n\}$ converges strongly to a solution $\bar{x} = \Pi_\Gamma u$. □

Corollary 3.2. *Let H be a real Hilbert space, $F : H \longrightarrow H$ be pseudomonotone and Lipschitz continuous operator, and $\{T_i\}_{i=1}^M$ be a finite family of quasi nonexpansive mappings of H into itself. Let $\{u_n\}$, $\{t_n\}$, $\{y_n\}$, $\{w_n\}$, and $\{z_n\}$ be sequences generated by Algorithm 3 and $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$ and let $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1)$-$(A4)$ and condition $(B1)$ of Algorithm 3. Suppose $\Gamma = VI(C, F) \bigcap \Omega \neq \emptyset$, where $\Omega = \{\bar{x} \in (\bigcap_{i=1}^M F(T_i) \bigcap (\bigcap_{i=1}^N (A_i^{-1}0))$ such that $L\bar{x} \in (\bigcap_{i=1}^N (B_i^{-1}0))\}$. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $\bar{x} = \Pi_\Gamma u$.*

Proof. Let $E = H$, thus by Theorem (3.1), we have that the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $\bar{x} = \Pi_\Gamma u$. □

4. Numerical illustration

In this section, we provide numerical experiments to demonstrate the advantages of the suggested method and compare with some known strongly convergent algorithms, including the Algorithm 3.1 introduced by Tang and Gibali [43]] (shortly, AlgAvi), and Algorithms 3.3 presented by Okeke et al. [36](shortly, AlgOke). All the programs are implemented in MATLAB R2023b on a personal computer.

Example 4.1. Let $E_1 = E_2 = L_2([0,1])$, $C = D = \{x \in L_2[0,1] : \langle a, x\rangle \leq b\}$, where $a = t^2 + 1$ and $b = 1$, with norm $||x|| = \sqrt{\int_0^1 |x(t)|^2 dt}$ and inner product $\langle x, y\rangle = \int_0^t x(t)y(t)dt$, for all $x, y \in L_2([0,1])$, $t \in [0,1]$. Define metric projection P_C as follows:

$$P_C(x) = \begin{cases} x, & \text{if } x \in C, \\ \\ \frac{b-\langle a,x\rangle}{||a||_{L_2}} a + x, & \text{otherwise}. \end{cases} \tag{4.1}$$

Let $F : L_2[0,1] \to L_2[0,1]$ be defined by $F(x(t)) = e^{-||x||} \int_0^t x(s)ds$, for all $x \in L_2[0,1]$, $t, s \in [0,1]$. Then, F is a pseudomonotone and uniformly continuous mapping. Let $A, B, L : L_2([0,1]) \to L_2([0,1])$ be operators defined as follows:

$$Lx(t) = \int_0^1 x(t)dt, \quad Ax(t) = 5x(t), \quad \text{and } Bx(t) = 4x(t)$$

for all $x \in L_2([0,1])$ and $t \in [0,1]$. Then A is bounded and linear, A and B are maximal monotone operators with resolvents $Q_\mu^A x(t) = \frac{x(t)}{1+5\mu}$ and $\Psi_\mu^B x(t) = \frac{x(t)}{1+4\mu}$, $\mu > 0$, respectively. Furthermore, we define the mappings $T : L_2([0,1]) \to L_2([0,1])$ by $T(x(t)) = \int_0^1 \frac{x(t)}{2} dt$, $T(x(t)) = x(t)$ is relatively nonexpansive mapping. We assume also that $\alpha_n = \frac{1}{5n+1}$, $\beta_n = 0.5 - \alpha_n$ and in addition for Algorithm 3.1 introduced by Tang and Gibali [43]] (shortly, AlgAvi), Algorithms 3.3 presented by Okeke et al. [36] (shortly, AlgOke), we take $\beta_n = \frac{1}{5n+1}$, $\theta_n = 0.5 - \beta_n$, $\rho_n = \frac{5n-1}{2n}$ and $D(x(t)) = 3x(t)$. Then, we let the iteration terminate when $||x_{n+1} - x_n|| \leq \epsilon$ where $\epsilon = 10^{-8}$. The numerical experiments are listed in Table 1. Also, we illustrate the efficiency of strong convergence of the proposed Algorithm 3 in comparison with convergence of Algorithm [43, Algorithm 3.1](shortly, AlgAvi) and [36, Algorithm](shortly, AlgOke) in Figure 1.

Table 1. Comparison of Algorithm 3, [43, Algorithm 3.1] (shortly, AlgAvi) and [36, Algorithm] (shortly, AlgOke).

		Algorithm 3	AlgOke [36]	AlgAvi [43]
$x_0 = 3t,\ x_1 = t^6$	Iterations	18	24	73
	CPU Time (s)	1.1707	1.3586	2.7914
$x_0 = t^3,\ x_1 = t^2 + 1$	Iterations	17	23	72
	CPU Time (s)	0.8441	1.2749	2.7630
$x_0 = 2t,\ x_1 = t^4 + t^2$	Iterations	17	23	71
	CPU Time (s)	0.9420	1.4623	3.0487
$x_0 = t^4,\ x_1 = t^8 + 4t^5 + 2t$	Iterations	18	24	74
	CPU Time (s)	0.9658	1.4264	3.5148

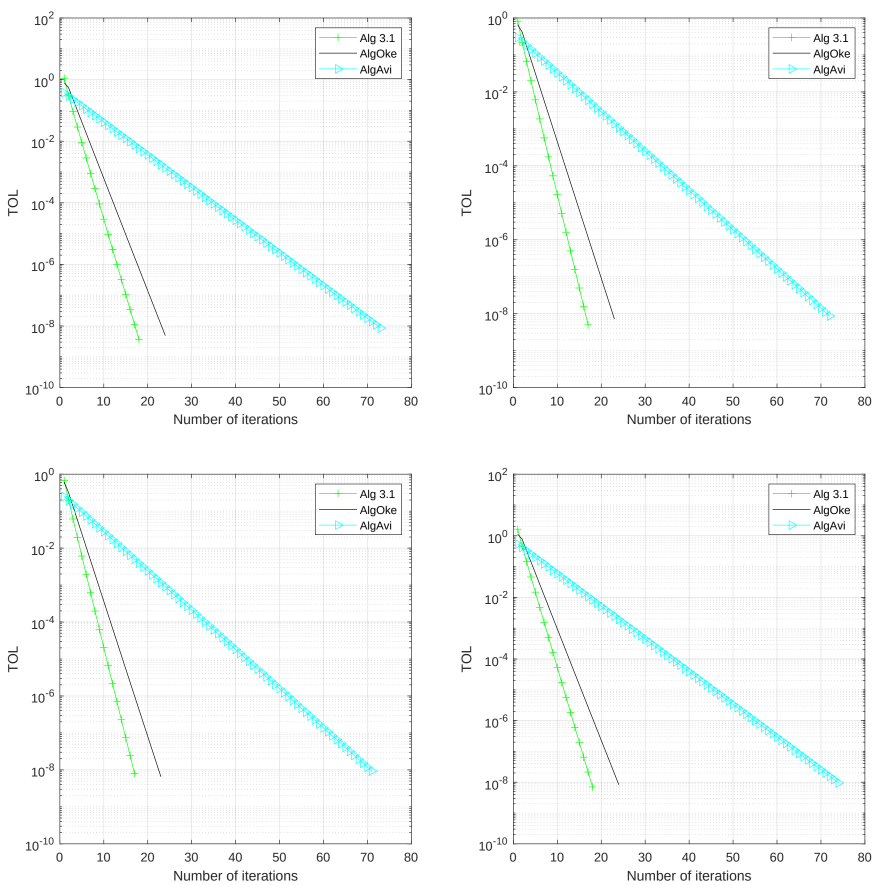

Figure 1. The error plotting of Comparison of Algorithm 3, [43, Algorithm 3.1](shortly, AlgAvi) and [36, Algorithm](shortly, AlgOke) for Example 4.1.

5. Conclusion

This paper introduces a new inertial Tseng's extragradient algorithm with a self adaptive step size for approximating common element of the set of solutions of split common null point and pseudomonotone variational inequality problem as well as common fixed point of a finite family of quasi nonexpansive mappings in uniformly smooth and 2 - uniformly convex Banach space. Furthermore, we prove a strong convergence theorem of our algorithm without prior knowledge of Lipschitz constant of the operator under some mild assumptions. we presented some numerical examples in order to illustrates the performance of our proposed algorithm and compare it with some existing ones in the literature. Our result generalize and improve many existing result in the literature.

References

[1] Y.I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces,* Panamer. Math. J., 1994, 4, 39–54.

[2] Y.I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications. In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,* Kartsatos, A.G., Ed.; Lecture Notes Pure Appl. Math.; Dekker: New York, NY, USA, 1996, 178, 15–50.

[3] B. Ali and J.T. Ajio:*Convergence theorem for equilibrium, split null point and common fixed point problems in Banach spaces,* J. Adv. Math. Stud.,2023, 16(2), 167–189.

[4] K. Aoyama and F. Kohsaka,*Strongly relatively nonexpansive sequences generated by firmly nonexpansive like mappings,* Fixed Point Theory Appl. 2014, 95.

[5] F.E. Browder, *Nonlinear maximal monotone operators in Banach spaces,* Math. Ann., 1968, 175, 89–113.

[6] C. Byrne, Y. Censor, A. Gibali and S. Reich, *Weak and strong convergence of algorithms for the split common null point problem,* J. Nonlinear Convex Anal., 2012, 13(4) , 759–775 .

[7] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction,* Inverse Problems, 2004, 20(1), 103–120.

[8] C. Byne, *Iterative oblique projection onto convex sets and the split feasibility problem,* Inverse Probl.,**18**(2002), 441–453.

[9] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem,* Numer. Algorithms, 2012, 59, 301–323.

[10] Y. Censor and T. Elfving: *A multiprojection algorithm using Bregman projections in a product space,* Numerical Algorithms, 1994, 8(2), 221–239.

[11] Y. Censor, T. Bortfeld, B. Martin and T. Trofimov, *A unified approach for inversion problem in intensity-modolated radiation therapy,* Phys. Med. Biol., 2006, 51, 2353–2365.

[12] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space,* J Optim Theory Appl. 2011, 148, 318–335.

[13] L. C. Ceng, A. Petrusel, X. Qin and J. C. Yao, *Pseudomonotone variational inequalities and fixed point,* Fixed Point Theory, 2021, 22, 543–558.

[14] S.S. Chang, J. K. Kim and X. R. Wang, *Modified block iterative algorithm for solving convex feasibility problems in Banach spaces,* Journal of Inequalities and Applications, vol. 2010, Article ID 869684, 14 pages, 2010.

[15] C.E. Chidume, *Geometric properties of Banach spaces and nonlinear iterations,* Lecture Notes in Mathematics, Springer Verlag Series, London, 1965, (2009), ISBN: 978-1-84882-189-7

[16] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems,* Kluwer, Dordrecht, 1990.

[17] A.A. Goldstein, *Convex programming in Hilbert space* Bull. Am. Math. Soc. 1964, 70, 709–710.

[18] P. Hartman and G. Stampacchia, *On some nonlinear elliptic differential functional equations,* Acta Math., 1966, 115, 271–310.

[19] B.S. He, *A class of projection and contraction methods for monotone variational inequalities,* Appl. Math. Optim., 1997, 35, 69–76.

[20] L.O. Jolaoso, A. Taiwo, T.O. Alakoya and O.T. Mewomo, *A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods,* J. Opt. Theory Appl, 2020, 185, 744–766.

[21] L. O. Jolaoso, Y. Shehu and J. C. Yao, *Strongly Convergent Inertial Proximal Point Algorithm Without On-line Rule.*, J Optim Theory Appl, 2024, 200, 555–584. https://doi.org/10.1007/s10957-023-02355-5

[22] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type Algorithm in a banach space,* SIAM Journal on Optimization, 2002, 13(3), 938–945.

[23] A.A. Khan, S. Migorski and M. Sama, *Inverse problems for multivalued quasi variational inequalities and noncoercive variational in equalities with noisy data,* Optimization, 2019, 8(10), 1897–1931.

[24] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space,* Abstr. Appl. Anal., 2004, 239–249.

[25] G. M. Korpelevich, *An extragradient method for solving saddle points and for other problems,* Ekon. Mat. Metody, 1976, 12, 747–756.

[26] E.S. Levitin and B.T. Polyak, *Constrained minimization problems,* USSR Comput. Math. Math. Phys., 1966, 6, 1–50.

[27] L.J. Lin, M.F. Yang, Q.H. Ansari and G. Kassay, *Existence results for Stampacchia and Minty type implicit variational inequalities with multivalued maps,* Nonlinear Anal. Theory Methods Appl., 2005, 61, 1–19.

[28] J.L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., 1967, 20, 493–517.

[29] G. Lopez, V. Martin-Marquez, F. Wang and H.K. Xu, *Solving the split feasibility problem without prior knowledge of matrix norms,* Inverse Probl., 2012, 28, 085004.

[30] P.E Mainge, *Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints,* Eur. J. Oper. Res.,2010, 205, 501–506.

[31] P. E Mainge, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems.* SIAM J. control optim. 2008,47, 1499–1515.

[32] P. E Mainge, *Strong convergence of projected subgradient methods for non-smooth and non-strictly convex minimization,* Set- valued Anal., 2008, 16, 899–912.

[33] R. Maluleka, G.C. Ugwunnadi and M. Aphane, *Inertial method for solving pseudomonotone variational inequality and fixed Point problems in Banach spaces,* Axioms 2023, 12,960. https://doi.org/10.3390/axioms12100960.

[34] J. Mashregi and M. Nasri, *Forcing strong convergence of Korpelevich's method in Banach spaces with applications in game theory,* Nonlinear Anal., 2010, 72, 2086–2099.

[35] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in Banach spaces,* J. Approx. Theory, 2005, 134, 257–266.

[36] C.C. Okeke, A.U. Bello, L.O. Jolaoso and K.C. Ukandu, *Inertial method for split null point problem with pseudomonotone variational inequality problems,* Numerical Algebra, Control and Optimization, 2021, 12(4), 815–836.

[37] B.T. Polyak, *Some methods of speeding up the convergence of the iteration methods,* USSR Comput. Math. Phys., 1964, 4, 1–17.

[38] S. Reich, *A weak convergence theorem for the alternating method with Bregman distance. In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type,* Kartsatos, A.G., Ed.; Marcel Dekker: New York, NY, USA, 1996, 313–318.

[39] Y. Shehu and A. Gibali, *New inertial relaxed method for solving split feasibilities,* Optimization Letters, 2021, 15, 2109–2126.

[40] D. J. Wen, *Modified Krasnoselski Mann type iterative algorithm with strong convergence for hierarchical fixed point problem and split monotone variational inclusions*, J. Comp. Appl. Math. 2021, doi.org/10.1016/j.cam.2021.113501.

[41] M.V. Solodov, B.F. Svaiter, *A new projection method for variational inequality problems,* SIAM J. Control Optim., 1999 37, 765–776.

[42] W. Takahashi, *The split common null point problem in Banach space,* Arch. Math., 2015, 104, 357–365.

[43] Y. Tang, A. Gibali, *New self adaptive step size algorithms for solving split variational inclusion problems and its applications*, Numer. Algorithm, 2020,83, 305–331.

[44] P. Tseng, *A modified Forward-Backward splitting method for maximal monotone mappings,* SIAM J Control Optim., 2000, 38, 431–446.

[45] H.K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Anal. 16(1991), 1127–1138.

[46] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. Lond. Math. Soc., 2002, 66, 240–256.

[47] Y. Yao, A. Adamu and Y. Shehu, *Forward–Reflected–Backward Splitting Algorithms with Momentum: Weak, Linear and Strong Convergence Results.* J Optim Theory Appl, 2024, 201, 1364–1397. https://doi.org/10.1007/s10957-024-02410-9