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Local Bifurcation Cyclicity for a Non-Polynomial System*

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Abstract In this paper, we propose a class of general non-polynomial analytic oscillator models, and study the limit cycle bifurcation at the nilpotent singularity or elementary center-focus. By Taylor expansion, two specific systems from the original model are transformed into two equivalent infinite polynomial systems, and the highest order of fine focus as the nilpotent Hopf bifurcation or Hopf bifurcation point is determined respectively. At the same time, the local bifurcation cyclicities and center problems for two systems are solved respectively. To our knowledge, such dynamic properties are rarely analyzed in many non-polynomial models.

Keywords Non-polynomial system, quasi-Lyapunov constant, nilpotent singularity, Hopf bifurcation

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1. Introduction

In this paper, we study the following weak perturbation nonlinear oscillator model:

$$\ddot{x} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x + h(x, \dot{x}), \tag{1.1}$$

where $\kappa_1 > 0$, $\kappa_2 > 0$ and $h(x, \dot{x})$ as a perturbation part is any smooth function on $\mathbb{R}^2$. Its background comes from a classic case which often appears in college physics textbooks: a class of overdamped ball motion models on a rotating ring [24]. The trajectory of the ball can be described by the following second-order ordinary differential equation.

$$mr\ddot{x} = -b\dot{x} - mg \sin x + mr\omega^2 \sin x \cos x, \tag{1.2}$$

where x is the swing angle, $r\ddot{x}$ is the acceleration, mg is the gravity, $mr\omega^2 \sin x \cos x$ is the lateral centrifugal force, the $b\dot{x}$ is the tangential damping force. This model shows rich dynamics properties, including various bifurcations [24].

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For the undamped case, i.e., when $b = 0$, the energy Hamilton function corresponding to model (1.2) can be written as follows:

$$H = \frac{\dot{x}^2}{2} - \frac{g}{r}\cos x + \frac{\omega^2}{2}\cos^2 x. \tag{1.3}$$

In fact, the existence of viscous damping in the actual background makes the target ball in model (1.2) be affected by nonlinear damping, which is similar to the Van der Pol oscillator model [5] and Rayleigh oscillator model [23], described by

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x, \quad \text{and} \qquad \ddot{x} = \mu(1 - \dot{x}^2)\dot{x} - x,$$

respectively, where μ is a scalar parameter representing the strength of the damping. Therefore, the term of damping force $b\dot{x}$ can be extended in the model (1.2), and we can propose the general model (1.1).

Furthermore, letting $\dot{x} = y$, and taking $h(x, \dot{x})$ as a specific polynomial function, we can transform model (1.1) into the non-polynomial analytic system as follows,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x + h(x, y), \end{cases} \tag{1.4}$$

where $h(x, y) = ax + by + \sum_{2 \le i+j \le n} b_{ij} x^i y^j$, and $a, b, b_{ij} \in \mathbb{R}$, $n \in \mathbb{Z}$. Motivated by many research works on the cyclicity of piecewise smooth systems or non-polynomial Hamiltonian systems, e.g. [11, 12, 27], we will study such a non-polynomial system on the local bifurcation cyclicity, that is, the maximum number of small amplitude limit cycles that can bifurcate in the vicinity of equilibrium.

For the trigonometrical functions in some practical models, the approximations: $\sin x \approx x$ and $\cos x \approx 1$ are usually used when x is small. However, such approximations are inappropriate in solving the local bifurcation cyclicity of the original system. It is easily checked that the Jacobian matrix of (1.4) at its origin

$$J = \begin{pmatrix} 0 & 1 \\ a + \kappa_2 - \kappa_1 & b \end{pmatrix} \tag{1.5}$$

has a pair of conjugate imaginary eigenvalues if and only if $b = 0, a + \kappa_2 - \kappa_1 < 0$, namely the origin is a Hopf bifurcation point. Furthermore, if and only if $b = 0$, and $a + \kappa_2 - \kappa_1 = 0$, the Jacobian matrix (1.5) has double zero eigenvalues, i.e., the origin is a nilpotent singularity.

In this paper, we consider the two above categories of equilibria for system (1.4). For the former, to solve its Hopf bifurcation cyclicity in a planar polynomial vector field of degree $n \ge 2$, the general approach is to determine the highest order of fine focus by computing the focal values or Lyapunov constants and finding the center conditions. There have been some classic methods, and the reader can see [13, 14, 17, 21, 25]. While the quadratic case has been completely solved, that is, Hopf bifurcation cyclicity 3 was proven by Bautin [4], while for $n > 2$, this problem is still open. More recent new progress can be found in [9, 26] and references therein. Here to overcome the difficulty arising from such non-polynomial functions, we will adopt Taylor expansion to transform the original model into its equivalent infinite polynomial systems, then apply the method proposed in [17] to calculate the focal values.

For the latter, as a degenerate case, the nilpotent singularity is much more complicated than elementary ones in the topological structure of phase orbits near its neighborhood. Consider the following generic planar systems with a nilpotent singularity,

$$\begin{cases} \dot{x} = y + \sum\limits_{k+j=2}^{\infty} a_{kj}x^k y^j = \Phi(x,y), \\ \dot{y} = \sum\limits_{k+j=2}^{\infty} b_{kj}x^k y^j = \Psi(x,y), \end{cases} \tag{1.6}$$

where $x, y, a_{kj}, b_{kj} \in \mathbb{R}\,(k, j \in \mathbb{N})$, and the functions $\Phi(x,y), \Psi(x,y)$ are analytic in the neighborhood of the origin. Lyapunov provided the sufficient and necessary conditions as the center-focus type for the origin of the system (1.6) in [2].

Similar to the former nondegenerate case, calculating the focal value and detecting nilpotent center are needed. There also exist several available and classical ways, such as the normal form theory [22], Lyapunov function [6] and Poincare return map [1]. Some good results can be seen in [3,7,8]. It is worth mentioning that the integral factor method of calculating the quasi-Lyapunov constants proposed in [19], is convenient to compute the higher order focal values and solve the center-focus problem of the three-order nilpotent critical point. In view of this, several planar cubic system have been investigated recently, e.g. [15,16,18]. Here we extend this method to solve the nilpotent Hopf bifurcation for the above system (1.4) via Taylor's expansion.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic concepts and the quasi-Lyapunov constant method used for nilpotent Hopf bifurcation of planar Lyapunov systems. In Section 3, the original model with a three-order nilpotent singularity is transformed into an equivalent infinite polynomial system by Taylor's expansion, and the highest order of the nilpotent focus is determined. At the same time, the nilpotent Hopf bifurcation cyclicity is solved. In Section 4, for one specific system with an elementary center-focus, similarly the highest order of the focus is determined and the Hopf bifurcation cyclicity is solved. Local bifurcation cyclicity for a non-polynomial system has been less studied, and this problem is interesting.

2. Preliminary knowledge on nilpotent singularity

In this section, we give the definition and method on symbolic computation of quasi-Lyapunov constant for the nilpotent singularity. For more details, see [19,20].

Lemma 2.1 ([19])**.** *The origin of system (1.6) is a third-order monodromic critical point if and only if*

$$b_{20} = 0,\ (2a_{20} - b_{11})^2 + 8b_{30} < 0. \tag{2.1}$$

Without loss of generality, we assume that $a_{20} = \mu,\ b_{20} = 0,\ b_{11} = 2\mu,\ b_{30} = -2$. Otherwise, by letting

$$2a_{20} + b_{11} = 4\lambda\mu, (2a_{20} - b_{11})^2 + 8b_{30} = -16\lambda^2, \tag{2.2}$$

and making the transformation

$$\xi = \lambda x, \eta = \lambda y + \frac{1}{4}(2a_{20} - b_{11})\lambda x^2, \tag{2.3}$$

we can convert (1.6) into the following form, i.e., a Lyapunov system [20]:

$$\begin{cases} \dot{x} = y + \mu x^2 + \sum\limits_{k+2j=3}^{\infty} a_{kj}x^k y^j = X(x,y), \\ \dot{y} = -2x^3 + 2\mu xy + \sum\limits_{k+2j=4}^{\infty} b_{kj}x^k y^j = Y(x,y), \end{cases} \tag{2.4}$$

where $x, y, \mu, a_{kj}, b_{kj} \in \mathbb{R}\,(k, j \in \mathbb{N})$. By using the transformation of generalized polar coordinates:

$$x = r\cos\theta, y = r^2\sin\theta, \tag{2.5}$$

system (2.4) can be converted into

$$\tfrac{\mathrm{d}r}{\mathrm{d}\theta} = \tfrac{-\cos\theta[\sin\theta(1-2\cos^2\theta)+\mu(\cos^2\theta+2\sin^2\theta)]}{2(\cos^4\theta+\sin^2\theta)} r + o(r). \tag{2.6}$$

When r is sufficiently small, $\frac{\mathrm{d}\theta}{\mathrm{d}t} < 0$. So the successor function $\Delta(h)$ of Eq.(2.6) in the neighborhood of the origin is written as

$$\Delta(h) = r(-2\pi, h) - h = \sum_{m=2}^{\infty} v_m(-2\pi)h^m, \tag{2.7}$$

where the quantity of $v_{2k}(-2\pi)$ is called the k-th focal value at the origin, $k = 1, 2, \cdots$.

Further, we give the calculation formula of quasi-Lyapunov constant.

Lemma 2.2 ([19]). *For system (2.4), any positive integer s and a given number sequence $\{c_{0\beta}\}, \beta \geqslant 3$, we can derive successively and uniquely the terms of the following formal series with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$,*

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta \tag{2.8}$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty} \omega_m(s,\mu)x^m, \tag{2.9}$$

where $s\mu = 0$. And if $\alpha \neq 0$, $c_{\alpha\beta}$ is determined by the following recursive formula:

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha} \sum\nolimits_{k+j=2}^{\alpha+\beta-1} \Phi_{kj}, \tag{2.10}$$

and for any positive integer $m \geqslant 3$, ω_m is determined by the following recursive formula:

$$\omega_m = \sum\nolimits_{k+j=2}^{m-1} [\Phi_{kj}]_{\alpha=m+1,\beta=-1}, \tag{2.11}$$

where

$$\Phi_{kj} = [k-(s+1)(\alpha-k)]a_{kj}c_{\alpha-k,\beta-j+1} + [j-(s+1)(\beta-j+2)]b_{kj}c_{\alpha-k-1,\beta-j+2}$$

and when $\alpha < 0$ or $\beta < 0$, we set $c_{\alpha\beta} = 0$.

Remark 2.1. About the calculation of the above formulas (2.10) and (2.11) for each ω_m, only the coefficients of polynomials of degree $m-1$ or less are involved in system (2.4). Particularly, by choosing appropriate s and number sequence $\{c_{0\beta}\}$ ($\beta \geqslant 3$), we can make $\omega_{2i+1}(s,\mu)=0$, then let

$$\omega_{2i+4}(s,\mu)=(2i-4s-1)\lambda_i \tag{2.12}$$

where λ_i is said the i-th quasi-Lyapunov constants of the origin of system (2.4), with $i=1,2,\cdots$.

Lemma 2.3 ([20]). *For system (2.4), and any positive integer m, the following assertion holds: $v_{2m}(-2\pi)\sim\sigma_m\lambda_m$, namely*

$$v_{2m}(-2\pi)=\sigma_m\lambda_m+\sum_{k=1}^{m-1}\xi_m^{(k)}\lambda_m, \tag{2.13}$$

where $\xi_m^{(k)}(k=1,2,\cdots,m-1)$ are polynomial functions of coefficients of system (2.4), and

$$\sigma_m=\frac{1}{2}\int_0^{2\pi}\frac{(1+\sin^2\theta)\cos^{2m+4}\theta}{(\cos^4\theta+\sin^2\theta)^{s+2}}v_1^{2m-4s-1}(\theta)\mathrm{d}\theta>0. \tag{2.14}$$

Then the relation between $v_{2m}(-2\pi)$ and $\sigma_m\lambda_m$ is called algebraic equivalence.

Consider the perturbed system of system (2.4):

$$\dot{x}=X(x,y),\quad \dot{y}=\delta_1x+\delta_2y+Y(x,y), \tag{2.15}$$

where $0<|\delta_1|,|\delta_2|\ll h\ll 1$, and X, Y are given by system (2.4). Obviously, when $\delta_1<0$, in a neighborhood of the origin, there exists one elementary focus at the origin and two complex critical points of system (2.15). And when $\delta_1\to 0$ and $\delta_2\to 0$, the three points coincide to become a third-order critical point. Thus, similar to the proof of Theorem 4.7 in [20], we have the following same conclusion.

Lemma 2.4 ([20]). *Suppose that the first m focus values depend on $m-1$ parameters $\gamma=\{\gamma_1,\gamma_2,\cdots,\gamma_{m-1}\}$, expressed as $v_{2j}=v_j(-2\pi,\gamma)$, $j=2,4,\cdots,2m$. And when $\delta_1=\delta_2=0$, if there exists $\gamma=\gamma^{(0)}=\{\gamma_1^{(0)},\gamma_2^{(0)},\cdots,\gamma_{m-1}^{(0)}\}$ such that*

$$\begin{aligned}&v_{2k}(\gamma^{(0)})\neq 0,\quad v_j(\gamma^{(0)})=0,\quad j=2,4,\cdots,2m-2,\\&\det\left[\frac{\partial(v_2,v_4,\cdots,v_{2m-2})}{\partial(\gamma_1^{(0)},\gamma_2^{(0)},\cdots,\gamma_{m-1}^{(0)})}(\gamma^{(0)})\right]\neq 0,\end{aligned} \tag{2.16}$$

then the origin of the perturbed system (2.15) has exactly m limit cycles.

3. Nilpotent Hopf bifurcation and center problem

In this section, we will consider the nilpotent Hopf bifurcation and center problem of the system (1.4). Actually, we let $a=b=0$ and $\kappa_1=\kappa_2=K>0$ in system (1.4), yielding the origin as a three-order nilpotent singularity, and take $h(x,y)$ a mixed polynomial of degree 4 as follows,

$$\dot{x}=y,\ \ \dot{y}=-K\sin x(1-\cos x)+b_{02}y^2+Y_3+Y_4, \tag{3.1}$$

where

$$Y_3 = b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \;\; Y_4 = b_{40}x^4 + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \tag{3.2}$$

Since the system (3.1) contains trigonometric functions and is a non-polynomial analytic system, Lemma 2.2 cannot be directly applied to determine the quasi-Lyapunov constant of the origin unless the system is polynomial transformed. According to Remark 2.1, by performing Taylor expansion at the origin on the right side of the system to the terms of degree 18, we can strictly determine the first $\omega_m, m \leq 18$, i.e., we take

$$\begin{aligned}\sin x(1-\cos x) = {} & \tfrac{x^3}{2} - \tfrac{x^5}{8} + \tfrac{x^7}{80} - \tfrac{17x^9}{24192} + \tfrac{31x^{11}}{1209600} - \tfrac{x^{13}}{1520640} \\ & + \tfrac{5461x^{15}}{435891456000} - \tfrac{257x^{17}}{1394852659200} + o(x^{18}).\end{aligned} \tag{3.3}$$

Obviously, by introducing the transformation:

$$\xi = \frac{\sqrt{K}}{2}x, \quad \eta = \frac{\sqrt{K}}{2}y, \tag{3.4}$$

system (3.1) is converted into the following form with the same as system (2.4) :

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x^3 + \frac{2b_{02}y^2}{\sqrt{K}} + \frac{2x^5}{K} - \frac{4x^7}{5K^2} + \frac{34x^9}{189K^3} - \frac{124x^{11}}{4725K^4} + \frac{4x^{13}}{1485K^5} \\ \quad - \frac{43688x^{15}}{212837625K^6} + \frac{514x^{17}}{42567525K^7} + \frac{4Y_3}{K} + \frac{8Y_4}{K\sqrt{K}} + o(x^{18}), \end{cases} \tag{3.5}$$

where ξ, η is still referred to as x, y.

Applying the powerful symbolic computation function of Mathematica system and the recursive formulas in Lemma 2.2, where c_{kj}, ω_m in (2.10) (2.11) can be found in the website: https://github.com/wql2001399/wql. We obtain the first 18 quantities as follows:

$$\begin{array}{ll} \omega_3 = 0, & \omega_4 = 0, \\ \omega_5 = 0, & \omega_6 = -\frac{4}{3K}b_{21}(4s-1), \\ \omega_7 = 3(1+s)c_{03}, & \omega_8 = -\frac{24}{5K}b_{03}(4s-3), \\ \omega_9 = -\frac{16}{K^{3/2}}b_{13}(s-1), & \omega_{10} = \frac{32}{35K^3}b_{13}\left(11b_{02}K + 6b_{40}\right), \\ \omega_{11} = \frac{121}{16}(\varphi_1 + c_{05}), & \omega_{12} = \frac{2176}{105K^3}b_{13}(b_{22} + \varphi_2), \\ \omega_{13} = \frac{192}{5K^{3/2}}b_{13}(\varphi_3 - c_{04}), & \omega_{14} = \frac{4352}{231K^3}b_{13}(b_{04} + \varphi_4), \\ \omega_{15} = \frac{35}{4}(\varphi_5 + c_{07}), & \omega_{16} = -\frac{64}{36240584216265K^{12}}b_{13}b_{40}\varphi_6, \\ \omega_{17} = \frac{3072}{35K^{3/2}}b_{13}(\varphi_7 - c_{06}), & \omega_{18} = \frac{128}{391371129097499801 25K^{15}}b_{13}b_{40}\varphi_8, \end{array} \tag{3.6}$$

where

$$
\begin{aligned}
\varphi_1 &= (3388b_{12}b_{13}K^2 + 484b_{13}K^2 + 3888b_{13}b_{40}^2)/(1815\,K^{9/2}),\\
\varphi_2 &= (24684b_{12}b_{40}K^2 + 23595b_{40}K^2 + 135648b_{40}^3)/(45254\,K^3),\\
\varphi_3 &= (64713220b_{12}^2K^4 + 21156245b_{12}K^4 + 164395440b_{12}b_{40}^2K^2\\
&\quad + 71743320b_{40}^2K^2 + 284562720b_{40}^4 + 1991176K^4)/(44801460\,K^6),\\
\varphi_4 &= (87553180b_{12}b_{40}K^4 + 74303075b_{40}K^4 + 448842240b_{12}b_{40}^3K^2\\
&\quad + 2269650240b_{40}^3K^2 + 7705663488b_{40}^5)/(186174956K^6),
\end{aligned}
\tag{3.7}
$$

and φ_5, φ_6, φ_7, φ_8 are given in Appendix A. In the above expression of each ω_k, we have already let $\omega_{k-1} = 0$, $k = 3, 4, \cdots, 18$.

In particular, in order to make $\omega_{2i+1} = 0$, $i = 3, 4, \cdots, 8$, we can let $c_{03} = 0$, $c_{04} = \varphi_3$, $c_{05} = -\varphi_1$, $c_{06} = -\varphi_3$, $c_{07} = \varphi_5$ and $s = 1$ at the same time. Thus, the corresponding first 7 quasi-Lyapunov constants can be calculated from the expression (2.12), then the following conclusion is obtained.

Theorem 3.1. *For the system (3.5), we get the first 7 quasi-Lyapunov constants of the origin as follows:*

$$
\begin{aligned}
\lambda_1 &= \tfrac{4}{3K}b_{21},\\
\lambda_2 &= \tfrac{24}{5K}b_{03},\\
\lambda_3 &= \tfrac{32}{35K^3}b_{13}\left(11b_{02}K + 6b_{40}\right),\\
\lambda_4 &= \tfrac{2176}{315K^3}b_{13}(b_{22} + \varphi_2),\\
\lambda_5 &= \tfrac{4352}{1155K^3}b_{13}(b_{04} + \varphi_4),\\
\lambda_6 &= -\tfrac{64}{253684089513855K^{12}}b_{13}b_{40}\varphi_6,\\
\lambda_7 &= \tfrac{128}{35223401618774982 1125K^{15}}b_{13}b_{40}\varphi_8,
\end{aligned}
\tag{3.8}
$$

where for each λ_k, we have already let $\omega_{k-1} = 0$, $k = 1, 2, \cdots, 7$.

From Theorem 3.1, we have the following theorem.

Theorem 3.2. *The order of the origin of the system (3.5) as a nilpotent fine focus is 7 if and only if*

$$
b_{21} = b_{03} = 0,\ b_{02} = -\tfrac{6}{11K}b_{40},\ b_{22} = -\varphi_2,\ b_{04} = -\varphi_4,\ \varphi_6 = 0,\ b_{13}b_{40} \neq 0. \tag{3.9}
$$

Proof. (i) Obviously, when the conditions (3.8) hold, from Theorem 3.1, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 0$. Then, $\lambda_7 \neq 0$, i.e., $\varphi_8 \neq 0$ should be proved. In fact, under the condition $\varphi_6 = 0$ in (3.8), we can obtain $\varphi_8 \neq 0$ by calculating the resultant of φ_8 and φ_6 with respect to b_{12} as follows:

$$
\text{Resultant}[\varphi_8, \varphi_6, b_{12}] = K^{12}(8064b_{40}^2 + 1573K^2)H_{20} \neq 0, \tag{3.10}
$$

where $H_{20} = \sum_{i=0}^{10} q_i b_{40}^{2i}$ is a polynomial only in b_{40}^2 with all $q_i > 0$. Thus the origin of the system (3.5) is a nilpotent fine focus of degree 7.

Furthermore, we can find one group of valid critical values such that the conditions (3.9) hold, for example,

$$b_{13} = b_{40} = 1,\ b_{12} = \frac{3(6326595715K^4+181712433408K^2+615046625280)}{13428580K^2(1573K^2+8064)}, \tag{3.11}$$

where $K = K_0 \approx \pm 7.0302$ are two real roots of the following equation: $3144305649254400+1217227620787200K^2+73347310051488K^4-2008406304773K^6=0$.

(ii) Conversely, from $\lambda_1 = \lambda_2 = \cdots = \lambda_6 = 0$, $\lambda_7 \neq 0$ in Theorem 3.1 and the resultant (3.10), the necessity is easily proven. □

From Theorem 3.2, we have the following theorem.

Theorem 3.3. *The first 7 quasi-Lyapunov constants all disappear, and the origin of the system (3.1) is a center if and only if one of the following two groups of conditions hold:*

$$\begin{aligned} &\text{(i) } b_{21} = b_{03} = b_{13} = 0, \\ &\text{(ii) } b_{21} = b_{03} = b_{22} = b_{40} = b_{04} = b_{02} = 0, s = 1. \end{aligned} \tag{3.12}$$

Proof. For the case (i), we obtain the original system

$$\dot{x} = y,\ \dot{y} = b_{02}y^2 + b_{04}y^4 + b_{12}xy^2 + b_{22}x^2y^2 + b_{40}x^4 - K\sin x(1-\cos x), \tag{3.13}$$

which defines a vector field symmetric with respect to the x-axis. For the case (ii),

$$\dot{x} = y, \quad \dot{y} = b_{12}xy^2 + b_{13}xy^3 - K\sin x(1-\cos x), \tag{3.14}$$

which defines a vector field symmetric with respect to the y-axis. From these, we obtain the origin of the system (3.1) is a center, namely the sufficiency of two conditions in (3.12) is proven.

On the other hand, from Theorem 3.2, the necessity is obvious. □

Furthermore, from Theorems 3.2 and 3.3, we have the following corollary.

Corollary 3.1. *The origin of the system (3.1) is a nilpotent fine focus of order 7 at most, and the first seven focal values are respectively:*

$$v_{2j}(-2\pi) = \sigma_j\lambda_j, 1 \leq j \leq 7, \tag{3.15}$$

where each $\sigma_j > 0$, *and in the above expression of each* v_{2k}, *we have already let* $v_{2k-2} = 0,\ k = 1, 2, \cdots, 7$.

Next, we will apply the condition (3.9) to discuss the nilpotent Hopf bifurcation of the origin of the system (3.1), and obtain the following conclusion.

Theorem 3.4. *Under appropriate perturbations, system (3.1) can generate at most 7 small-amplitude limit cycles in the neighborhood of the origin via nilpotent Hopf bifurcation.*

Proof. We first set the parameters $a = \delta_1$, $b = \delta_2$ and $\kappa_1 = \kappa_2$ in the original system (1.4), which makes the system (3.1) become the perturbed form as (2.15).

Under the condition (3.9), from Theorem 3.1 and Corollary 3.1, we have $v_{14} \neq 0$, and easily calculate the Jacobian determinant of the focal values $v_2, v_4, \cdots, v_{12}$ with respect to the variables $b_{02}, b_{21}, b_{12}, b_{03}, b_{22}, b_{04}, b_{40}$:

$$J = \frac{\partial(v_2,v_4,v_6,v_8,v_{10},v_{12})}{\partial(b_{02},b_{21},b_{12},b_{03},b_{22},b_{04},b_{40})} = \frac{8589934592}{8398039524249375K^{20}}\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6 b_{13}^4 b_{40}\varphi_9, \quad (3.16)$$

where $\varphi_9 = 216576138240b_{12}b_{40}^2K^2 + 42246312680b_{12}K^4 + 1845139875840b_{40}^4 + 545137300224b_{40}^2K^2 + 18979787145K^4$. Further, under the condition $\varphi_6 = 0$ in (3.8), we can verify $\varphi_9 \neq 0$ by calculating

$$\mathrm{Resultant}[\varphi_9, \varphi_6, b_{12}] = K^4(8064b_{40}^2 + 1573K^2)H_8 \neq 0, \quad (3.17)$$

where $H_8 = \sum_{i=0}^{4} r_i b_{40}^{2i}$ is a polynomial only in b_{40}^2 with all $r_i > 0$. Thus under the conditions (3.8), we have that $J \neq 0$ holds.

By applying Lemma 2.4, we know that certain appropriate perturbations can generate exactly 7 small-amplitude limit cycles at neighborhood of the origin for system (3.1), and Theorem 3.3 implies that the highest order of the origin as the fine focus is 7, yielding at most 7 small-amplitude limit cycles via nilpotent Hopf bifurcation. Therefore the proof is completed. □

4. Hopf bifurcation and center problem

In this section, we consider the Hopf bifurcation and center problem at the origin as an elementary singularity for system (1.4). When taking $h(x, y)$ as a combination of the linear terms and a complete homogeneous polynomial of degree 3, we have system (1.4) as follows,

$$\dot{x} = y, \quad \dot{y} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x + ax + by + Y_2, \quad (4.1)$$

where $Y_2 = b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3$. For investigating Hopf bifurcation, we apply the singular point quantity method given in [17]. Similarly, we need to transform the non-polynomial analytic system (4.1) into the polynomial form. Thus also by performing Taylor expansion at the origin on the right side of the system, we take the following form to the terms of degree 16,

$$\begin{aligned}\kappa_2 \sin x \cos x - \kappa_1 \sin x = {} & x(\kappa_2 - \kappa_1) + \tfrac{1}{6}x^3(\kappa_1 - 4\kappa_2) + x^5\left(\tfrac{2\kappa_2}{15} - \tfrac{\kappa_1}{120}\right) \\ & + x^7\left(\tfrac{\kappa_1}{5040} - \tfrac{4\kappa_2}{315}\right) + x^9\left(\tfrac{2\kappa_2}{2835} - \tfrac{\kappa_1}{362880}\right) + x^{11}\left(\tfrac{\kappa_1}{39916800} - \tfrac{4\kappa_2}{155925}\right) \\ & + x^{13}\left(\tfrac{4\kappa_2}{6081075} - \tfrac{\kappa_1}{6227020800}\right) + x^{15}\left(\tfrac{\kappa_1}{1307674368000} - \tfrac{8\kappa_1}{638512875}\right) + o(x^{16}).\end{aligned}$$

Then utilizing the condition of its Jacobian matrix (1.5) at the origin as Hopf bifurcation point, we set

$$b = 0, \quad a = \kappa_1 - \kappa_2 - p^2, \quad (4.2)$$

where $p > 0$. Then the matrix J has a pair of imaginary eigenvalues $\pm\mathbf{i}p$ with $\mathbf{i} = \sqrt{-1}$. Furthermore, by introducing a matrix P which transforms J into a diagonal one, we obtain

$$P^{-1}JP = \begin{pmatrix} \mathbf{i}p & 0 \\ 0 & -\mathbf{i}p \end{pmatrix} \text{ with } P = \begin{pmatrix} -\mathbf{i}/p & \mathbf{i}/p \\ 1 & 1 \end{pmatrix}. \tag{4.3}$$

In fact, using the transformation: $(x, y)' = P(z, w)'$, and after a time scaling: $T = \mathbf{i}pt$, system (4.1) can become the following complex symmetrical system,

$$\begin{aligned} \tfrac{\mathrm{d}z}{\mathrm{d}T} &= z + \textstyle\sum_{m=2}^{13} Z_m + o(|z|^{16}) = Z, \\ \tfrac{\mathrm{d}w}{\mathrm{d}T} &= -w - \textstyle\sum_{m=2}^{13} W_m - o(|w|^{16}) = -W, \end{aligned} \tag{4.4}$$

where

$$Z_m = \sum\nolimits_{k+j=m} \tilde{a}_{kj} z^k w^j, \quad W_m = \sum\nolimits_{k+j=m} \tilde{b}_{kj} w^k z^j, \ k, j \in \mathbb{N},$$

which are given in the website: https://github.com/wql2001399/wql. Actually, we have $w = \overline{z}$ and $\tilde{b}_{kj} = \overline{\tilde{a}_{kj}}$, thus systems (4.4) and (4.1) are called conjugate. When there exists no misunderstanding, $\tilde{a}_{kj}$ and $\tilde{b}_{kj}$ are still written as a_{kj} and b_{kj} for system (4.4).

Lemma 4.1 ([17]). *For system (4.4), when taking $c_{11} = 1, c_{20} = c_{02} = 0, c_{kk} = 0, k = 2, 3, \cdots$, we can derive successively and uniquely the terms of the following formal series:*

$$F(z, w) = zw + \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} z^\alpha w^\beta,$$

such that

$$\frac{\mathrm{d}F}{\mathrm{d}T} = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}. \tag{4.5}$$

If $\alpha \neq \beta$, $c_{\alpha\beta}$ is determined by the following recursive formula:

$$c_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta} [(\alpha - k + 1)a_{k,j-1} - (\beta - j + 1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1}, \tag{4.6}$$

and for any positive integer m, μ_m is determined by the following recursive formula:

$$\mu_m = \sum_{k+j=3}^{2m+2} [(m - k + 2)a_{k,j-1} - (m - j + 2)b_{j,k-1}]c_{m-k+2,m-j+2}. \tag{4.7}$$

When $\alpha = \beta > 0$ or $\alpha < 0$ or $\beta < 0$, we set $c_{\alpha\beta} = 0$. And μ_m is called the m-th singular point quantity at the origin of the system, $m = 1, 2, \cdots$.

Remark 4.1. Here the calculation of the above formulas $c_{\alpha\beta}$ in (4.6) and μ_m in (4.7), only the coefficients of polynomials of degree $2m + 1$ or less are involved in system (4.4), this is to say, for the origin of system (4.1) or (4.4) as a Hopf bifurcation point, we can determine strictly the first 7 singular point quantities, i.e., $\mu_m, m \leq 7$.

Applying also the symbolic computation function of Mathematica system and the recursive formulas in Lemma 4.1, where c_{kj}, μ_m in (4.6) (4.7) can be found in the website: https://github.com/wql2001399/wql. We obtain the first 7 singular point quantities as follows:

$$\begin{aligned} u_1 &= -\frac{\mathbf{i}}{p^3}\left(3b_{03}p^2+b_{21}\right),\\ u_2 &= \frac{\mathbf{i}}{p^5}b_{03}\left(2b_{12}p^2+6b_{30}+\kappa_1-4\kappa_2\right),\\ u_3 &= \frac{\mathbf{i}}{12p^7}b_{03}\left(72b_{03}^2p^4+8b_{12}^2p^2+3b_{12}p^2+9b_{30}+18\kappa_2\right),\\ u_4 &= \frac{\mathbf{i}}{45p^9}b_{03}\left(90b_{03}^2p^4+36b_{03}^2b_{12}p^4+4b_{12}^3p^2+10b_{12}^2p^2+3b_{12}p^2+9b_{30}\right),\\ u_5 &= -\frac{\mathbf{i}}{28350p^{11}}b_{03}\left(9b_{03}^2p^2+b_{12}^2\right)f_5,\\ u_6 &= -\frac{\mathbf{i}}{198450p^{11}}b_{03}\left(9b_{03}^2p^2+b_{12}^2\right)f_6,\\ u_7 &= -\frac{\mathbf{i}}{228614400p^{13}}b_{03}\left(9b_{03}^2p^2+b_{12}^2\right)f_7, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} f_5 &= 3744b_{03}^2b_{12}p^2+336b_{12}^3-100b_{12}^2+23b_{12}+30,\\ f_6 &= 480b_{12}^3+1250b_{12}^2+941b_{12}+210,\\ f_7 &= 17350b_{12}^2+21535b_{12}+4902. \end{aligned} \tag{4.9}$$

In the above expression of each μ_k, we have already let $\mu_{k-1}=0$, $k=1,2,\cdots,7$.

From the above expressions of seven singular point quantities, we have

Theorem 4.1. *The order of the origin of system (4.4) as a fine focus is 7 if and only if*

$$\begin{aligned} & b_{21} = -3b_{03}p^2,\\ & \kappa_1 = 4\kappa_2-2b_{12}p^2-6b_{30},\\ & \kappa_2 = -\tfrac{1}{18}(72b_{03}^2p^4+8b_{12}^2p^2+3b_{12}p^2+9b_{30}),\\ & b_{30} = -\tfrac{p^2}{9}(90b_{03}^2p^2+36b_{03}^2b_{12}p^2+4b_{12}^3+10b_{12}^2+3b_{12}),\\ & f_5=f_6=0,\ b_{03}\neq 0\ . \end{aligned} \tag{4.10}$$

Proof. (i) According to the first 7 singular point quantities in (4.8), considering the conditions (4.10), we have got naturally $\mu_1=\mu_2=\cdots=\mu_6=0$. And under the condition $f_6=0$ in (4.10), we can verify $f_7\neq 0$ easily. Then, $\mu_7\neq 0$ hold, namely the origin of the system (4.4) is a fine focus of order 7.

(ii) Conversely, if $\mu_1=\mu_2=\cdots=\mu_6=0$ and $\mu_7\neq 0$, the necessity of (4.10) is easily proved. □

And more we have the following theorem.

Theorem 4.2. *The first 7 focal values all disappear, and the origin of the system (4.1) is a center if and only if the following conditions holds*

$$a-\kappa_1+\kappa_2<0,\ b=0,\ b_{21}=b_{03}=0, \tag{4.11}$$

Proof. For case (4.11), we obtain the original system

$$\dot{x}=y,\quad \dot{y}=-\kappa_1\sin x+\kappa_2\sin x\cos x+ax+b_{30}x^3+b_{12}xy^2, \tag{4.12}$$

which defines a vector field symmetric with respect to the x-axis. Thus we obtain the origin of the system (4.1) is a center, namely the sufficiency of (4.11) is proven.

On the other hand, from Theorem 4.1, the necessity is obvious. □

Furthermore, from Theorems 4.1 and 4.2, applying the relationship between the focal values and the singular point quantities given in [17], we obtain the following corollary.

Corollary 4.1. *The origin of the system (4.1) is a fine focus of order 7 at most, and the first seven focal values are respectively:*

$$v_{2j+1}(2\pi) = \mathbf{i}\pi\mu_j, 1 \leq j \leq 7, \tag{4.13}$$

in the above expression of each v_{2k+1}*, we have already let* $v_{2k-1} = 0,\ k = 1, 2, \cdots, 7$*.*

Next, setting the parameters $b = \delta$ with $0 < |\delta| \ll 1$ in the condition (4.2) of Hopf bifurcation, we make system (4.1) become the perturbed form as follows,

$$\dot{x} = y, \quad \dot{y} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x - p^2 x + \delta y + Y_2, \tag{4.14}$$

where $\kappa_1 = \kappa_2$. Thus by calculating based on the perturbation of linear part, we obtain the strong focus value $v_1 = \pi\delta/p + o(\delta)$.

Further, under the condition (4.10), we can verify that there exists a valid critical value such that $v_{15} \neq 0$, and more by calculating the Jacobian determinant of the seven focal values $v_1, v_3, v_3, v_5, \cdots, v_{13}$ with respect to the variables $\delta, b_{21}, k_1, k_2, b_{30}, b_{12}, b_{03}$, we obtain:

$$\begin{aligned} J|_{(4.10)} &= \tfrac{\partial(v_1,v_3,v_5,v_7,v_9,v_{11},v_{13})}{\partial(\delta, b_{21},k_1,k_2,b_{30},b_{12},b_{03})}|_{(4.10)} \\ &= \tfrac{828780187627401112309649 3\pi^{19}}{1482934826295546825600 0p^{151}} \neq 0. \end{aligned} \tag{4.15}$$

From these, applying Theorem 4.2 in [10], also similar to the proof of Theorem 3.4, we have the following theorem.

Theorem 4.3. *Under appropriate perturbations, system (4.1) can generate at most 7 small-amplitude limit cycles in the neighborhood of the origin via Hopf bifurcation.*

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Appendix A

$\varphi_5 = (343119513033220b_{12}^3 b_{13} K^6 + 1469667810663360b_{12}^2 b_{13} b_{40}^2 K^4 +$
$1955179134481 65b_{12}^2 b_{13} K^6 + 5939090697909600b_{12} b_{13} b_{40}^4 K^2 +$
$1501138725373980b_{12} b_{13} b_{40}^2 K^4 + 42006515691063b_{12} b_{13} K^6 +$
$17610229477169280b_{13} b_{40}^6 + 6153888182904000b_{13} b_{40}^4 K^2 +$
$420403067003304b_{13} b_{40}^2 K^4 + 3420033941720b_{13} K^6)/(28222959736125\, K^{21/2}),$

$\varphi_6 = 262057127270400b_{12}^2 b_{40}^2\ K^4 + 51118038342800b_{12}^2 K^6 +$
$4465238499532800b_{12} b_{40}^4 K^2 + 1319232265542080b_{12} b_{40}^2 K^4 +$
$45931084890900b_{12} K^6 - 31443056492544 00b_{40}^6 - 1217227620787200b_{40}^4 K^2 -$
$73347310051488b_{40}^2 K^4 + 2008406304773K^6,$

$\varphi_7 = (678482043601753190800b_{12}^4 K^8 + 3884942131492729507200b_{12}^3 b_{40}^2 K^6 +$
$517189699488844181800b_{12}^3 K^8 + 2609551698705757935360 0b_{12}^2 b_{40}^4 K^4 +$
$6233342449673781842400b_{12}^2 b_{40}^2 K^6 - 162288561532429917585b_{12}^2 K^8 +$
$20196582677991794511360 0b_{12} b_{40}^6 K^2 + 64924146621553440513600b_{12} b_{40}^4 K^4 +$
$3489125940475096466160b_{12} b_{40}^2 K^6 + 249995603386955140 00b_{12} K^8 -$
$490992871049590659 75b_{13}^2 K^9 + 109928226192273375667200b_{40}^8 +$
$57294337601814507532800b_{40}^6 K^2 + 9373013654929833585600b_{40}^4 K^4 +$
$426858341519447416920b_{40}^2 K^6 + 1599971861676512896K^8)/$
$(1366241032485817488 00\, K^{12}),$

$\varphi_8 = 368746108044652800000b_{12}^3 b_{40}^2 K^6 + 719292693395633500000b_{12}^3 K^8 +$
$15114514871207164723200b_{12}^2 b_{40}^4 K^4 + 3940653216762166132800b_{12}^2 b_{40}^2 K^6 +$
$131093805037993715500b_{12}^2 K^8 + 15109678503793749934080 0b_{12} b_{40}^6 K^2 +$
$50911925011665130828800b_{12} b_{40}^4 K^4 + 3389291976826728886320b_{12} b_{40}^2 K^6 +$
$63784109539965564805b_{12} K^8 - 48262391626301846323200b_{40}^8 -$
$16626718893566768025600b_{40}^6 K^2 - 608046494501652307 20b_{40}^4 K^4 +$
$16146122585334814314 0b_{40}^2 K^6 + 3618635711488746817K^8,$

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