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Correspondence Between Renormalized and Entropy Solutions to the Parabolic Initial-Boundary Value Problem Involving Variable Exponents and Measure Data

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Abstract We study the initial-boundary value parabolic problem involving variable exponent under the generalized Leray-Lions conditions. We clarify the definitions of entropy and renormalized solutions to such parabolic problems, and we establish the equivalence between these definitions of entropy and renormalized solutions to the parabolic problems with the Leray-Lions operator and with measure data.

Keywords Capacity, diffuse measure data, entropy solution, exponential Lebesgue space, parabolic equation, renormalized solution, soft measure, variable Laplacian

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1. Introduction

We consider the parabolic problem with measure data

$$\frac{\partial b\left(u\right)}{\partial t}-div\left(a\left(x,\ t,\ u,\ \nabla u\right)\right)+d\left(x\right)\left|u\right|^{p(x)-2}u=\mu,\quad \mu\in \mathrm{M}_0\left(Q\right), \tag{1.1}$$

$$b\left(u\right)\left(x,\ 0\right)=b\left(\varphi\left(x\right)\right),\quad x\in\Omega, \tag{1.2}$$

$$u|_{\partial\Omega\times(0,\ \infty)}=0, \tag{1.3}$$

where $Q\overset{def}{\equiv}\Omega\times(0,\ T)$, $(x,\ t)\in Q$, $\Omega\subseteq R^n$, $n\geq 3$ is a smooth Lipschitz domain and $\partial\Omega$ is a Lipschitz boundary of an open set Ω; $p\in P^{\log}(\Omega)$ is a log-Holder continuous function such that $1<p_m=\inf\{p(x),x\in\Omega\}$ and $p_S=\sup\{p(x),x\in\Omega\}<\infty$. We assume the function φ is measurable and such that $b(\varphi)$ belongs to $L^1(\Omega)$.

We consider the parabolic equation (1.1) under generalized Leray-Lions conditions. This type of problem was studied in weighted Orlicz–Sobolev spaces in [11]. In the case of constant exponents, the existence result was obtained for the obstacle parabolic problem associated with the equation with a Leray-Lions operator [18] and some a priori estimates were obtained in [1]. The existence of renormalized solutions for nonlinear elliptic equations with variable exponents with L^1 data was studied in [5], and the existence and uniqueness of the renormalized solution for parabolic

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equations under the Leray-Lions conditions [15] were established for constant exponents in [7]. The solvability of parabolic problem with L^1 data was investigated in [3], and for elliptic equations in [4, 5]. The concept of a renormalized solution was introduced by R.J. Perna, and P.L. Lions [15, 21], who studied the existence and uniqueness of solutions for some class of parabolic problems treated in [21].

Recently, results on the solvability of nonlinear parabolic equations with both singularities and unbounded lower-order terms were obtained by T.T. Dang, and G. Orlandi [6]. The authors did not impose the coercivity assumption on the main differential operator of convection-diffusion type (non-atomic measure), and the equation has the convective term with an unbounded coefficient in the Lorentz class. L. Zhao and S. Zheng studied the local Besov regularity of to the elliptic variational inequality with double-phase Orlicz growth. They proved that the fractional differentiability of the differential is reflected by additional differentiability assumptions on the obstacle term and the external force, under some regularity on the coefficient [27, 28]. M. Bendahmane and P. Wittbold [4] investigated the existence and uniqueness of the renormalized solution to a parabolic problem with L^1 data by employing some results of abstract semigroup theory to show the solvability of the parabolic problem with the singular right-hand side. J.X. Yin, J.K. Li, Y.Y. Ke investigated positive solutions of the variable exponent Laplacian equation by applying the Krasnoselskii fixed point theorem on the cone in [24]. Q. H. Zhang showed the existence of solutions to a problem with a variable exponent operator under the Caratheodory conditions. The author used the strict monotony condition to deal with the limit of the approximate solutions [15,26]. The monotone operators were studied by J.L. Lions [15].

As an example of the problem (1.1)-(1.3), we can consider a variable Laplacian problem

$$\frac{\partial b\left(u\right)}{\partial t}-\Delta_{p(\cdot)}u=\mu, \quad \mu\in \mathrm{M}_0\left(Q\right), \tag{1.4}$$

$$b\left(u\right)\left(x,\ 0\right)=b\left(\varphi\left(x\right)\right), \quad x\in\Omega, \tag{1.5}$$

$$u|_{\partial\Omega\times(0,\ \infty)}=0. \tag{1.6}$$

By changing the unknown $v=b\left(u\right)$ and $\Psi=b^{-1}$, we obtain the generalized porous medium operator [1] $\partial_t v-\Delta_{p(\cdot)}\Psi\left(v\right)$ with a strictly increasing function Ψ. This type of problem often appears in models that describe the properties of fluids and processes of diffusion.

The presence of measure data presents additional complications. The class of problems covered by our conditions is Leray-Lions operators [15, 21] in divergent form $A\left(u\right)=-\nabla\cdot a\left(x,\ u,\ \nabla u\right)$. The presence of a measure datum μ compels us to work in the framework of entropy and renormalized solutions. The natural condition of the measure data is that these measures do not charge the sets of null capacity.

In the present paper, we establish that a function u is an entropy solution to the initial boundary parabolic problem (1.1)-(1.3) under the Leray-Lions conditions if and only if this function u is a renormalized solution to the same initial boundary parabolic problem (1.1)-(1.3) under the same conditions.

2. Preliminary information

2.1. Variable exponent Lebesgue spaces

A function $\alpha : \Omega \to R,\ \Omega \subset R^n$ is said to be globally log-Holder continuous if and only if there are two constants c_l and c_∞ such that

$$|\alpha(x) - \alpha(y)| \le \frac{c_l}{\log\left(e + |x-y|^{-1}\right)} \tag{2.1}$$

and

$$|\alpha(x) - \alpha(\infty)| \le \frac{c_\infty}{\log(e + |x|)} \tag{2.2}$$

for all $x,\ y \in \Omega$.

Definition 2.1. The functional class $P^{\log}(\Omega)$ is a collection of all variable exponents $p : \Omega \to (1,\ \infty)\,,\ p \in P(\Omega)$ such that p^{-1} is globally log-Holder continuous.

A modular $\rho_{p(\cdot)}(u)$ of $p \in P^{\log}(\Omega)$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \tag{2.3}$$

and the variable exponent Lebesgue space norm $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is defined by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0\ :\ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.4}$$

Let $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ with $q(x) = \frac{p(x)}{p(x)-1},\ p_m > 1$. Then the following integral inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p_m} + \frac{1}{q_m}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)} \tag{2.5}$$

is called the generalized Holder inequality.

For all $u \in L^{p(\cdot)}(\Omega)$, we have

$$\min\left(\|u\|^{p_m}_{L^{p(\cdot)}(\Omega)},\ \|u\|^{p_s}_{L^{p(\cdot)}(\Omega)}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|^{p_m}_{L^{p(\cdot)}(\Omega)},\ \|u\|^{p_s}_{L^{p(\cdot)}(\Omega)}\right). \tag{2.6}$$

For each number $m > 0$, the truncation operator $T_m\ :\ R \to R$ is defined by

$$T_m(s) = \max\{-m,\ \min\{m,\ s\}\} \tag{2.7}$$

for all $s \in R$.

Let $u,\ u_k \in L^{p(\cdot)}(\Omega)\,,\ k = 1,\ 2,\ ....$ Then

$$\lim_{k\to\infty} \|u - u_k\|_{L^{p(\cdot)}(\Omega)} = 0$$

if and only if

$$\lim_{k\to\infty} \rho_{p(\cdot)}(u - u_k) = 0.$$

Assume $p : \Omega \to (1,\ \infty)\,, p \in P(\Omega)$. Then the following statements are equivalent:

1) $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and $\rho_{p(\cdot)}(u) \leq 1$;

2) $\|u\|_{L^{p(\cdot)}(\Omega)} = 1$ and $\rho_{p(\cdot)}(u) = 1$.

Let $p : \Omega \to (1, \infty), p \in P(\Omega)$. Then $L^{p(\cdot)}(\Omega)$ is a separable reflexive linear space, for which the Poincare inequality

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\, diam\,(\Omega)\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \tag{2.8}$$

holds for all $u \in W^{p(\cdot)}_{1,\,0}(\Omega)$, where the constant c depends only on the dimension and the log-Holder constant. So, we have that there is a constant c depending only on the dimension and the log-Holder constant such that

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \leq \|u\|_{\in W^{p(\cdot)}_{1,\,0}(\Omega)} \leq (1 + c\, diam\,(\Omega))\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in W^{p(\cdot)}_{1,\,0}(\Omega)$. We remark that the Poincare inequality

$$\rho_{p(\cdot)}(u) \leq c\, \rho_{p(\cdot)}(\nabla u)$$

does not hold even in the case where p is continuous and has a maximum or minimum.

Let Ω have a finite measure and $p_1, p_2 \in P(\Omega)$. Then the embedding $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$ is continuous if and only if the inequality $p_1(x) \leq p_2(x)$ holds for almost all $x \in \Omega$.

For each number $m > 0$, the truncation operator $T_m : R \to R$ is defined by

$$T_m(s) = \max\{-m,\ \min\{m,\ s\}\} \tag{2.9}$$

for all $s \in R$.

Let $T_m(u) \in L^{p(\cdot)}\left((0,\ T),\ W^{p(\cdot)}_{1,\,0}(\Omega)\right),\ m > 0$ for a measurable function u. To formalize the definition of the renormalized solution, we need to widen the notion of the gradient operator. We define the generalization of the gradient $\nabla u : Q \to R^n$ of u by taking

$$\nabla T_m(u) = 1_{\{|u|<m\}} \nabla u \tag{2.10}$$

almost everywhere in Q, for each $m > 0$.

2.2. Variable exponent capacity and measures

To cope with entropy and renormalized solutions, we introduce the concept of variable exponent parabolic capacity associated with parabolic differential operators in variable exponent spaces. In differential problems with measure data, measures that do not charge sets of null capacity play an essentially important role since such measures have well-known decomposition properties.

Definition 2.2. We define a space $V^{p(\cdot)} = W^{p(\cdot)}_{1,\,0}(\Omega) \cap L^2(\Omega)$ equipped with the norm $\|u\|_{V^{p(\cdot)}} = \|u\|_{W^{p(\cdot)}_{1,\,0}(\Omega)} + \|u\|_{L^2(\Omega)}$ and a space

$$W^{p(\cdot),\,q(\cdot)} = \left\{u \in L^{p(\cdot)}\left((0,\ T),\ V^{p(\cdot)}\right),\ \partial_t u \in L^{q(\cdot)}\left((0,\ T),\ \left(V^{p(\cdot)}\right)^*\right)\right\} \tag{2.11}$$

with the norm $\|u\|_{W^{p(\cdot),\,q(\cdot)}} = \|u\|_{L^{p(\cdot)}\left((0,\,T),\,V^{p(\cdot)}\right)} + \|\partial_t u\|_{L^{q(\cdot)}\left((0,\,T),\,\left(V^{p(\cdot)}\right)^*\right)}$. Also, we define the space

$$\tilde{W}^{p(\cdot),\,q(\cdot)} = \left\{\begin{array}{l} u \in L^{p(\cdot)}\left((0,\ T),\ W^{p(\cdot)}_{1,\,0}(\Omega)\right) \cap L^{\infty}\left((0,\ T),\ L^2(\Omega)\right), \\ \partial_t u \in L^{q(\cdot)}\left((0,\ T),\ W^{q(\cdot)}_{-1}(\Omega)\right) \end{array}\right\}, \tag{2.12}$$

where

$$L^{p(\cdot)}\left((0,\ T),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right) \equiv L^{p_m}\left((0,\ T),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right)$$

and

$$L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right) \equiv L^{q_S}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right).$$

We define a parabolic capacity of an open subset E of Q by

$$Cap_{p(\cdot)}(E) = \inf\left\{\|u\|_{W^{p(\cdot),\ q(\cdot)}} :\ u \in W^{p(\cdot),\ q(\cdot)},\ u \geq 1_E \ a.e.\ in\ Q\right\}$$

and a parabolic capacity of a Borelian subset B of Q by

$$Cap_{p(\cdot)}(B) = \inf\left\{Cap_{p(\cdot)}(E),\ \ E\ is\ open\ in\ Q,\ B \subset E\right\}.$$

Let $\alpha \in C^{\infty}(R \times R^n)$ and $u \in W^{p(\cdot),\ q(\cdot)}$. Then $\alpha u \in W^{p(\cdot),\ q(\cdot)}$ and the inequality

$$\|\alpha u\|_{W^{p(\cdot),\ q(\cdot)}} \leq c(\alpha)\|u\|_{W^{p(\cdot),\ q(\cdot)}}$$

holds with constant $c(\alpha)$ depending only on α. In the distributional sense, we have

$$\frac{\partial(\alpha u)}{\partial t} = \frac{\partial \alpha}{\partial t}u + \alpha\frac{\partial u}{\partial t},$$

where $\frac{\partial \alpha}{\partial t}u \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right)$.

Let $\{E_i\}$ be a family of Borelian sets in Q. Then the parabolic capacity has the subadditive property

$$Cap_{p(\cdot)}\left(\bigcup_i E_i\right) \leq \sum_i Cap_{p(\cdot)}E_i.$$

The sets, which are contained in the section of parabolic cylinder $\Omega \times \{\tau\}$, $\tau \in [0,\ T]$ and have null capacity satisfy the following description: for any Borelian set E in Ω, the necessary and sufficient condition for $Cap_{p(\cdot)}(E \times \{\tau\}) = 0$ is that the Borelian set E is null measure. Also, we have a more general result: for any Borelian set E in Ω, and any subinterval $(t_1,\ t_2) \subset [0,\ T]$, the equality $Cap_{p(\cdot)}(E \times (t_1,\ t_2)) = 0$ holds if and only if the elliptic capacity of the set E vanishes, ${Cap_{p(\cdot)}}^{ellip}(E) = 0$.

The set $\mathrm{M}_0(Q)$ consists of all bounded measures μ on the σ-algebra of all Borelian subsets of Q such that $\mu(E) = 0$ for all subsets $E \subset Q$ with zero capacity, namely $Cap_{p(\cdot)}(E) = 0$. Assume $\mu \in \mathrm{M}_0(Q)$. Then, for all $\phi \in C_C^{\infty}(Q)$, there is a decomposition

$$(\Upsilon_1,\ \Upsilon_2,\ F) \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right) \times L^{p(\cdot)}\left((0,\ T),\ V^{p(\cdot)}\right) \times L^1(Q)$$

such that

$$\int_Q \phi d\mu = \int_Q F\phi dxdt + \int_{[0,\ T]}\langle\Upsilon_1,\ \phi\rangle dt - \int_{[0,\ T]}\left\langle\frac{\partial\phi}{\partial t},\ \Upsilon_2\right\rangle dt.$$

Now, we formulate the definition of the renormalized solution for the parabolic partial differential equation with the measure data.

Definition 2.3. Let $\mu \in \mathrm{M}_0(Q)$. Then, a measurable function u is said to be a renormalized solution to the parabolic problem (1.1)-(1.3) if and only if there is a μ-decomposition $\Upsilon_1 \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right),\ \Upsilon_2 \in L^{p(\cdot)}\left((0,\ T),\ V^{p(\cdot)}\right)$ and $F \in L^1(Q)$ such that

$$b(u) - \Upsilon_2 \in L^{\infty}\left((0,\ T),\ L^1(\Omega)\right), \tag{2.13}$$

$$T_m\left(b(u) - \Upsilon_2\right) \in L^{p(\cdot)}\left((0,\ T),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right),\ m \geq 0, \tag{2.14}$$

$$\lim_{k\to\infty}\int_{\{(x,\ t)\in Q\ :\ k\leq|b(u)-\Upsilon_2|\leq k+1\}}|\nabla u|^{p(x)}\,dxdt = 0, \tag{2.15}$$

and the integral identity

$$\begin{aligned}
&\int_Q \phi\frac{\partial h\left(b(u) - \Upsilon_2\right)}{\partial t}dxdt \\
&+\int_Q a(x,\ t,\ u,\ \nabla u)\,h'\left(b(u) - \Upsilon_2\right)\nabla\phi dxdt \\
&+\int_Q \phi a(x,\ t,\ u,\ \nabla u)\,h''\left(b(u) - \Upsilon_2\right)\nabla\left(b(u) - \Upsilon_2\right)dxdt \\
&+\int_Q \phi d(x)\,|u|^{p(x)-2}\,uh'\left(b(u) - \Upsilon_2\right)dxdt \\
=&\int_Q \phi F h'\left(b(u) - \Upsilon_2\right)dxdt \\
&+\int_Q \phi\Theta_1 h''\left(b(u) - \Upsilon_2\right)\nabla\left(b(u) - \Upsilon_2\right)dxdt \\
&+\int_Q \Theta_1 h'\left(b(u) - \Upsilon_2\right)\nabla\phi dxdt,
\end{aligned} \tag{2.16}$$

$$h\left(b(u) - \Upsilon_2\right)(0) = h\left(b(\varphi)\right) \in L^1(\Omega) \tag{2.17}$$

for all C^1-function $h \in W^{2,\,\infty}(R)$ such that h' has a compact support, and all $\phi \in C_C^{\infty}(Q),\ \Theta_1 \in \left(L^{q(\cdot)}(Q)\right)^n$.

Definition 2.4. Let $\mu \in \mathrm{M}_0(Q)$. Let $\tilde{T}_k(s) = \int_{[0,\ s]} T_k(t)\,dt$ for all $s \in [0,\ T]$. Then, a measurable function u is said to be an entropy solution to the parabolic problem (1.1)-(1.3) if and only if there is a μ-decomposition $\Upsilon_1 \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right),\ \Upsilon_2 \in L^{p(\cdot)}\left((0,\ T),\ V^{p(\cdot)}\right)$ and $F \in L^1(Q)$ such that

$$T_m\left(b(u) - \Upsilon_2\right) \in L^{p(\cdot)}\left((0,\ T),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right),\ m \geq 0, \tag{2.18}$$

$$t \in [0,\ T] \to \int_\Omega \tilde{T}_m\left(b(u) - \Upsilon_2 - \psi\right)(x,\ t)\,dx$$

is almost everywhere a continuous function, $m \geq 0$ and $\psi \in \mathrm{B}$, where

$$\mathrm{B} =$$

$$\left\{u \in L^{p(\cdot)}\left((0,T),W_{1,0}^{p(\cdot)}(\Omega)\right) \cap L^{\infty}(Q),\ \partial_t u \in L^{q(\cdot)}\left((0,T),W_{-1}^{q(\cdot)}(\Omega)\right) + L^1(Q)\right\}$$

and

$$\begin{aligned}
&\int_{\Omega} \tilde{T}_m \left(b\left(u\right) - \Upsilon_2 - \psi\right)\left(T,\ x\right) dx \\
&- \int_{\Omega} \tilde{T}_m \left(b\left(\varphi\right) - \psi\left(0,\ x\right)\right) dx \\
&+ \int_{[0,\ T]} \left\langle \partial_t \psi,\ T_m\left(b\left(u\right) - \Upsilon_2 - \psi\right)\right\rangle dt \\
&+ \int_{Q} a\left(x,\ t,\ u,\ \nabla u\right) \nabla T_m\left(b\left(u\right) - \Upsilon_2 - \psi\right) dxdt \qquad (2.19)\\
&+ \int_{Q} d\left(x\right) \left|u\right|^{p(x)-2} u T_m\left(b\left(u\right) - \Upsilon_2 - \psi\right) dxdt \\
\leq &\int_{Q} F T_m\left(b\left(u\right) - \Upsilon_2 - \psi\right) dxdt \\
&+ \int_{Q} \Theta_1 \nabla\left(T_m\left(b\left(u\right) - \Upsilon_2 - \psi\right)\right) dxdt
\end{aligned}$$

for all $m \geq 0$ and $\psi \in \mathrm{B}$, where

$$\langle \cdot,\ \cdot\rangle :\ W_{-1}^{q(\cdot)}\left(\Omega\right) + L^1\left(\Omega\right) \times W_{1,\ 0}^{p(\cdot)}\left(\Omega\right) \cap L^{\infty}\left(\Omega\right) \to R$$

is the duality pairing.

We remark that since $\Upsilon_2 \in L^{p(\cdot)}\left(\left(0,\ T\right),\ W_{1,\ 0}^{p(\cdot)}\left(\Omega\right)\right)$, the identity (2.15) is equivalent to

$$\lim_{k\to\infty} \int_{\{(x,\ t)\in Q:\quad k\leq |b(u)-\Upsilon_2|\leq k+1\}} \left|\nabla\left(u - \Upsilon_2\right)\right|^{p(x)} dxdt = 0. \qquad (2.20)$$

The main incentive to study renormalized entropy solutions is that they can be correctly defined under rather lighter regularity conditions on the structural coefficients than classical solutions. Some information on entropy solutions can be found in [11, 18]. Generally, the existence and uniqueness of the entropy solutions can be shown by establishing the existence and uniqueness of solutions to approximation problems, which are regularizations of the initial problem with singular data. Similar approximation arguments are applicable for dealing with the solvability of the parabolic problem in the renormalized sense [4, 6]. Since the definition of the renormalized solutions is more demanding than the notion of distributional solution [15], for renormalized solutions, one can show its uniqueness under more general conditions than classical case but stricter than in the distributional framework [21].

2.3. Hypotheses and conditions

Throughout this article, we suppose that the following e hypotheses on the coefficients hold true.

1) A function $p \in P^{\log}\left(\Omega\right)$ is log-Holder continuous and such that $1 < p_m$, and $p_S < \infty$;

2) $b : R \to R,\ b \in C^1\left(R\right)$ is a strictly increasing function such that $0 < \inf b'\left(\tau\right) \leq b'\left(\tau\right) \leq \sup b'\left(\tau\right) < \infty,\ b\left(0\right) = 0$;

3) $a : \Omega \times (0, T) \times R \times R^n \to R^n$ such that $a(\cdot, \cdot, \eta, \xi)$ is measurable in Q for each $(\eta, \xi) \in R \times R^n$ and $a(x, t, \cdot, \cdot)$ is continuous on $R \times R^n$ for almost every (x, t) in Q;

4) $a(x, t, \eta, \xi)\xi \geq \nu |\xi|^{p(x)}$ for all $\xi \in R^n$;

5) $|a(x, t, \eta, \xi)| \leq \alpha |\xi|^{p(x)-1} + \beta |\eta|^{p(x)-1} + \gamma(x, t)$ for all $\xi \in R^n$ and $\eta \in R$;

6) $(a(x, t, \eta, \xi_1) - a(x, t, \eta, \xi_2))(\xi_1 - \xi_2) > 0$ for almost all $(x, t) \in Q$, and all $\xi_1, \xi_2 \in R^n$, $\xi_1 \neq \xi_2$, with some constants $\nu, \alpha, \beta > 0$, and nonnegative function $\gamma \in L^{q(\cdot)}(Q)$;

7) $d : \Omega \to R$ is a measurable function satisfying $0 < d_m$ and $d_S < \infty$;

8) the given measure is $\mu \in \mathrm{M}_0(Q)$.

We denote

$$L^{p(\cdot)}\left((0, T), W_{1,0}^{p(\cdot)}(\Omega)\right)$$
$$= \left\{u : (0, T) \to W_{1,0}^{p(\cdot)}(\Omega) \textit{ is measurable } \int_{[0,T]} \|u(t)\|_{W_{1,0}^{p(\cdot)}(\Omega)}^{p_m} dt < \infty\right\}.$$

We remark that all terms of equality (2.16) of the definition of the renormalized solution are correctly defined. So, we assume that m is a positive number such that

$$\sup p(h') \subset [-m, m] \tag{2.21}$$

and we obtain

$$h(b(u) - \Upsilon_2) = h(T_m(b(u) - \Upsilon_2)) \in L^{p(\cdot)}\left((0, T), W_{1,0}^{p(\cdot)}(\Omega)\right)$$

and $\frac{\partial h(b(u)-\Upsilon_2)}{\partial t}$ is a distribution over Q. In the sense of almost everywhere in Q, we have an identity

$$a(x, t, u, \nabla u) h'(b(u) - \Upsilon_2)$$
$$= a\left(x, t, b^{-1}(T_m(v) + \Upsilon_2), \nabla\left(b^{-1}(T_m(v) + \Upsilon_2)\right)\right) h'(T_m(b(u) - \Upsilon_2)),$$

where $b(u) - \Upsilon_2 = v$ and $u = b^{-1}(T_m(v) + \Upsilon_2)$ over $\{|b(u) - \Upsilon_2| \leq m\}$. Applying 2) and 5), we obtain

$$\begin{aligned} &|a(x, t, u, \nabla u) h'(T_m(b(u) - \Upsilon_2))| \\ &\leq \|h'\|_{L^\infty(R)} \Big(\alpha \left|\nabla b^{-1}(T_m(v) + \Upsilon_2)\right|^{p(x)-1} \\ &\quad + \beta \left|b^{-1}(T_m(v) + \Upsilon_2)\right|^{p(x)-1} + \gamma(x, t)\Big) \\ &\leq \|h'\|_{L^\infty(R)} \left(\frac{\alpha}{\inf\limits_{\tau \in R} b'(\tau)} \left|\nabla b^{-1}(T_m(v) + \Upsilon_2)\right|^{p(x)-1}\right. \\ &\quad \left. + \beta \left|b^{-1}(T_m(v) + \Upsilon_2)\right|^{p(x)-1} + \gamma(x, t)\right). \end{aligned}$$

Applying 5) and (2.14), we obtain

$$a(x, t, u, \nabla u) h'(b(u) - \Upsilon_2) \in \left(L^{q(\cdot)}(Q)\right)^n.$$

Next, since $u = b^{-1}(T_m(v) + \Upsilon_2)$ we obtain the identities

$$a(x,\ t,\ u,\ \nabla u)\, h''(b(u) - \Upsilon_2)\, \nabla(b(u) - \Upsilon_2)$$
$$= a\left(x,\ t,\ b^{-1}(T_m(v) + \Upsilon_2),\ \nabla b^{-1}(T_m(v) + \Upsilon_2)\right)$$
$$\times h''(b(u) - \Upsilon_2)\, \nabla T_m(b(u) - \Upsilon_2) \in L^1(Q).$$

Also, we conclude

$$F h'(b(u) - \Upsilon_2) \in L^1(Q),$$
$$\Theta_1 h''(b(u) - \Upsilon_2)\, \nabla T_m(b(u) - \Upsilon_2) \in L^1(Q),$$

and

$$\Theta_1 h'(b(u) - \Upsilon_2) \in \left(L^{q(\cdot)}(Q)\right)^n.$$

Since

$$\frac{\partial h(b(u) - \Upsilon_2)}{\partial t} \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right) + L^1(Q)$$

and

$$h(b(u) - \Upsilon_2) \in L^{p(\cdot)}\left((0,\ T),\ W_{1,\,0}^{p(\cdot)}(\Omega)\right),$$

we deduce

$$h(b(u) - \Upsilon_2) \in C\left((0,\ T),\ L^1(\Omega)\right),$$

thus the identity

$$h(b(u) - \Upsilon_2)(0) = h(b(\varphi))$$

is correctly defined.

3. Equivalence between entropy and renormalized solutions

This section focuses on establishing the main result, stated in the form of the following theorem.

Theorem 3.1. *Let $\mu \in \mathrm{M}_0(Q)$. If conditions 1)-8) hold then each entropy solution to the initial-bounded problem (1.1)-(1.3) is a renormalized solution to the initial-bounded problem (1.1)-(1.3), and each renormalized solution to the initial-bounded problem (1.1)-(1.3) is an entropy solution to the initial-bounded problem (1.1)-(1.3).*

Proof. For each $m \geq 0$, we define a function

$$h_m(s) = s,\ |s| \leq m, \tag{3.1}$$
$$\sup p(h'_m) \subset [-m-1,\ m+1], \tag{3.2}$$
$$\|h''_m\|_{L^\infty(R)} \leq 1. \tag{3.3}$$

We assume that a function ς_ε is continuous on $[0,\ \infty)$, $\varsigma_\varepsilon(t) = 1$, $t \in [0,\ \tau]$, $\tau \in (0,\ T)$, and $\varsigma_\varepsilon(t) = 0$, $t \in [\tau + \varepsilon,\ \infty]$, and ς_ε is linear on $[\tau,\ \tau + \varepsilon]$. We denote $v = b(u) - \Upsilon_2$, and we take a test function $\phi = \varsigma_\varepsilon T_k(v - \psi)$, where $\psi \in \mathrm{B}$. We have

$$\int_{[0,\ T]} \varsigma_\varepsilon \left\langle \frac{\partial(h_m(v))}{\partial t},\ T_k(v - \psi) \right\rangle dt$$

$$
\begin{aligned}
&+\int_Q \varsigma_\varepsilon a(x,\ t,\ u,\ \nabla u)\, h_m'(v)\, \nabla(T_k(v-\psi))\, dxdt \\
&+\int_Q \frac{1}{b'(u)} \varsigma_\varepsilon a(x,\ t,\ u,\ \nabla u)\, h_m''(v)\, T_k(v-\psi)\, \nabla v dxdt \\
&+\int_Q \varsigma_\varepsilon d(x)\, |u|^{p(x)-2}\, u h_m'(v)\, T_k(v-\psi)\, dxdt \\
=&\int_Q \varsigma_\varepsilon F h_m'(v)\, T_k(v-\psi)\, dxdt \\
&+\int_Q \frac{1}{b'(u)} \varsigma_\varepsilon T_k(v-\psi)\, \Theta_1 h_m''(v)\, \nabla v dxdt \\
&+\int_Q \varsigma_\varepsilon \Theta_1 h_m'(v)\, \nabla(T_k(v-\psi))\, dxdt.
\end{aligned}
$$

We obtain

$$
\frac{\partial(h_m(v))}{\partial t} \in L^{q(\cdot)}\left((0,\ T),\ W_{-1}^{q(\cdot)}(\Omega)\right) + L^1(Q),
$$

$$
T_k(v-\psi) \in L^{p(\cdot)}\left((0,\ T),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right) \cap L^\infty(Q)
$$

and

$$
\left\langle \frac{\partial(h_m(v))}{\partial t},\ T_k(v-\psi) \right\rangle \in L^1(0,\ T),
$$

and

$$
a(x,\ t,\ u,\ \nabla u)\, \nabla(T_k(v-\psi)) \in L^1(Q).
$$

Since $h_m''(s) = 0,\ s \notin [m,\ m+1]$, we have

$$
\begin{aligned}
&\frac{1}{b'(u)} a(x,\ t,\ u,\ \nabla u)\, h_m''(v)\, T_k(v-\psi)\, \nabla v \\
=&a(x,\ t,\ u,\ \nabla u)\, h_m''(v)\, T_k(v-\psi)\, \nabla(T_{m+1}(v)) \in L^1(Q)
\end{aligned}
$$

and

$$
\frac{1}{b'(u)} T_k(v-\psi)\, \Theta_1 h_m''(v)\, \nabla v = T_k(v-\psi)\, \Theta_1 h_m''(v)\, \nabla T_{m+1}(v) \in L^1(Q).
$$

Taking the limit as $\varepsilon \to 0$ we obtain $\lim \varsigma_\varepsilon = 1_{[0,\ \tau]}$ and

$$
\begin{aligned}
&\int_{[0,\ \tau]} \left\langle \frac{\partial(h_m(v))}{\partial t},\ T_k(v-\psi) \right\rangle dt \\
&+\int_{[0,\ \tau]\times\Omega} a(x,\ t,\ u,\ \nabla u)\, h_m'(v)\, \nabla(T_k(v-\psi))\, dxdt \\
&+\int_{[0,\ \tau]\times\Omega} \frac{1}{b'(u)} a(x,\ t,\ u,\ \nabla u)\, h_m''(v)\, T_k(v-\psi)\, \nabla v dxdt \\
&+\int_{[0,\ \tau]\times\Omega} d(x)\, |u|^{p(x)-2}\, u h_m'(v)\, T_k(v-\psi)\, dxdt \\
=&\int_{[0,\ \tau]\times\Omega} F h_m'(v)\, T_k(v-\psi)\, dxdt
\end{aligned}
$$

$$+\int_{[0,\ \tau]\times\Omega}\frac{1}{b'(u)}T_k(v-\psi)\,\Theta_1 h''_m(v)\,\nabla v dxdt$$
$$+\int_{[0,\ \tau]\times\Omega}\Theta_1 h'_m(v)\,\nabla\left(T_k(v-\psi)\right)dxdt.$$

We assume that $m\geq k+\|\psi\|_{L^\infty(Q)}$ and obtain $T_k(v-\psi)=T_k(h_m(v)-\psi)$.

$$\begin{aligned}&\textstyle\int_{[0,\ \tau]}\left\langle\frac{\partial(h_m(v))}{\partial t},\ T_k(v-\psi)\right\rangle dt\\&=\textstyle\int_\Omega\tilde{T}_k\left(\bar{h}_m(v)(\tau)-\psi(\tau)\right)dx\\&-\textstyle\int_\Omega\tilde{T}_k\left(h_m(b(\varphi))-\psi(0)\right)dx\\&+\textstyle\int_{[0,\ \tau]}\left\langle\frac{\partial\psi}{\partial t},\ T_k(v-\psi)\right\rangle dt,\end{aligned}$$

where $\bar{h}$ is continuous function $\bar{h}(v):[0,\ \tau]\to L^1(\Omega)$ and equals $h(v)$ almost everywhere on $[0,\ T]$. By Lebesgue's dominated convergence theorem, we have

$$\int_\Omega\tilde{T}_k\left(\bar{h}_m(v)(\tau)-\psi(\tau)\right)dx\overset{m\to\infty}{\longrightarrow}\int_\Omega\tilde{T}_k(v-\psi)(\tau)\,dx$$

and

$$\int_\Omega\tilde{T}_k\left(h_m(b(\varphi))-\psi(0)\right)dx\overset{m\to\infty}{\longrightarrow}\int_\Omega\tilde{T}_k\left(b(\varphi)-\psi(0)\right)dx.$$

Next, we calculate

$$\begin{aligned}&\lim_{m\to\infty}\textstyle\int_{[0,\ \tau]\times\Omega}a(x,\ t,\ u,\ \nabla u)\,h'_m(v)\,\nabla\left(T_k(v-\psi)\right)dxdt\\&=\textstyle\int_{[0,\ \tau]\times\Omega}a(x,\ t,\ u,\ \nabla u)\,\nabla\left(T_k(v-\psi)\right)dxdt,\end{aligned}$$

similarly

$$\begin{aligned}&\lim_{m\to\infty}\textstyle\int_{[0,\ \tau]\times\Omega}d(x)\,|u|^{p(x)-2}\,uh'_m(v)\,T_k(v-\psi)\,dxdt\\&=\textstyle\int_{[0,\ \tau]\times\Omega}d(x)\,|u|^{p(x)-2}\,uT_k(v-\psi)\,dxdt.\end{aligned}$$

We have

$$\lim_{m\to\infty}\int_{[0,\ \tau]\times\Omega}Fh'_m(v)\,T_k(v-\psi)\,dxdt=\int_{[0,\ \tau]\times\Omega}FT_k(v-\psi)\,dxdt$$

and

$$\int_{[0,\ \tau]\times\Omega}\Theta_1 h'_m(v)\,\nabla\left(T_k(v-\psi)\right)dxdt\overset{m\to\infty}{\longrightarrow}\int_{[0,\ \tau]\times\Omega}\Theta_1\nabla\left(T_k(v-\psi)\right)dxdt.$$

Since $|h''_m(s)|\leq 1$ and $h''_m(s)\neq 0$ only for $|s|\in[m,\ m+1]$, we have

$$\left|\int_{[0,\ \tau]\times\Omega}\frac{1}{b'(u)}a(x,\ t,\ u,\ \nabla u)\,h''_m(v)\,T_k(v-\psi)\,\nabla v dxdt\right|$$
$$+\left|\int_{[0,\ \tau]\times\Omega}\frac{1}{b'(u)}T_k(v-\psi)\,\Theta_1 h''_m(v)\,\nabla v dxdt\right|$$

$$\leq c\int_{\{m\leq|v|\leq m+1\}}\left(\left(\alpha\left|\nabla u\right|^{p(x)-1}+\beta\left|u\right|^{p(x)-1}+\gamma\right)\left|\nabla v\right|+\left|\Theta_1\right|\left|\nabla v\right|\right)dxdt$$
$$\leq c\int_{\{m\leq|v|\leq m+1\}}\left(\left|\nabla u\right|^{p(x)}+\left|u\right|^{p(x)}+\gamma^{q(x)}+\left|\Theta_1\right|^{q(x)}+\left|\nabla v\right|^{p(x)}\right)dxdt$$
$$\leq c\int_{\{m\leq|v|\leq m+1\}}\left(\left|\nabla u\right|^{p(x)}+\left|u\right|^{p(x)}+\gamma^{q(x)}+\left|\Theta_1\right|^{q(x)}+\left|\nabla \Upsilon_2\right|^{p(x)}\right)dxdt,$$

hence $\left|\nabla v\right|^{p(x)}\leq const\left(\left|\nabla u\right|^{p(x)}+\left|\nabla\Upsilon_2\right|^{p(x)}\right)$. We can take the limit as m tends to infinity so the inequality

$$\begin{aligned}&\int_\Omega\tilde{T}_m\left(b\left(u\right)-\Upsilon_2-\psi\right)\left(T,\ x\right)dx\\&-\int_\Omega\tilde{T}_m\left(b\left(\varphi\right)-\psi\left(0,\ x\right)\right)dx\\&+\int_{[0,\ T]}\left\langle\partial_t\psi,\ T_m\left(b\left(u\right)-\Upsilon_2-\psi\right)\right\rangle dt\\&+\int_{[0,\ \tau]\times\Omega}a\left(x,\ t,\ u,\ \nabla u\right)\nabla T_m\left(b\left(u\right)-\Upsilon_2-\psi\right)dxdt\\&+\int_{[0,\ \tau]\times\Omega}d\left(x\right)\left|u\right|^{p(x)-2}uT_m\left(b\left(u\right)-\Upsilon_2-\psi\right)dxdt\\\leq&\int_{[0,\ \tau]\times\Omega}FT_m\left(b\left(u\right)-\Upsilon_2-\psi\right)dxdt\\&+\int_{[0,\ \tau]\times\Omega}\Theta_1\nabla\left(T_m\left(b\left(u\right)-\Upsilon_2-\psi\right)\right)dxdt\end{aligned}$$

holds for $\tau\in(0,\ T)$. Taking the limit as $\tau\to T$, we prove that an entropy solution is a renormalized solution.

Conversely, we assume that u is a renormalized solution so we have

$$\lim_{k\to\infty}\int_{\{(x,\ t)\in Q\ :\ k\leq|b(u)-\Upsilon_2|\leq k+1\}}a\left(x,\ t,\ u,\ \nabla u\right)\nabla udxdt=0.$$

For each $l\geq 1$, we take an element ϖ_l from $L^\infty\left(\Omega\right)\cap W_{1,0}^{p(\cdot)}\left(\Omega\right)$ such that $\left\|\varpi_l\right\|_{L^\infty(\Omega)}\leq m$ and $\lim\limits_{l\to\infty}\varpi_l=T_m\left(\varphi\right)$ almost everywhere in Ω, and $\lim\limits_{l\to\infty}\frac{\left\|\varpi_l\right\|_{L^{p(\cdot)}(\Omega)}}{l}=0$.

For all $m\geq 0$ and all $l\geq 1$, there exists the unique solution $T_m\left(w\right)_l\in L^{p(\cdot)}\left(\left(0,\ T\right),\ W_{1,\ 0}^{p(\cdot)}\left(\Omega\right)\right)\cap L^\infty\left(Q\right)$ to each problem

$$\frac{\partial T_m\left(w\right)_l}{\partial t}+l\left(T_m\left(w\right)_l-T_m\left(w\right)\right)=0,\tag{3.4}$$

in $D'\left(Q\right)$, and

$$T_m\left(w\right)_l\left(x,\ 0\right)=\varpi_l,\ x\in\Omega.\tag{3.5}$$

We have $T_m\left(w\right)_l\overset{L^{p(\cdot)}\left(\left(0,\ T\right),\ W_{1,\ 0}^{p(\cdot)}(\Omega)\right)}{\underset{l\to\infty}{\longrightarrow}}T_m\left(w\right)$ and $T_m\left(w\right)_l\overset{a.e.\ in\ Q}{\underset{l\to\infty}{\longrightarrow}}T_m\left(w\right)$ such that $\left\|T_m\left(w\right)\right\|_{L^\infty(\Omega)}\leq m$ and $\frac{\partial T_m(w)_l}{\partial t}\in L^{p(\cdot)}\left(\left(0,\ T\right),\ W_{1,\ 0}^{p(\cdot)}\left(\Omega\right)\right)$ for all $l\geq 1$.

We denote the duality pairing between $W_{1,\,0}^{p(\cdot)}(\Omega)+L^{\infty}(\Omega)$ and $W_{-1}^{q(\cdot)}(\Omega)+L^{1}(\Omega)$ by $\langle\cdot,\,\cdot\rangle$. We assume $m\ge 0$, $\varpi_k\in L^{p(\cdot)}\left((0,\,T),\,W_{1,\,0}^{p(\cdot)}(\Omega)\right)\cap L^{\infty}(Q)$, $\varpi\in L^{p(\cdot)}\left((0,\,T),\,W_{1,\,0}^{p(\cdot)}(\Omega)\right)\cap L^{\infty}(Q)$, $\frac{\partial\varpi_k}{\partial t}\in L^{q(\cdot)}\left((0,\,T),\,W_{-1}^{q(\cdot)}(\Omega)\right)+L^{1}(Q)$, and $v_k=h(\varpi_k)$. We calculate

$$\begin{aligned}&\int_{[0,\,T]}\int_{[0,\,t]}\left\langle\frac{\partial\varpi_k}{\partial t},\,\xi(\varpi_k)\left(T_m(\varpi_k)-T_m(\varpi)_l\right)\right\rangle dsdt\\
=&\frac{1}{2}\int_{[0,\,T]}\int_{\Omega}\xi(\varpi_k)^2\left|v_k-T_m(v)_l\right|^2dxdt\\
&-\frac{T}{2}\int_{[0,\,T]}\xi(\varpi_k)^2\left|v_k-T_m(v)_l\right|^2\Big|_{t=0}dx\\
&-\frac{1}{2}\int_{[0,\,T]}\int_{\Omega}\xi(\varpi_k)^2\left|v_k-T_m(v_k)\right|^2dxdt\\
&+\frac{T}{2}\int_{[0,\,T]}\xi(\varpi_k)^2\left|v_k-T_m(v_k)\right|^2\Big|_{t=0}dx\\
&+\int_{[0,\,T]}\int_{[0,\,t]}\int_{\Omega}\frac{\partial\left(T_m(v)_l\right)}{\partial t}\xi(\varpi_k)\left(v_k-T_m(v)_l\right)dxdsdt,\end{aligned}$$

hence

$$\int_{[0,\,t]}s-T_m(s)\,dsdt=\frac{1}{2}\left|t-T_m(t)\right|^2.$$

Since $v_k\underset{*-weak\ L^{\infty}(Q)}{\overset{L^2(Q)}{\longrightarrow}}v$, $v_k|_{t=0}\overset{L^2(\Omega)}{\longrightarrow}\varphi$, taking the limit as $k\to\infty$ and taking the lower limit as $l\to\infty$, we conclude

$$\begin{aligned}&\lim_{l\to\infty}\lim_{k\to\infty}\int_{[0,\,T]}\int_{[0,\,t]}\left\langle\frac{\partial\varpi_k}{\partial t},\,\xi(\varpi_k)\left(T_m(\varpi_k)-T_m(\varpi)_l\right)\right\rangle dsdt\\
=&\liminf_{l\to\infty}l\int_{\Omega}\int_{[0,\,T]}\int_{[0,\,t]}\xi(\varpi)\left(T_m(v)-T_m(v)_l\right)\left(v-T_m(v)_l\right)dsdtdx.\end{aligned}$$

And

$$\left(T_m(v)-T_m(v)_l\right)\left(v-T_m(v)_l\right)\ge 0,$$

thus, for $\varpi_k=b_k(u_k)-(\Upsilon_2)_k$ and $\xi\in W_1^{\infty}(R)$ being a nonnegative function with compact support, we obtain

$$\liminf_{l\to\infty}\lim_{k\to\infty}\int_{[0,\,T]}\int_{[0,\,t]}\left\langle\frac{\partial\varpi_k}{\partial t},\,\xi(\varpi_k)\left(T_m(\varpi_k)-T_m(\varpi)_l\right)\right\rangle dsdt\ge 0.$$

We take $\varpi=b(u)-\Upsilon_2$ and obtain

$$\begin{aligned}&\int_Q a(x,\,t,\,u,\,\nabla u)\nabla\left(T_k\left(\varpi-T_m(\varpi)_l\right)\right)dxdt\\
\le&\int_Q\tilde{T}_k\left(b(\varphi)-b(\varpi_l)\right)dxdt\\
&+\int_Q FT_k\left(\varpi-T_m(\varpi)_l\right)dxdt\end{aligned}$$

$$+\int_Q \Theta_1 \nabla \left(T_k \left(\varpi - T_m \left(\varpi\right)_l\right)\right) dxdt.$$

Taking the limit as $l \to \infty$ we have

$$\begin{aligned}&\int_Q a\left(x,\ t,\ u,\ \nabla u\right) \nabla \left(T_k \left(\varpi - T_m \left(\varpi\right)\right)\right) dxdt \\ \leq &\int_Q \tilde{T}_k \left(b\left(\varphi\right) - b\left(\varpi\right)\right) dxdt \\ &+ \int_Q F T_k \left(\varpi - T_m \left(\varpi\right)\right) dxdt \\ &+ \int_Q \Theta_1 \nabla \left(T_k \left(\varpi - T_m \left(\varpi\right)\right)\right) dxdt.\end{aligned}$$

Applying condition 4), we have

$$\begin{aligned}&\nu \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} |\nabla u|^{p(x)} dxdt \\ \leq &\int_Q a\left(x,\ t,\ u,\ \nabla u\right) \nabla \left(T_k \left(\varpi - T_m \left(\varpi\right)\right)\right) dxdt \\ &+ \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} a\left(x,\ t,\ u,\ \nabla u\right) \cdot \nabla \Upsilon_2 dxdt \\ \leq &k \int_\Omega \left|b\left(\varphi\right) - T_m\left(b\left(\varphi\right)\right)\right| dx + \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2|\}} |F| \, dxdt \\ &+ \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} |\Theta_1| \, |\nabla \varpi| \, dxdt \\ &+ \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} \left(\alpha \, |\nabla u|^{p(x)-1} + \beta \, |u|^{p(x)-1} + \gamma\right) |\nabla \Upsilon_2| \, dxdt.\end{aligned}$$

Since $|\nabla \varpi| \leq |\nabla u| + |\nabla \Upsilon_2|$ there is a constant c independent of k such that

$$\begin{aligned}&\nu \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} |\nabla u|^{p(x)} dxdt \\ \leq &k \int_{\{(x,\ t) \in Q\ :\ k \leq |b(\varphi)|\}} \left|b\left(\varphi\right)\right| dx \\ &+ k \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2|\}} |F| \, dxdt \\ &+ \frac{\nu}{2} \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} |\nabla u|^{p(x)} dxdt \\ &+ c \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} \left(|\Theta_1|^{q(x)} + |u|^{p(x)} + \gamma^{q(x)} + |\nabla \Upsilon_2|^{p(x)}\right) dxdt.\end{aligned}$$

Taking the limit as $k \to \infty$, from the inequality

$$\begin{aligned}&\frac{\nu}{2} \int_{\{(x,\ t) \in Q\ :\ k \leq |b(u) - \Upsilon_2| \leq k+m\}} |\nabla u|^{p(x)} dxdt \\ \leq &k \int_{\{(x,\ t) \in Q\ :\ k \leq |b(\varphi)|\}} \left|b\left(\varphi\right)\right| dx\end{aligned}$$

$$+ k \int_{\{(x,\,t)\in Q\,:\,k\le|b(u)-\Upsilon_2|\}} |F|\,dxdt$$
$$+ c \int_{\{(x,\,t)\in Q\,:\,k\le|b(u)-\Upsilon_2|\le k+m\}} \left(|\Theta_1|^{q(x)} + |u|^{p(x)} + \gamma^{q(x)} + |\nabla\Upsilon_2|^{p(x)}\right) dxdt$$

follows $\int_{\{(x,\,t)\in Q\,:\,k\le|b(u)-\Upsilon_2|\le k+1\}} |\nabla u|^{p(x)}\,dxdt \overset{k\to\infty}{\longrightarrow} 0$.

Now, we prove that an entropy solution and a renormalized solution coincide. We denote $\widehat{a}\,(x,\ t,\ \xi) \equiv a\,(x,\ t,\ u\,(x,\ t)\,,\ \xi)$ such that $\widehat{a}\,(x,\ t,\ \xi) \le \alpha\,|\xi|^{p(x)-1} + \widehat{\gamma}\,(x,\ t)$, where $\widehat{\gamma}\,(x,\ t) = \gamma\,(x,\ t) + \beta\,|u\,(x,\ t)|^{p(x)-1}$ for almost all $(x,\ t) \in Q$. We consider the simplified problem

$$\frac{\partial b\left(\widehat{u}\right)}{\partial t} - div\left(\widehat{a}\,\left(x,\ t,\ \nabla\,\widehat{u}\right)\right) + d\,(x)\left|\widehat{u}\right|^{p(x)-2}\widehat{u} = \mu,\ \mu \in \mathrm{M}_0\,(Q)\,, \tag{3.6}$$

$$b\left(\widehat{u}\right)(x,\ 0) = b\,(\varphi\,(x))\,,\ x \in \Omega, \tag{3.7}$$

$$\widehat{u}\Big|_{\partial\Omega\times(0,\,\infty)} = 0. \tag{3.8}$$

We assume that $\widehat{u}$ is a renormalized solution to the simplified problem and show that $u = \widehat{u}$, where u is an entropy solution. We take $T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)$ as a test function, where $\widehat{\varpi} = b\left(\widehat{u}\right) - \Upsilon_2$, and we obtain

$$\begin{aligned}
&\int_\Omega \tilde{T}_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)(T,\ x)\,dx \\
&- \int_\Omega \tilde{T}_m\,(b\,(\varphi) - h_k\,(b\,(\varphi)))\,dx \\
&+ \int_{[0,\,T]} \left\langle \partial_t\left(h_k\left(\widehat{\varpi}\right)\right),\ T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)\right\rangle dt \\
&+ \int_Q \widehat{a}\,(x,\ t,\ \nabla u)\,\nabla T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right) dxdt \\
&+ \int_Q d\,(x)\,|u|^{p(x)-2}\,uT_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right) dxdt \\
\le &\int_Q FT_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right) dxdt \\
&+ \int_Q \Theta_1\nabla\left(T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)\right) dxdt.
\end{aligned}$$

Each function h_k is bounded by $k+1$ such that

$$T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)$$
$$= T_m\left(T_{m+k+1}\,(\varpi) - h_k\left(\widehat{\varpi}\right)\right) \in L^{p(\cdot)}\left((0,\ T)\,,\ W^{p(\cdot)}_{1,\,0}\,(\Omega)\right) \cap L^\infty\,(Q)\,.$$

From the definition of a renormalized solution, we have

$$\int_Q \frac{\partial h_k\left(\widehat{\varpi}\right)}{\partial t} T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right) dxdt$$

$$+\int_Q \widehat{a}\left(x,\ t,\ \nabla\widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$+\int_Q \frac{1}{b'\left(\widehat{u}\right)}\widehat{a}\left(x,\ t,\ \nabla\widehat{u}\right)h_k''\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\nabla\widehat{\varpi}\,dxdt$$
$$+\int_Q d\left(x\right)\left|\widehat{u}\right|^{p(x)-2}\widehat{u}\,h_k'\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$=\int_Q Fh_k'\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$+\int_Q \frac{1}{b'\left(\widehat{u}\right)}\Theta_1h_k''\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\nabla\widehat{\varpi}\,dxdt$$
$$+\int_Q \Theta_1h_k'\left(\widehat{\varpi}\right)\nabla\left(T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\right)dxdt.$$

Since $|h_m''(s)|\le 1$ and $h_m''(s)=0$ for $|s|\notin[m,\ m+1]$, we obtain

$$\Bigg|\int_Q \frac{1}{b'\left(\widehat{u}\right)}\Theta_1h_k''\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\nabla\widehat{\varpi}\,dxdt$$
$$-\int_Q \frac{1}{b'\left(\widehat{u}\right)}\widehat{a}\left(x,\ t,\ \nabla\widehat{u}\right)h_k''\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\nabla\widehat{\varpi}\,dxdt\Bigg|$$
$$\le\frac{c_1}{\inf b'(u)}m\int_{\left\{k\le\left|\widehat{\varpi}\right|\le k+1\right\}}\left(|\Theta_1|^{q(x)}+\left|\nabla\widehat{u}\right|^{p(x)}+\widehat{\gamma}^{q(x)}+|\nabla\Upsilon_2|^{p(x)}\right)dxdt$$
$$\le w_1(k),$$

where $w_1(k)\overset{k\to\infty}{\longrightarrow}0$. So, we have

$$\int_Q \frac{\partial h_k\left(\widehat{\varpi}\right)}{\partial t}T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$\ge w_1(k)+\int_Q Fh_k'\left(\widehat{\varpi}\right)T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$+\int_Q \Theta_1h_k'\left(\widehat{\varpi}\right)\nabla\left(T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\right)dxdt$$
$$+\int_Q \widehat{a}\left(x,\ t,\ \nabla\widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$

and we estimate

$$\int_Q\left(\widehat{a}\left(x,\ t,\ \nabla u\right)-\widehat{a}\left(x,\ t,\ \nabla\widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\right)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$\le\int_Q\left(1-h_k'\left(\widehat{\varpi}\right)\right)FT_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$+\int_Q\Theta_1\left(1-h_k'\left(\widehat{\varpi}\right)\right)\nabla\left(T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)\right)dxdt$$

$$+\int_{\Omega}\tilde{T}_m\left(b\left(\varphi\right)-h_k\left(b\left(\varphi\right)\right)\right)dx+w_1\left(k\right).$$

Next, we deduce

$$\int_Q\left(\widehat{a}\ (x,\ t,\ \nabla u)-\widehat{a}\ \left(x,\ t,\ \nabla\ \widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\right)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$=\int_{\left\{(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|\le k\right\}}\left(\widehat{a}\ (x,\ t,\ \nabla u)-\widehat{a}\ \left(x,\ t,\ \nabla\ \widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\right)$$
$$\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$+\int_{\left\{(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|>k\right\}}\widehat{a}\ (x,\ t,\ \nabla u)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt$$
$$-\int_{\left\{(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|>k\right\}}\widehat{a}\ \left(x,\ t,\ \nabla\ \widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt.$$

We have $h_k\left(\widehat{\varpi}\right)=\widehat{\varpi}$, $h_k'\left(\widehat{\varpi}\right)=1$ and calculate

$$\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)$$
$$=1_{\left\{\left|\varpi-h_k\left(\widehat{\varpi}\right)\right|\le m\right\}}\left(\tfrac{1}{b'(u)}\nabla\varpi-h_k'\left(\widehat{\varpi}\right)\tfrac{1}{b'\left(\widehat{u}\right)}\nabla\ \widehat{\varpi}\right)$$
$$=1_{\left\{\left|\varpi-\widehat{\varpi}\right|\le m\right\}}\left(\tfrac{1}{b'(u)}\nabla\varpi-\tfrac{1}{b'\left(\widehat{u}\right)}\nabla\ \widehat{\varpi}\right)$$
$$=1_{\left\{\left|u-\widehat{u}\right|\le m\right\}}\left(\tfrac{1}{b'(u)}\nabla\varpi-\tfrac{1}{b'\left(\widehat{u}\right)}\nabla\ \widehat{\varpi}\right).$$

So, we estimate

$$\left|\int_{\left\{(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|>k\right\}}\widehat{a}\ (x,\ t,\ \nabla u)\nabla T_m\left(\varpi-h_k\left(\widehat{\varpi}\right)\right)dxdt\right|$$
$$\le\int_{\left\{(x,\ t)\in Q\ :\ k<\left|\widehat{\varpi}\right|,\ \left|\varpi-h_k\left(\widehat{\varpi}\right)\right|\le m\right\}}\frac{1}{b'\left(u\right)}\left|\widehat{a}\ \left(x,\ t,\ \nabla\ \widehat{u}\right)\nabla\varpi\right|dxdt$$
$$+\int_{\left\{(x,\ t)\in Q\ :\ k<\left|\widehat{\varpi}\right|,\ \left|\varpi-h_k\left(\widehat{\varpi}\right)\right|\le m\right\}}\frac{1}{b'\left(\widehat{u}\right)}h_k'\left(\widehat{\varpi}\right)\widehat{a}\ \left(x,\ t,\ \nabla\ \widehat{u}\right)\nabla\ \widehat{\varpi}\ dxdt$$
$$\le\int_{\left\{(x,\ t)\in Q\ :\ k-m\le\left|\widehat{\varpi}\right|\le k+m+1\right\}}\frac{1}{b'\left(u\right)}\left(\alpha\left|\nabla u\right|^{p(x)-1}+\widehat{\gamma}\ (x,\ t)\right)\left|\nabla\varpi\right|dxdt$$
$$+\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ k\le\left|\widehat{\varpi}\right|\le k+1,\\ k-m\le\left|\widehat{\varpi}\right|\le k+m+1\end{array}\right\}}\frac{1}{b'\left(\widehat{u}\right)}$$
$$\left(\alpha\left|\nabla u\right|^{p(x)-1}+\widehat{\gamma}\ (x,\ t)\right)\left|\nabla\ \widehat{\varpi}\right|dxdt$$
$$\le c\int_{\left\{(x,\ t)\in Q\ :\ k-m\le\left|\widehat{\varpi}\right|\le k+m+1\right\}}\left(\left|\nabla u\right|^{p(x)}+\widehat{\gamma}^{\,q(x)}+\left|\nabla\Upsilon_2\right|^{p(x)}\right)dxdt$$

$$+ c\int_{\left\{(x,\,t)\in Q\,:\,k\leq\left|\widehat{\varpi}\right|\leq k+1\right\}}\left|\nabla\,\widehat{u}\right|^{p(x)}dxdt = w_2\left(k\right),$$

where $w_2\left(k\right)\overset{k\to\infty}{\longrightarrow}0$ since the function $\widehat{u}$ is a renormalized solution. Then, we have

$$\left|\int_{\left\{(x,\,t)\in Q\,:\,\left|\widehat{\varpi}\right|>k\right\}}\widehat{a}\,\left(x,\ t,\ \nabla u\right)h_k'\left(\widehat{\varpi}\right)\nabla T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)dxdt\right|$$

$$\leq\frac{1}{\inf b'\left(u\right)}\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ k\leq\left|\widehat{\varpi}\right|\leq k+1,\\ \left|\varpi - h_k\left(\widehat{\varpi}\right)\right|\leq m\end{array}\right\}}$$

$$\left(\alpha\left|\nabla u\right|^{p(x)-1}+\widehat{\gamma}\,\left(x,\ t\right)\right)\left(\left|\nabla\varpi\right|+\left|\nabla\,\widehat{\varpi}\right|\right)dxdt$$

$$\leq c\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ k\leq\left|\widehat{\varpi}\right|\leq k+1,\\ k-m\leq\left|\varpi\right|\leq m+k+1\end{array}\right\}}$$

$$\left(\left|\nabla u\right|^{p(x)}+\left|\nabla\,\widehat{u}\right|^{p(x)}+\widehat{\gamma}^{\,q(x)}+\left|\nabla\Upsilon_2\right|^{p(x)}\right)dxdt = w_3\left(k\right),$$

where $w_3\left(k\right)\overset{k\to\infty}{\longrightarrow}0$.

Thus, we conclude

$$\int_Q\left(\widehat{a}\,\left(x,\ t,\ \nabla u\right)-\widehat{a}\,\left(x,\ t,\ \nabla\,\widehat{u}\right)h_k'\left(\widehat{\varpi}\right)\right)\nabla T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)dxdt$$

$$\geq\frac{1}{\sup b'(u)}\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|\leq k,\\ \left|b\left(u\right)-b\left(\widehat{u}\right)\right|\leq m\end{array}\right\}}$$

$$\left(\widehat{a}\,\left(x,\ t,\ \nabla u\right)-\widehat{a}\,\left(x,\ t,\ \nabla\,\widehat{u}\right)\right)\left(\nabla u-\nabla\,\widehat{u}\right)dxdt - w_4\left(k\right),$$

where $w_4\left(k\right)\overset{k\to\infty}{\longrightarrow}0$. So

$$\frac{1}{\sup b'\left(u\right)}\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right|\leq k,\\ \left|b\left(u\right)-b\left(\widehat{u}\right)\right|\leq m\end{array}\right\}}$$

$$\left(\widehat{a}\,\left(x,\ t,\ \nabla u\right)-\widehat{a}\,\left(x,\ t,\ \nabla\,\widehat{u}\right)\right)\left(\nabla u-\nabla\,\widehat{u}\right)dxdt$$

$$\leq m\int_Q\left|F\right|\left|1-h_k'\left(\widehat{\varpi}\right)\right|dxdt$$

$$+\int_Q\left|\Theta_1\right|\left|1-h_k'\left(\widehat{\varpi}\right)\right|\left|\nabla\left(T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)\right)\right|dxdt$$

$$+\int_Q\left|b\left(\varphi\right)-h_k\left(b\left(\varphi\right)\right)\right|dx + w_5\left(k\right),$$

where $w_5\left(k\right)\overset{k\to\infty}{\longrightarrow}0$. We have

$$\int_Q\left|\Theta_1\right|\left|1-h_k'\left(\widehat{\varpi}\right)\right|\left|\nabla\left(T_m\left(\varpi - h_k\left(\widehat{\varpi}\right)\right)\right)\right|dxdt$$

$$\leq c \int_{\{(x,\, t)\in Q\,:\, k-m\leq |\varpi|\leq m+k+1\}} \left(|\Theta_1|^{q(x)} + |\nabla u|^{p(x)} + |\nabla \Upsilon_2|^{p(x)} \right) dxdt$$
$$+ c \int_{\left\{(x,\, t)\in Q\,:\, k\leq \left|\widehat{\varpi}\right| \leq k+1\right\}} \left(|\Theta_1|^{q(x)} + \left|\nabla \widehat{u}\right|^{p(x)} + |\nabla \Upsilon_2|^{p(x)} \right) dxdt.$$

Therefore, we have

$$\int_{\left\{\begin{array}{l}(x,\ t)\in Q\ :\ \left|\widehat{\varpi}\right| \leq k, \\ \left|b(u) - b\left(\widehat{u}\right)\right| \leq m\end{array}\right\}} \left(\widehat{a}\ (x,\ t,\ \nabla u) - \widehat{a}\ \left(x,\ t,\ \nabla \widehat{u}\right)\right) \left(\nabla u - \nabla \widehat{u}\right) dxdt$$

$$\leq w_6\,(k) \overset{k\to\infty}{\longrightarrow} 0.$$

By Fatou's lemma, we have

$$\int_{\left\{(x,\ t)\in Q\,:\, \left|b(u)-b\left(\widehat{u}\right)\right|\leq m\right\}} \left(\widehat{a}\ (x,\ t,\ \nabla u) - \widehat{a}\ \left(x,\ t,\ \nabla \widehat{u}\right)\right) \left(\nabla u - \nabla \widehat{u}\right) dxdt$$

$$\leq 0.$$

Hence

$$\left(\widehat{a}\ (x,\ t,\ \nabla u) - \widehat{a}\ \left(x,\ t,\ \nabla \widehat{u}\right)\right) \left(\nabla u - \nabla \widehat{u}\right) \geq 0$$

on $\left\{(x,\ t)\in Q\ :\ \left|b(u) - b\left(\widehat{u}\right)\right| \leq m\right\}$. Therefore, we obtain

$$\left(\widehat{a}\ (x,\ t,\ \nabla u) - \widehat{a}\ \left(x,\ t,\ \nabla \widehat{u}\right)\right) \left(\nabla u - \nabla \widehat{u}\right) = 0$$

almost everywhere on $\left\{(x,\ t)\in Q\ :\ \left|b(u) - b\left(\widehat{u}\right)\right| \leq m\right\}$, and since $b(u) - b\left(\widehat{u}\right)$ is almost everywhere finite, we deduce

$$\left(\widehat{a}\ (x,\ t,\ \nabla u) - \widehat{a}\ \left(x,\ t,\ \nabla \widehat{u}\right)\right) \left(\nabla u - \nabla \widehat{u}\right) = 0$$

almost everywhere on Q. Thus, we conclude $\nabla u - \nabla \widehat{u} = 0$ almost everywhere on Q. We have $T_1\left(T_{m+1}(\varpi) - T_{m+1}\left(\widehat{\varpi}\right)\right) \in L^{p(\cdot)}\left((0,\ T),\ W^{p(\cdot)}_{1,\,0}(\Omega)\right)$ so

$$T_1\left(T_{m+1}(\varpi) - T_{m+1}\left(\widehat{\varpi}\right)\right)$$
$$= \begin{cases} 0 \ on\ \left\{|\varpi| \leq m+1,\ \left|\widehat{\varpi}\right| \leq m+1\right\} \\ \frac{1}{b'(u)}\nabla\varpi 1_{\left\{\left|\varpi - T_{m+1}\left(\widehat{\varpi}\right)\right|\leq 1\right\}}\ on\ \left\{|\varpi| \leq m+1,\ \left|\widehat{\varpi}\right| > m+1\right\} \\ -\frac{1}{b'\left(\widehat{u}\right)}\nabla \widehat{\varpi}\, 1_{\left\{\left|\widehat{\varpi} - T_{m+1}(\varpi)\right|\leq 1\right\}}\ on\ \left\{|\varpi| > m+1,\ \left|\widehat{\varpi}\right| \leq m+1\right\} \end{cases}$$

and

$$\int_Q \left|T_1\left(T_{m+1}(\varpi) - T_{m+1}\left(\widehat{\varpi}\right)\right)\right|^{p(x)} dxdt$$
$$\leq \tfrac{1}{\inf b'(u)} \int_{\{m\leq|\varpi|\leq m+1\}} |\nabla\varpi|^{p(x)}\, dxdt + \tfrac{1}{\inf b'\left(\widehat{u}\right)} \int_{\left\{m\leq\left|\widehat{\varpi}\right|\leq m+1\right\}} \left|\nabla \widehat{\varpi}\right|^{p(x)} dxdt.$$

Thus, we obtain

$$T_1\left(T_{m+1}\left(\varpi\right)-T_{m+1}\left(\widehat{\varpi}\right)\right)\underset{L^{p(\cdot)}\left((0,\,T),\,W_{1,\,0}^{p(\cdot)}(\Omega)\right)}{\overset{m\to\infty}{\longrightarrow}}0.$$

So

$$T_1\left(T_{m+1}\left(\varpi\right)-T_{m+1}\left(\widehat{\varpi}\right)\right)\underset{a.e.\ Q}{\overset{m\to\infty}{\longrightarrow}}T_1\left(\varpi-\widehat{\varpi}\right),$$

and $T_1\left(u-\widehat{u}\right)=T_1\left(\varpi-\widehat{\varpi}\right)=0$. Hence $u=\widehat{u}$. Thus, the function u is a renormalized solution.

□

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