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Double Phase Phenomena in Navier Boundary Problems with Degenerated $(p(.), q(.))$-Operators

Abdessamad El Katit[1,†], Abdelrachid El Amrouss[2] and Fouad Kissi[2]

Abstract In this paper, we are interested in some results of the existence of multiple solutions for Navier boundary value problem involving degenerated $(p(.), q(.))$-Biharmonic and $(p(.), q(.))$-Laplacian operators. Our approach is based on variational method and critical point theory.

Keywords Weighted variable exponent Lebesgue-Sobolev spaces, degenerated $(p(.), q(.))$-Biharmonic operator, Navier boundary problem

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1. Introduction

In this paper, we consider the following nonlinear boundary value problem

$$\begin{cases} \Delta\left(\omega(x)(|\Delta u|^{p(x)-2}\Delta u + |\Delta u|^{q(x)-2}\Delta u)\right) - \mathcal{L}(u) = \lambda f(x,u) & in\ \Omega, \\ u = \Delta u = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain of $\mathbb{R}^N$, the variable exponents $p, q : \mathbb{R}^N \to (1, \infty)$ are continuous functions with $1 < p(x) < q(x)$, the weight ω is non-negative locally integrable function on Ω, λ is a real parameter and $f(x, s)$ is continuous on $\bar{\Omega} \times \mathbb{R}$ and

$$\mathcal{L}(u) := \operatorname{div}\left(\omega(x)(|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u)\right).$$

In the last few years, elliptic equations with variable exponents have been widely performed and have got a considerable amount of attention . They have contributed to the progress in elasticity theory and electrorheological fluids dynamics (see [6,34]).

Problems involving p-Laplacian and (p, q)-Laplacian operators in bounded or unbounded domains have been studied by many authors, for instance [1, 3, 14, 20–22, 29, 30, 32]. However, very few works have concerned $p(x)$-Laplacian and $p(x)$-Biharmonic type problems with singular weights i.e, with not bounded weights or not separated from zero in Ω (see [11, 23, 24, 28]). In that case, the above operators are called degenerated operators. We note that similar degeneracy can be physically connected with the equilibrium of continuous anisotropic media [7].

The study of double phase phenomena in partial differential equations is crucial because it extends the understanding of complex systems where material properties

† the corresponding author.
Email address: abdessamad.elkatit@ump.ac.ma(A. El katit), elamrouss@hotmail.com(A. R. El Amrouss), kissifouad@hotmail.com(F. Kissi)
[1] Department of mathematics, Mohamed first University, El qods, 60050, Morocco.
[2] LaMao, Faculty of sciences, El qods, 60050, Morocco

exhibit phase transitions, such as in elasticity or fluid dynamics. In such systems, the governing equations have coefficients that vary according to two different phases. The double phase framework is also significant in understanding models where energy densities switch between two distinct behaviors.

In the constant case, the authors in [9] have treated the following bifurcation problem involving degenerated p-Laplacian

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = \lambda b(x)|u|^{p-2}u + f(\lambda, x, u) \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega. \end{cases} \tag{1.2}$$

Also, for weight w satisfying the Muckenhoupt condition, the following fourth order elliptic equation has been treated

$$\begin{cases} \Delta\left(\omega(x)(|\Delta u|^{p-2}\Delta u + |\Delta u|^{q-2}\Delta u)\right) \\ -\operatorname{div}\left(\omega(x)(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)\right) = f(x) - div(G(x)) \text{ in } \Omega, \\ u = \Delta u = 0 \quad on\ \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^{p'}(\Omega, \omega^{-1/(p-1)})$ and $G \in \left[L^{p'}(\Omega, \omega^{-1/(q-1)})\right]^N$ (see [5]). The author has shown the existence of unique solution in the weighted Sobolev space $W^{2,p}(\Omega, \omega) \cap W_0^{1,p}(\Omega, \omega)$.

In the variable case, the Dirichlet problem involving degenerated p(x)-Laplacian

$$\begin{cases} -\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right) = \mu g(x)|u|^{p(x)-2}u + f(\lambda, x, u, \nabla u) \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega, \end{cases}$$

has been studied (see [25]), where $w, w^{-1/(p(x)-1)}$ are locally integrable functions on Ω and w satisfies the hypothesis $(\Im)$ mentioned in (2.4).

The novelty of this paper is in extending the problem (1.3) using critical point theory (see [2, 10]), which requires a particular kind of weight depending on the variable exponent q, and a more complicated non-linearity. For this reason further analysis has to be realized.

The problem (1.1) is stated in the weighted Sobolev space $X = W^{2,q(x)}(\Omega, \omega) \cap W_0^{1,q(x)}(\Omega, \omega)$. A function $u \in X$ is a weak solution to (1.1) if and only if for every v in X, we have

$$\begin{aligned} &\int_\Omega \omega(x)(|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v)\, dx \\ &+ \int_\Omega \omega(x)(|\nabla u|^{p(x)-2}\nabla u\nabla v + |\nabla u|^{q(x)-2}\nabla u\nabla v) \\ =&\lambda \int_\Omega f(x, u) v dx. \end{aligned}$$

In order to prove the existence and multiplicity of solutions for the problem (1.1), we assume that the weight w belongs to the class $\tilde{A}_{q(.)}(\Omega)$ defined in (2.3), and satisfies the hypothesis $(\Im)$ mentioned in (2.4). Furthermore, we consider the following conditions on f

($\mathcal{F}_0$) $|f(x,s)| \leq C(1+|s|^{\alpha(x)-1})$ a.e $x \in \Omega$, and for all $s \in \mathbb{R}$ where $C \geqslant 0$, $\alpha(x) \in C_+(\Omega)$ and $\alpha(x) < q^*_{\theta,2}(x)$ for all $x \in \bar{\Omega}$, where

$$q_\theta(x) = \frac{q(x)\theta(x)}{1+\theta(x)}, \qquad q^*_{\theta,2}(x) = \begin{cases} \frac{Nq_\theta(x)}{N-2q_\theta(x)} & \text{if } 2q_\theta(x) < N, \\ \infty & \text{if } 2q_\theta(x) \geq N, \end{cases}$$

with $\theta(x)$ is given in ($\Im$).

($\mathcal{F}_1$) There exist $x_0 \in \Omega$ and $\rho_0, l > 0$ such that $B(x_0,\rho_0) \subset \Omega$ and $F(x,s) \geq 0$ for $x \in B(x_0,\rho_0)$ and $s \in [0,l[$, $F(x,l) > 0$ for $x \in B(x_0,\rho_0/2)$.

($\mathcal{F}_2$) $f(x,s) = o(|s|^{\tau-1})$ uniformly for a.e $x \in \Omega$ with $q^+ < \tau < q^*_{\theta,2}(x)$ for all $x \in \bar{\Omega}$,

Our main results are given by the following two theorems:

Theorem 1.1. *Suppose that* ($\mathcal{F}_0$) *and* ($\mathcal{F}_1$) *hold and* $\alpha^+ < q^-$. *Then there exists* $\lambda^* > 0$ *such that for each* $\lambda \in (\lambda^*, \infty)$, *the problem* (1.1) *admits at least one nontrivial weak solution* $u_{1,\lambda}$ *satisfying* $E_\lambda(u_{1,\lambda}) < 0$.

Theorem 1.2. *Assume that* ($\mathcal{F}_0$)-($\mathcal{F}_2$) *are satisfied. Then for each* $\lambda \in (\lambda^*, \infty)$, *the problem* (1.1) *has a second nontrivial weak solution* $u_{2,\lambda}$ *fulfilling* $E_\lambda(u_{2,\lambda}) > 0$.

This paper is organized as follows. In Section 2, we recall some backgrounds about the weighted generalized Lebesgue–Sobolev space. In Section 3, we state and prove some auxiliary results about the weighted generalized Sobolev space and discuss the equivalent norms in the space $W^{2,q(x)}(\Omega,\omega) \cap W_0^{1,q(x)}(\Omega,\omega)$.

In Section 4, we provide the proof of our main results. Finally we give an example to illustrate our results.

2. Preliminaries

For the suitability of readers, we remind some backgrounds about the weighted variable exponent Lebesgue-Sobolev spaces. By a weight $\omega(.)$, we always mean a non-negative locally integrable function on Ω. Set

$$C_+(\Omega) = \{\varrho \in C(\bar{\Omega}) \ : \ \varrho(x) > 1 \textit{ for all } x \in \bar{\Omega}\},$$
$$\varrho^+ = \max_{\bar{\Omega}} \varrho(x), \quad \varrho^- = \min_{\bar{\Omega}} \varrho(x), \quad \textit{for } \varrho \in C_+(\Omega).$$

For a measurable positive weight $\omega(.)$ and exponent $q(.)$ in $C_+(\Omega)$, we introduce the weighted variable exponent Lebesgue space $L^{q(x)}(\Omega,\omega)$ composed of measurable real-valued functions u such that

$$\int_\Omega \omega(x)|u(x)|^{q(x)}dx < \infty,$$

equipped with the norm

$$|u|_{L^{q(x)}(\Omega,\omega)} = \inf\left\{\mu > 0 \ : \ \int_\Omega \omega(x)\left|\frac{u(x)}{\mu}\right|^{q(x)} dx \leqslant 1\right\},$$

then the space $L^{q(x)}(\Omega,\omega)$ endowed with the above norm is reflexive and Banach space (see [25]).

At the time that $\omega(x)\equiv 1$, we have $L^{q(x)}(\Omega,\omega)\equiv L^{q(x)}(\Omega)$ and we employ the notation $L^{q(x)}(\Omega)$ in the place of $L^{q(x)}(\Omega,\omega)$.

For $m\in\mathbb{N}^*$, the weighted variable exponent Sobolev space $W^{m,q(x)}(\Omega,\omega)$ is defined by

$$W^{m,q(x)}(\Omega,\omega)=\left\{u\in L^{q(x)}(\Omega)\ :\ D^{\alpha}u\in L^{q(x)}(\Omega,\omega),|\alpha|\leqslant m\right\},$$

with $\alpha\in\mathbb{N}^*$. We can define the norm on $W^{m,q(x)}(\Omega,\omega)$ by

$$\|u\|_{W^{m,q(x)}(\Omega,\omega)}=|u|_{L^{q(x)}(\Omega)}+\sum_{1\leqslant|\alpha|\leqslant m}|D^{\alpha}u|_{L^{q(x)}(\Omega,\omega)}.$$

Define the class $LH(\mathbb{R}^N)$ of globally log-Hölder continuous functions composed of measurable functions $h:\mathbb{R}^N\to[1,\infty)$ with $1<h^-\leqslant h(x)\leqslant h^+<\infty$, where $h^-=\operatorname{ess\,inf}_{x\in\mathbb{R}^N}h(x)$ and $h^+=\operatorname{ess\,sup}_{x\in\mathbb{R}^N}h(x)$ satisfying the following:

$$|h(x)-h(y)|\leq\frac{C}{-log|x-y|},|x-y|<1/2,\tag{2.1}$$

$$|h(x)-h(y)|\leq\frac{C}{log(e+|x|)},|y|\geqslant|x|.\tag{2.2}$$

We note that for $r\in C_+(\Omega)$, smooth functions are not in general dense in the Sobolev space $W^{1,r(x)}(\Omega)$. However, when the exponent $r(.)$ fulfills (2.1) for $x,y\in\Omega$, then smooth functions are dense in variable exponent Sobolev space. As a result there is no confusion in defining the Sobolev space $W_0^{1,r(x)}(\Omega)$ as the completion of $C_0^{\infty}(\Omega)$ with respect to the norm of $W^{1,r(x)}(\Omega)$.

Throughout this paper, we assume that q belongs to the class of globally log-Hölder continuous functions $LH(\mathbb{R}^N)$.

Proposition 2.1. *([12]) The space* $(L^{q(x)}(\Omega),|u|_{q(x)})$ *is separable, reflexive and uniformly convex Banach, and its conjugate space is* $L^{p(x)}(\Omega)$*, where* $\frac{1}{q(x)}+\frac{1}{p(x)}=1,\forall x\in\bar{\Omega}$*. For any* $v\in L^{q(x)}(\Omega)$ *and* $w\in L^{p(x)}(\Omega)$*, we have*

$$\left|\int_{\Omega}vwdx\right|\leqslant\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|v|_{L^{q(x)}(\Omega)}|w|_{L^{p(x)}(\Omega)}\leqslant 2|v|_{L^{q(x)}(\Omega)}|w|_{L^{p(x)}(\Omega)}.$$

Denote $B(x,r)$ the open ball centered at x of radius r. Let us define the class $\widetilde{A}_{q(.)}(\Omega)$ to contain those weights $\omega(.)$ which fulfill the following condition

$$\sup_{x\in\Omega,r>0}\left(\frac{1}{|\tilde{B}(x,r)|}\int_{\tilde{B}(x,r)}|\omega(y)|^{q(y)}dy\right)\left(\frac{1}{|\tilde{B}(x,r)|}\int_{\tilde{B}(x,r)}\frac{dy}{|\omega(y)|^{\frac{q(y)}{q(y)-1}}}\right)^{q^--1}<\infty,\tag{2.3}$$

where $|B(x,r)|$ is the N-dimensional Lebesgue measure of $B(x,r)$ and $\tilde{B}(x,r)=B(x,r)\cap\Omega$.

Remark 2.1. We point out that if $w\in\tilde{A}_{q(.)}(\Omega)$, then it necessarily satisfies the following condition

$$(w1)\ \ w\in L^1_{loc}(\Omega)\ \textit{and}\ w^{-1/(q(x)-1)}\in L^1_{loc}(\Omega).$$

This yields that $L^{q(x)}(\Omega,\omega) \subset L^1_{loc}(\Omega)$, which makes sense to talk about weak derivatives in $L^{q(x)}(\Omega,\omega)$.

Here, define the Hardy-Littlewood maximal function, Mf, for a locally integrable f on Ω by

$$Mf(x) = \sup_{r>0} \frac{1}{|\tilde{B}(x,r)|} \int_{\tilde{B}(x,r)} |f(y)|dy.$$

Proposition 2.2. *([25]) Let $\phi \in C_0^\infty(\Omega)$ and let a multi-index γ be fixed. If $\omega(.)$ satisfies $(w1)$, then the formula*

$$L_\gamma(u) = \int_\Omega u D^\gamma \phi dx, \quad u \in L^{q(x)}(\Omega,\omega),$$

defines a continuous linear functional L_γ on $L^{q(x)}(\Omega,\omega)$.

Define the modular $\rho : L^{q(x)}(\Omega,\omega) \to \mathbb{R}$, by $\rho(u) = \int_\Omega \omega(x)|u(x)|^{q(x)}$. As $u \in L^{q(x)}(\Omega,\omega)$ i.e. $\omega^{\frac{1}{q(x)}} u \in L^{q(x)}(\Omega)$ and $\left|\omega^{\frac{1}{q(x)}}|u|\right|_{L^{q(x)}(\Omega)} = |u|_{L^{q(x)}(\Omega,\omega)}$, then in view of [[12],theorem 1.3], we have the following Lemma.

Lemma 2.1. *For each $u_n, u \in L^{q(x)}(\Omega,\omega)$, we have*

(1) $|u|_{L^{q(x)}(\Omega,\omega)} > 1$ *then* $|u|^{q^-}_{L^{q(x)}(\Omega,\omega)} \leqslant \rho(u) \leqslant |u|^{q^+}_{L^{q(x)}(\Omega,\omega)}$;

(2) $|u|_{L^{q(x)}(\Omega,\omega)} < 1$ *then* $|u|^{q^+}_{L^{q(x)}(\Omega,\omega)} \leqslant \rho(u) \leqslant |u|^{q^-}_{L^{q(x)}(\Omega,\omega)}$;

(3) $\lim_{n\to+\infty} |u_n - u|_{L^{q(x)}(\Omega,\omega)} = 0$ *if and only if* $\lim_{n\to+\infty} \rho(u_n - u) = 0$.

Let us define the norm on X as follows:

$$\|u\| = \inf\left\{\mu > 0 \ : \ \int_\Omega \omega(x) \left|\frac{\Delta u(x)}{\mu}\right|^{q(x)} dx \leqslant 1\right\}.$$

Remark 2.2. Setting $J(u) = \int_\Omega \omega(x)|\Delta u|^{q(x)}dx$, we also have

$$\min(\|u\|^{q^-}, \|u\|^{q^+}) \leqslant J(u) \leqslant \max(\|u\|^{q^-}, \|u\|^{q^+}).$$

In order to ensure certain properties of the space X, we suppose that $w(.)$ fulfills the following

$$(\Im) \ \ \omega^{-\theta(x)} \in L^1(\Omega) \text{ for some } \theta \in C_+(\Omega) \text{ with } \theta(x) \in \left(\frac{N}{q(x)}, \infty\right) \cap \left[\frac{1}{q(x)-1}, \infty\right). \tag{2.4}$$

Corollary 2.1. *([25]) For $q \in C_+(\Omega)$, if $(\omega1)$ and $(\Im)$ hold, then the estimate*

$$|u|_{L^{q(x)}(\Omega)} \leqslant c|\nabla u|_{L^{q(x)}(\Omega,\omega)}$$

holds for each $u \in C_0^\infty(\Omega)$, where c is a positive constant independent of u.

3. Auxiliary results

In this section, we state and prove some auxiliary results about the weighted generalized Sobolev space that will be useful in the proof of our main results.

Proposition 3.1. *If $\omega \in \tilde{A}_{q(.)}(\Omega)$, then $X = W^{2,q(x)}(\Omega,\omega) \cap W_0^{1,q(x)}(\Omega,\omega)$ is a reflexive and Banach space.*

Proof. Let $\{u_n\} \subset X$ be a Cauchy sequence. Hence $\{u_n\}$ is a Cauchy sequence in $W_0^{1,q(x)}(\Omega,\omega)$ and $\{D_i u_n\}$ is a Cauchy sequence in $W^{1,q(x)}(\Omega,\omega)$ which is a Banach space. As a result there exist $u \in W_0^{1,q(x)}(\Omega,\omega)$ and $v_i \in W^{1,q(x)}(\Omega,\omega)$ such that

$$\mathrm{D}_i u_n \to D_i u \text{ in } L^{q(x)}(\Omega,\omega) \;\text{ and }\; \nabla(D_i u_n) \to \nabla v_i \text{ in } L^{q(x)}(\Omega,\omega,\mathbb{R}^N),\; i = 1,\cdots,N.$$

By Remark 2.1, Proposition 2.2 and the following relation

$$\int_\Omega D_i u_n.\nabla\varphi\, dx = -\int_\Omega \nabla(D_i u_n).\varphi\, dx,$$

it follows that

$$\int_\Omega D_i u.\nabla\varphi\, dx = -\int_\Omega \nabla v_i.\varphi\, dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

As $\nabla v_i \in L^{q(x)}(\Omega,\omega,\mathbb{R}^N)$, we obtain that $\nabla(D_i u) = \nabla v_i$. Hence the Cauchy sequence (u_n) converges to u in X and thus X is a Banach space. Since $\mathrm{W}^{1,q(x)}(\Omega,\omega)$ is a reflexive space (see [25]), and there exists an isometry T defined as follows:

$$\begin{aligned} T : &X \longrightarrow W^{1,q(x)}(\Omega,\omega) \times \left(W^{1,q(x)}(\Omega,\omega)\right)^N = Y, \\ &u \to (u, \nabla u). \end{aligned}$$

This yields that $T(X)$ is a closed subspace of Y which is a reflexive space, hence both $T(X)$ and X are reflexive spaces. This completes the proof. ☐

Theorem 3.1. *Assume that $\omega(.)$ fulfills $(\Im)$ and belongs to the class $\widetilde{A}_{q(.)}(\Omega)$. Then in the space X, the norms $\|.\|_{W^{2,q(x)}(\Omega,\omega)}$ and $|\Delta\, .|_{L^{q(x)}(\Omega,\omega)}$ are equivalents.*

Proof. Let u in $\pi = C_0^\infty(\Omega)$ and define $R_\epsilon^i(u)(x) = \int_{|x-y|>\epsilon} C_N \frac{x_i - y_i}{|x-y|^{N+1}} u(y)dy$.

Let $k(x,y) = C_N \frac{x_i - y_i}{|x-y|^{N+1}}$. The authors in ([33]) have shown that k is a standard kernel and satisfies (a) and (b) of [[8], Proposition 4.3], then it follows from Proposition 4.3 that for every s with $2 \le s < \infty$, the operators R_ϵ^i are uniformly bounded on $L^s(\mathbb{R}^N)$ with respect to ϵ. Furthermore

$$R^i(u)(x) = \lim_{\epsilon\to 0^+} R_\epsilon^i(u)(x) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N} k_\epsilon(x,y)u(y)dy, \tag{3.1}$$

exists almost everywhere and $\lim_{\epsilon\to 0^+} R_\epsilon^i(u) = R^i(u)$ in $L^s(\mathbb{R}^N)$. In particular R^i is continuous on $L^{q(.)}(\mathbb{R}^N)$.

From [[8], Remark 4.4], the operator R^i defined in (3.1) is a Calderon Zygmund operator. Now, consider the extension $\hat{\omega}$ of ω defined by $\hat{\omega} = \omega$ in Ω and $\hat{\omega} = 0$ in

$\mathbb{R}^N \backslash \Omega$. Thus $\hat{\omega} \in \tilde{A}_{q(.)}(\mathbb{R}^N)$. From [15], it follows that R^i is a bounded operator in $L^{q(x)}(\mathbb{R}^N, \hat{\omega})$, i.e.

$$|R^i(u)|_{L^{q(x)}(\mathbb{R}^N,\hat{\omega})} \leqslant C|u|_{L^{q(x)}(\mathbb{R}^N,\hat{\omega})} \quad \forall\, u \in L^{q(x)}(\mathbb{R}^N, \hat{\omega}),$$

where C is a positive constant.

Since $|R^i(u)|_{L^{q(x)}(\mathbb{R}^N,\hat{\omega})} = |R^i(u)|_{L^{q(x)}(\Omega,\omega)}$ and $|u|_{L^{q(x)}(\mathbb{R}^N,\hat{\omega})} = |u|_{L^{q(x)}(\Omega,\omega)}$, it follows that R^i is a bounded operator in the space $L^{q(x)}(\Omega, \omega)$, i.e.

$$|R^i(u)|_{L^{q(x)}(\Omega,\omega)} \leqslant C|u|_{L^{q(x)}(\Omega,\omega)} \quad \forall\, u \in L^{q(x)}(\Omega, \omega).$$

In view of ([31]), we know that $\frac{\partial^2 u}{\partial x_i \partial x_j} = -R^i R^j(\Delta u)$. Consequently, for any u in $\pi \subset L^{q(x)}(\Omega, \omega)$, we get

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{L^{q(x)}(\Omega,\omega)} = \left|R^i R^j(\Delta u)\right|_{L^{q(x)}(\Omega,\omega)} \leqslant C|\Delta u|_{L^{q(x)}(\Omega,\omega)}, \tag{3.2}$$

Let Γ be a Newtonian potential. According to [13], we have

$$|\nabla u| = |D\Gamma * (\Delta u)| \leqslant C_1 I_1(\Delta u) = \int_\Omega \frac{\Delta u(y)}{|x-y|^{N-1}} dy,$$

where C_1 depends only on N. It is well known (see [4,26]) that $I_1(\Delta u) \leqslant C_2 \mathcal{M}(\Delta u)$, where C_2 only depends on Ω and N. Therefore $|\nabla u| \leqslant C_3 \mathcal{M}(\Delta u)$. In view of [[26], Theorem A], the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded in $L^{q(x)}(\Omega, \omega)$, hence for every u in $\pi \subset L^{q(x)}(\Omega, \omega)$, one has

$$|\nabla u|_{L^{q(x)}(\Omega,\omega)} \leqslant C_3|\mathcal{M}(\Delta u)|_{L^{q(x)}(\Omega,\omega)} \leqslant C_4|\Delta u|_{L^{q(x)}(\Omega,\omega)}, \quad C_4 > 0. \tag{3.3}$$

From Remark 2.1, Corollary 2.1 and (3.3), for every u in π, we get

$$|u|_{L^{q(x)}(\Omega)} \leqslant C_5|\nabla u|_{L^{q(x)}(\Omega,\omega)} \leqslant C_6|\Delta u|_{L^{q(x)}(\Omega,\omega)}, \tag{3.4}$$

where C_5, C_6 are two positive constants. Combine (3.2), (3.3) and (3.4), we can claim that

$$|\Delta u|_{L^{q(x)}(\Omega,\omega)} \leqslant \|u\|_{W^{2,q(x)}(\Omega,\omega)} \leqslant C_7|\Delta u|_{L^{q(x)}(\Omega,\omega)}, \tag{3.5}$$

Since π is dense in X, the inequality (3.5) holds for any u in X, thus the proof is completed. □

Lemma 3.1. *([12]) Let $\alpha, r \in C_+(\Omega)$. Assume that $\alpha(x) < r_2^*(x)$ for every $x \in \bar{\Omega}$. Then there is a continuous and compact embedding from $W^{2,r(x)}(\Omega) \cap W_0^{1,r(x)}(\Omega)$ into $L^{\alpha(x)}(\Omega)$.*

Next, we will prove a compact embedding theorem for the weighted variable exponent Sobolev space.

Theorem 3.2. *Assume that $\omega(.)$ satisfies $(\Im)$ and belongs to the class $\widetilde{A}_{q(.)}(\Omega)$, then we have the compact embedding*

$$X \hookrightarrow L^{\gamma(x)}(\Omega),$$

provided that $\gamma \in C_+(\Omega)$ and $\gamma(x) < q_{\theta,2}^*(x) = \frac{Nq_\theta(x)}{N-2q_\theta(x)}$ for all $x \in \bar{\Omega}$.

Proof. Firstly, the embedding $X \hookrightarrow W^{2,q_\theta(x)}(\Omega) \cap W_0^{1,q_\theta(x)}(\Omega)$ is continuous. Indeed, let u in X, we have

$$\int_\Omega |\Delta u|^{q_\theta(x)}dx \leqslant 2\left|\omega^{\frac{\theta(x)}{\theta(x)+1}}|\Delta u|^{q_\theta(x)}\right|_{L^{\frac{\theta(x)+1}{\theta(x)}}(\Omega)}\left|\omega^{-\frac{\theta(x)}{\theta(x)+1}}\right|_{L^{\theta(x)+1}(\Omega)}.$$

By ($\Im$) and Lemma 2.1, we have

$$\left|\omega^{-\frac{\theta(x)}{\theta(x)+1}}\right|_{L^{\theta(x)+1}(\Omega)} \leqslant \left(\int_\Omega \omega^{-\theta(x)}(x)dx+1\right)^{\frac{1}{\theta_-+1}} \leq K.$$

This implies that

$$\int_\Omega |\Delta u|^{q_\theta(x)}dx \leqslant K\left|\omega^{\frac{\theta(x)}{\theta(x)+1}}|\Delta u|^{q_\theta(x)}\right|_{L^{\frac{\theta(x)+1}{\theta(x)}}(\Omega)}. \tag{3.6}$$

Without loss of generality, we can assume that $\int_\Omega |\Delta u|^{q_\theta(x)} > 1$. If $\int_\Omega \omega(x)|\Delta u|^{q(x)}dx < 1$, then from (3.6) and Lemma 2.1, we have

$$\begin{aligned}|\Delta u|^{\frac{q^-\theta^-}{\theta^-+1}}_{L^{q_\theta(x)}(\Omega)} &\leqslant \int_\Omega |\Delta u|^{q_\theta(x)}dx\\ &\leqslant K\left(\int_\Omega \omega(x)|\Delta u|^{q(x)}dx\right)^{\frac{\theta^-}{1+\theta^-}}\\ &\leqslant K|\Delta u|^{\frac{q^-\theta^-}{\theta^-+1}}_{L^{q(x)}(\Omega,\omega)},\end{aligned}$$

i.e.

$$|\Delta u|_{L^{q_\theta(x)}(\Omega)} \leqslant K|\Delta u|_{L^{q(x)}(\Omega,\omega)}. \tag{3.7}$$

On the other hand, if $\int_\Omega \omega(x)|\Delta u|^{q(x)}dx > 1$, then from (3.6) and Lemma 2.1, we have

$$\begin{aligned}|\Delta u|^{\frac{q^-\theta^-}{\theta^-+1}}_{L^{q_\theta(x)}(\Omega)} &\leqslant \int_\Omega |\Delta u|^{q_\theta(x)}dx\\ &\leqslant K\left(\int_\Omega \omega(x)|\Delta u|^{q(x)}dx\right)^{\frac{\theta^+}{1+\theta^+}}\\ &\leqslant K|\Delta u|^{\frac{q^+\theta^+}{\theta^++1}}_{L^{q(x)}(\Omega,\varrho)},\end{aligned}$$

i.e.

$$|\Delta u|_{L^{q_\theta(x)}(\Omega)} \leqslant K|\Delta u|^{\zeta}_{L^{q(x)}(\Omega,\omega)}, \tag{3.8}$$

where $\zeta = \frac{q^+\theta^+}{\theta^++1}.\frac{\theta^-+1}{q^-\theta^-}$. From inequalities (3.7) and (3.8), we get $\Delta u \in L^{q_\theta(x)}(\Omega)$.

Following the same lines as above, one can also show that $\nabla u \in L^{q_\theta(x)}(\Omega)$, and $D^\alpha u \in L^{q_\theta(x)}(\Omega)$, with $|\alpha| = 2$. Thus, we conclude that $X \hookrightarrow W^{2,q_\theta(x)} \cap W_0^{1,q_\theta(x)}(\Omega,\omega)$ continuously. Furthermore, as $\gamma(x) < q^*_{\theta,2}(x)$, it follows from Lemma 2.2 that

$$W^{2,q_\theta(x)}(\Omega) \cap W_0^{1,q_\theta(x)}(\Omega) \hookrightarrow L^{\gamma(x)}(\Omega).$$

Therefore $X \hookrightarrow L^{\gamma(x)}(\Omega)$ compactly, this completes the proof. □

4. Proof of main results

For $u \in X$, the functional energy associated with problem (1.1) is defined by $E_\lambda(u) = T(u) - \lambda\Psi(u)$, where

$$T(u) = \int_\Omega \omega \left(\frac{1}{p(x)} |\Delta u|^{p(x)} + \frac{1}{q(x)} |\Delta u|^{q(x)} \right) + \int_\Omega \omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla u|^{q(x)} \right),$$

$$\Psi(u) = \int_\Omega F(x,u)dx.$$

The functional T is Gateaux differentiable, and for u,v in X,

$$< T^{'}(u), v > = < N_p(u), v > + < N_q(u), v > + < M_p(u), v > + < M_q(u), v >,$$

where

$$< N_r(u), v > = \int_\Omega \omega(x) |\Delta u|^{r(x)-2} \Delta u \Delta v \, dx,$$

$$< M_r(u), v > = \int_\Omega \omega(x) |\nabla u|^{r(x)-2} \nabla u \nabla v \, dx, \quad \text{for } r = p, q.$$

Now we shall prove that $T^{'}$ is continuous. For that purpose, we only prove that N_q is continuous. Similarly one can show that N_p, M_p and M_q are continuous. Let $\{u_n\}$ be a sequence of X converging to u in X. For $v \in X$, we have

$$\begin{aligned}
&|< N_q(u_n) - N_q(u), v >| \\
&= \left| \int_\Omega \omega(x) \left(|\Delta u_n|^{q(x)-2} \Delta u_n - |\Delta u|^{q(x)-2} \Delta u \right) \Delta v dx \right| \\
&\leqslant \left| \int_\Omega \omega(x)^{\frac{q(x)-1}{q(x)}} \left(|\Delta u_n|^{q(x)-2} \Delta u_n - |\Delta u|^{q(x)-2} \Delta u \right) \omega(x)^{\frac{1}{q(x)}} \Delta v dx \right| \\
&\leqslant 2 \left| \omega^{\frac{q(x)-1}{q(x)}} \left(|\Delta u_n|^{q(x)-2} \Delta u_n - |\Delta u|^{q(x)-2} \Delta u \right) \right|_{L^{q'(x)}(\Omega)} \left| w^{\frac{1}{q(x)}} \Delta v \right|_{L^{q(x)}(\Omega)} \\
&\leqslant 2 \left| |\Delta u_n|^{q(x)-2} \Delta u_n - |\Delta u|^{q(x)-2} \Delta u \right|_{L^{q'(x)}(\Omega,\omega)} |\Delta v|_{L^{q(x)}(\Omega,\omega)} \\
&= 2 \left| |\Delta u_n|^{q(x)-2} \Delta u_n - |\Delta u|^{q(x)-2} \Delta u \right|_{L^{q'(x)}(\Omega,\omega)} \|v\|.
\end{aligned}$$

Since $u_n \to u$ in X, i.e, $\Delta u_n \to \Delta u$ in $L^{q(x)}(\Omega, \omega)$, we get

$$|\Delta u_n|^{q(x)-2} \Delta u_n \to |\Delta u|^{q(x)-2} \Delta u \text{ in } L^{q'(x)}(\Omega, \omega).$$

Therefore, from the above inequality, it follows that N_q is continuous on X.

Proposition 4.1. *(a) The operator $T^{'} : X \to X^{'}$ is of (S_+) type.*
(b) If f satisfies the condition $(\mathcal{F}_0)$, then we have
(i) Ψ is a C^1 functional. (ii) The operator $\Psi^{'} : X \to X^{'}$ is completely continuous.

Proof. (a) Let $\{u_n\} \subset X$ be such that

$$u_n \rightharpoonup u \textit{ in } X \textit{ and } \limsup_{n\to+\infty} < T^{'}(u_n), u_n - u > \leq 0.$$

It follows that

$$\limsup_{n\to+\infty} < T^{'}(u_n) - T^{'}(u), u_n - u >= 0, \quad \limsup_{n\to+\infty} \int_\Omega \omega(x)\sigma(u_n, u) = 0, \qquad (4.1)$$

where

$$\sigma(u_n, u) = (|\Delta u_n|^{q(x)-2}\Delta u_n - |\Delta u|^{q(x)-2}\nabla u)(\Delta u_n - \Delta u),$$
$$\textit{and } \varphi(u_n, u) = |\Delta u_n| + |\Delta u|.$$

In the other part, we have

$$\int_{\{q(x)\geq 2\}} \omega(x)|\Delta u_n - \Delta u|^{q(x)}dx \leqslant k_1 \int_\Omega \omega(x)\sigma(u_n, u)dx,$$
$$\int_{\{1<q(x)<2\}} \omega(x)|\Delta u_n - \Delta u|^{q(x)}dx$$
$$\leqslant k_2 \int_{\{1<q(x)<2\}} \omega(x)\sigma(u_n, u)^{\frac{q(x)}{2}} \varphi(u_n, u)^{(2-q(x))\frac{q(x)}{2}},$$

Setting $\Omega_q = \{x \in \Omega : 1 < q(x) < 2\}$, by (4.1) we can consider $0 \leqslant \int_\Omega \omega(x)\sigma(u_n, u) < 1$.

If $\int_\Omega \omega(x)\sigma(u_n, u) = 0$, then since $\omega(x)\sigma(u_n, u) \geqslant 0$ in Ω. It follows that, $\omega(x)\sigma(u_n, u) = 0$.

If $0 < \int_\Omega \omega(x)\sigma(u_n, u)dx < 1$, thanks to Young's inequality

$$\int_{\Omega_q} \omega(x)|\Delta u_n - \Delta u|^{q(x)}dx$$
$$\leqslant k_2 \left(\int_{\Omega_q} \omega(x)\sigma(u_n, u)\right)^{\frac{1}{2}} \int_{\Omega_q} \omega(x)\sigma(u_n, u)^{\frac{q(x)}{2}} \left(\int_{\Omega_q} \omega(x)\sigma(u_n, u)\right)^{\frac{-1}{2}} \varphi(u_n, u)^{(2-q(x))\frac{q(x)}{2}}$$
$$\leqslant k_2 \left(\int_{\Omega_q} \omega(x)\sigma(u_n, u)\right)^{\frac{1}{2}} \int_{\Omega_q} \omega(x) \left(\sigma(u_n, u) \left(\int_{\Omega_q} \omega(x)\sigma(u_n, u)\right)^{\frac{-1}{q(x)}} + \varphi(u_n, u)^{q(x)}\right)$$
$$\leqslant k_2 \left(\int_{\Omega_q} \omega(x)\sigma(u_n, u)\right)^{\frac{1}{2}} \left(1 + \int_\Omega \omega(x)\varphi(u_n, u)^{q(x)}\right).$$

Hence $J(u_n - u) \to 0$ as $n \to +\infty$. By Remark 2.1, we conclude that $u_n \to u$ in X. Thus $T^{'}$ is of (S_+) type.

(b) (i) By condition $(\mathcal{F}_0)$, we have

$$|F(x, s)| \leqslant C(|s| + |s|^{\alpha(x)}) \leqslant C^{'} + C|s|^{\alpha(x)}, \quad C^{'} > 0.$$

Using the Nemytski operator properties and the above implies that Ψ is a C^1 function in $L^{\alpha(x)}(\Omega)$. Since there is a continuous embedding of X into $L^{\alpha(x)}(\Omega)$, then we derive $\Psi \in C^1(X, \mathbb{R})$, and for every $u, v \in X$

$$< \Psi^{'}(u), v >= \int_\Omega f(x, u(x))v(x)\, dx.$$

(ii) Let $\{u_n\} \subset X$ be such that $u_n \rightharpoonup u$ weakly. Using the compact embedding of X into $L^{\alpha(x)}(\Omega)$, there exists a subsequence, also denoted by $\{u_n\}$, such that $u_n \to u$

in $L^{\alpha(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytski operator defined from $L^{\alpha(x)}(\Omega)$ into $L^{\alpha'(x)}(\Omega)$ by $N_f(u) = f(.,u)$ is continuous. Hence $f(.,u_n) \to f(.,u)$ in $L^{\alpha'(x)}(\Omega)$. In view of Proposition 2.1 and the continuous embedding of X into $L^{\alpha(x)}(\Omega)$, we get

$$| < \Psi'(u_n) - \Psi'(u), v > | \leqslant d\,|f(.,u_n) - f(.,u)|_{L^{\alpha'(x)}(\Omega)}\,\|v\|. \quad d > 0$$

Consequently $\Psi'(u_n) \to \Psi'(u)$ in X'. Hence Ψ' is completely continuous. □

4.1. Proof of Theorem 1.1

The functional E_λ is coercive on X. Indeed by contradiction, let $K \in \mathbb{R}$ and let $\{u_n\} \subset X$ be such that $\|u_n\| \to \infty$ and $E_\lambda(u_n) \leq K$. For n large enough, $\|u_n\| > 1$.

As $\alpha(x) < q^*_{\theta,2}(x)$ for all $x \in \bar{\Omega}$, then $X \hookrightarrow L^{\alpha(x)}(\Omega)$ continuously and from $(\mathcal{F}_0)$, one has

$$\begin{aligned} K \geqslant E_\lambda(u_n) &= T(u_n) - \lambda C\left[\frac{1}{\alpha^-}\int_\Omega |u_n|^{\alpha(x)}dx + \int_\Omega |u_n|dx\right] \\ &\geqslant T(u_n) - \lambda C\left[\frac{1}{\alpha^-}|u_n|^{\bar{\alpha}}_{L^{\alpha(x)}(\Omega)}dx + |u_n|_{L^1(\Omega)}\right] \\ &\geqslant \frac{\|u_n\|^{q^-}}{q^+} - \lambda C\left[\frac{c_1}{\alpha^-}\|u_n\|^{\bar{\alpha}} + c_2\|u_n\|\right]. \end{aligned}$$

Since $q^- > \alpha^+ > 1$, passing to the limit, we get a contradiction. As the functional E_λ is weakly lower semi-continuous, it follows that E_λ admits a minimum point $u_{1,\lambda} \in X$. Then $u_{1,\lambda}$ is a weak solution to (1.1). Now it remains to show that $u_{1,\lambda}$ is nontrivial.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \leqslant \varphi \leqslant l$ in $B(x_0,\rho_0)$ and $\varphi \equiv l$ in $B(x_0,\frac{\rho_0}{2})$. From $(\mathcal{F}_1)$, we have

$$\begin{aligned} &E_\lambda(\varphi) \\ &= T(\varphi) - \lambda\int_\Omega F(x,\varphi(x))dx \\ &= T(\varphi) - \lambda\left[\int_{\Omega\setminus B(x_0,\rho_0)} F(x,\varphi) + \int_{B(x_0,\rho_0)\setminus B(x_0,\frac{\rho_0}{2})} F(x,\varphi) + \int_{B(x_0,\frac{\rho_0}{2})} F(x,l)\right] \\ &\leqslant T(\varphi) - \lambda\int_{B(x_0,\frac{\rho_0}{2})} F(x,l). \end{aligned}$$

Take $\lambda^* = \frac{T(\varphi)}{\int_{B(x_0,\frac{\rho_0}{2})} F(x,l)dx}$. Hence for every $\lambda > \lambda^*$, we have $E_\lambda(\varphi) < 0$. Therefore, the problem (1.1) has at least one nontrivial solution for each $\lambda > \lambda^*$. This completes the proof.

4.2. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall verify both of the geometrical condition and the condition of compactness of the mountain pass theorem [2].

There exist $\eta, r > 0$ such that $E_\lambda(u) \geqslant \eta$ for every $u \in X$ with $\|u\| = r$. Indeed, let $u \in X$ with $\|u\| < 1$. Using $(\mathcal{F}_2)$, it follows that

$$|F(x,s)| \leqslant \frac{\epsilon}{\tau}|s|^\tau, \ \ a.e \ x \in \Omega, \ \forall |s| < \delta. \tag{4.2}$$

Combining $(\mathcal{F}_0)$ with (4.2) yields that

$$|F(x,s)| \leqslant C(\epsilon)|s|^\tau \ for \ a.e \ x \in \Omega, \ \forall s \in \mathbb{R} \ where \ C(\epsilon) = \frac{\epsilon}{\tau} + \frac{C}{\delta^{\tau-\bar{\alpha}}}\left(\frac{1}{\delta^{\bar{\alpha}-1}} + \frac{1}{\alpha^-}\right).$$

As $\tau < q^*_{\theta,2}(x)$, then $X \hookrightarrow L^\tau(\Omega)$ continuously. Therefore

$$E_\lambda(u) \geqslant T(u) - \lambda C(\epsilon)\int_\Omega |u|^\tau dx \geq \frac{\|u\|^{q^+}}{q^+} - \lambda C^{'}(\epsilon)\|u\|^\tau.$$

Take $\gamma = \left(\frac{1}{2q^+\lambda C^{'}(\epsilon)}\right)^{\frac{1}{\tau-q^+}}$ and $r = \min\left(\gamma, \|u_{1,\lambda}\|\right)$. Then $\|u_{1,\lambda}\| > r$, $E_\lambda(u_{1,\lambda}) < 0$ and

$$E_\lambda(u) \geqslant \frac{1}{2q^+}r^{q^+} = \eta \quad \forall u \in X \ \|u\| = r.$$

Now, it remains to show that E_λ satisfies the Palais-Smale condition. Indeed, let $(u_n)_n$ be a Palais-Smale sequence of X, i.e, $|E_\lambda(u_n)| \leq c$ and $E^{'}_\lambda(u_n) \to 0$ in $X^{'}$. From Theorem 3.1, the functional E_λ is coercive, hence $(u_n)_n$ is a bounded sequence in X reflexive. We may assume, taking a subsequence if necessary that $u_n \rightharpoonup u$ weakly in X. Hence

$$\lim_{n\to\infty} < E^{'}_\lambda(u_n), u_n - u >= \lim_{n\to\infty}\Big(< T^{'}(u_n), u_n - u > -\lambda < \Psi^{'}(u_n), u_n - u >\Big) = 0. \tag{4.3}$$

In view of Proposition 4.1, the operator $\Psi^{'}$ is completely continuous and $T^{'}$ is of (S_+) type, hence from (4.3), we conclude that $u_n \to u$ in X, therefore $(u_n)_n$ has a convergent subsequence. According to mountain pass theorem, the functional E_λ admits a critical value $c \geqslant \eta$ which is characterized by

$$c = \inf_{\vartheta\in\Gamma}\sup_{t\in[0,1]} E_\lambda(\vartheta(t)), \ \ \Gamma = \{\vartheta \in C([0,1]; X) : \vartheta(0) = 0 \text{ and } \vartheta(1) = u_{1,\lambda}\}.$$

Since $E_\lambda(u_{1,\lambda}) < 0 < c = E_\lambda(u_{2,\lambda})$, it follows that $u_{2,\lambda} \neq u_{1,\lambda}$. Therefore, the problem (1.1) has at least two nontrivial solutions.

Example 4.1. Consider the problem (1.1) with f defined by

$$f(x,s) = \begin{cases} |s|^{\beta(x)-2}s \text{ if } \ |s| > 1, \\ |s|^{\gamma(x)-2}s \text{ if } \ |s| < 1, \end{cases}$$

where $\beta(.), \gamma(.) \in C_+(\Omega)$ and $\beta(x) < \gamma(x) < q^*_{\theta,2}(x)$ for any $x \in \bar{\Omega}$.

If $\beta^+ < q^-$, by Theorem 3.1, there exists $\lambda^* > 0$ such that when $\lambda > \lambda^*$, the problem (1.1) has at least one nontrivial solution.

If in addition $\gamma^- > q^+$, the problem (1.1) has at least two nontrivial weak solutions for any $\lambda > \lambda^*$ thanks to Theorem 3.2.

5. Conclusion

In this study, we introduced a class of weights $\tilde{A}_{q(.)}(\Omega)$ in order to assure some basic properties of the space $W^{2,q(.)}(\Omega,\omega) \cap W_0^{1,q(.)}(\Omega,\omega)$, along with the equivalence of norms, then by applying variational approach and critical point theory, we established the existence of at least two distinct nontrivial weak solutions for the fourth-order double phase problem (1.1) governed by singular operators, for any $\lambda > \lambda^*$ under some suitable hypotheses on the nonlinearity f.

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