Journal of Nonlinear Modeling and Analysis
Volume 7, Number 4, August 2025, 1482–1496
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1482

Existence and Uniqueness Results for Solutions to Fractional $p(\cdot,\cdot)$-Laplacian Problems with a Variable-Order Derivative

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Abstract This paper investigates a class of fractional problems involving the variable-order $p(\cdot,\cdot)$-Laplacian with homogeneous Dirichlet boundary conditions. Under suitable assumptions on the nonlinear term, we establish novel existence and uniqueness results for weak solutions. We achieve this by combining variational techniques with a result from the theory of monotone operators. Additionally, we reveal several interesting properties of the solution.

Keywords Fractional $p(\cdot,\cdot)$-Laplacian, uniqueness, monotone operator theory, variational methods

MSC(2010) 35A15, 35D30, 35J35, 35J60.

1. Introduction

In recent decades, there has been a notable surge in interest and significance surrounding nonlinear problems that involve nonlocal and fractional pseudo-differential operators. The exploration of these problems has been motivated by their wide-ranging applications across various fields of applied sciences. These applications encompass physics and engineering, population dynamics, finance, chemical reaction design, optimization, minimal surfaces, and game theory (as detailed in references [9, 16, 17, 24]). Moreover, differential equations and variational problems with variable exponents have gained great attention due to their strong physical relevance. As evidenced in references [1, 13, 22], such equations emerge in the mathematical modeling of fluid dynamics, including electrorheological and thermorheological fluids. They are also encountered in elastic mechanics, image restoration, and biology (as indicated in references [11, 20, 21, 30]). Notably, recent research on fractional $p(x,\cdot)$-Laplacian problems and the corresponding variational problems can be found in references [2, 4–6, 10, 14, 15, 26, 27].

In this current paper, we focus on establishing the existence and uniqueness of

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weak solutions for the subsequent problem:

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^{\kappa(\cdot,\cdot)} u + a(x)|u|^{p(x,x)-2}u = \mu f(x,u) \text{ in } \Omega, \\ \\ \qquad\qquad\qquad u = 0 \qquad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{P_μ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, the variable exponent $p(\cdot,\cdot) : \mathbb{R}^{2N} \to (1,\infty)$ and the variable fractional $\kappa(\cdot,\cdot)$-order $\kappa(\cdot,\cdot) : \mathbb{R}^{2N} \to (0,1)$, are continuous functions, with $N > \kappa(x,y)p(x,y)$ for all $(x,y) \in \mathbb{R}^{2N}$, they fulfill the following two conditions respectively:

$$p(\cdot,\cdot) \text{ is symmetric and } 1 < \inf_{(x,y)\in\mathbb{R}^{2N}} p(x,y) =: p^- \leq \sup_{(x,y)\in\mathbb{R}^{2N}} p(x,y) =: p^+ < \infty, \tag{1.1}$$

$$\kappa(\cdot,\cdot) \text{ is symmetric and } 0 < \inf_{(x,y)\in\mathbb{R}^{2N}} \kappa(x,y) =: \kappa^- \leq \sup_{(x,y)\in\mathbb{R}^{2N}} \kappa(x,y) =: \kappa^+ < 1, \tag{1.2}$$

$a : \overline{\Omega} \to [0,\infty)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\mu > 0$ is parameter and $\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)}$ denotes the variable fractional $\kappa(\cdot,\cdot)$-order fractional $p(\cdot,\cdot)$-Laplacian operator defined as

$$\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)} u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+\kappa(x,y)p(x,y)}} dy, \; x \in \mathbb{R}^N,$$

where $p.v.$ is employed as an abbreviation in the principal value sense.

Note that when $\kappa(\cdot,\cdot) = \kappa$ (constant), $\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)}$ becomes the fractional $p(x,\cdot)$-Laplacian operator and problem (P_μ) reduces to fractional $p(x,\cdot)$-Laplacian problem studied by M. Ait Hammou [2]. By employing the Berkovits topological degree theory, the author proved the existence of at least one weak solution for (P_μ).

The variable-order fractional derivatives extend the concept of constant-order fractional derivatives, first proposed by S. G. Samko and B. Ross [23]. In this approach, the derivative order can vary continuously based on dependent or independent variables, allowing for a better representation of memory effects over time or space [7]. C. F. Lorenzo and T. T. Hartley later applied this concept to model diffusion processes that respond to temperature fluctuations [18], which can also be used to describe temperature changes [19].

Very recently, considerable attention has been focused by many researchers about the existence of at least one or multiple solutions for $p(x,\cdot)$-Laplacian problems in the fractional variable-order case see (for example [8, 28, 29, 31]). In [8], R. Biswas and S. Tiwari considered the following fractional nonlocal Choquard problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \lambda|u(x)|^{\alpha(x)-2}u(x) + \left(\displaystyle\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy\right) f(x,u(x)) \text{ in } \Omega, \\ \\ \qquad\qquad u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

under suitable assumption on α, μ, s, f and by using the variational approach, the authors have established the existence of at least two distinct nontrivial weak solutions to the problem. In [29], the existence and uniqueness of weak solutions to variable-order fractional $p(x,\cdot)$-Laplacian equation have been discussed.

Inspired by the above facts and aforementioned research papers, the principal objective of this paper is to prove the existence and uniqueness of solutions through the application of variational techniques in conjunction with a result derived from the theory of monotone operators as presented in reference [25, Theorems 25.F]. Additionally, we elucidate certain properties of the solutions pertaining to problem (P_μ).

2. Preliminaries

Initially, we provide pertinent notations and fundamental outcomes concerning variable exponent Lebesgue spaces, which will serve as essential tools in establishing the principal theorems (see [12]). For $\xi \in C(\overline{\Omega}, (1, \infty))$, we write

$$\max_{x\in\overline{\Omega}} \xi(x) =: \xi^+ \text{ and } \min_{x\in\overline{\Omega}} \xi(x) =: \xi^-.$$

Let us establish the definition of the variable exponent Lebesgue space as follows:

$$L^{\xi(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u|^{\xi(x)} dx < \infty \right\}.$$

$L^{\xi(\cdot)}(\Omega)$ endowed with the norm

$$||u||_{\xi(\cdot)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u}{\tau} \right|^{\xi(x)} dx \le 1 \right\},$$

is a separable and reflexive Banach space. Let $L^{\widehat{\xi}(\cdot)}(\Omega)$ be the conjugate space of $L^{\xi(\cdot)}(\Omega)$ with $\widehat{\xi}(x) = \dfrac{\xi(x)}{\xi(x) - 1}$.

Then the following Hölder-type inequality holds.

Lemma 2.1 ([12]). *Let $u \in L^{\xi(\cdot)}(\Omega)$ and $v \in L^{\widehat{\xi}(\cdot)}(\Omega)$. Then*

$$\int_\Omega |uv| dx \le 2||u||_{\xi(\cdot)} ||v||_{\widehat{\xi}(\cdot)}.$$

On the space $L^{\xi(\cdot)}(\Omega)$, we define the modular function given by

$$\rho_{\xi(\cdot)}(u) = \int_\Omega |u|^{\xi(x)} dx.$$

Lemma 2.2 ([12]). *For any $u \in L^{\xi(\cdot)}(\Omega)$, we have*

$$\min \left(||u||^{\xi^-}_{\xi(\cdot)}, ||u||^{\xi^+}_{\xi(\cdot)} \right) \le \rho_{\xi(\cdot)}(u) \le \max \left(||u||^{\xi^-}_{\xi(\cdot)}, ||u||^{\xi^+}_{\xi(\cdot)} \right).$$

For any $x \in \mathbb{R}^N$, we put $p(x, x) =: \widetilde{p}(x)$ and $\kappa(x, x) =: \widetilde{\kappa}(x)$. Subsequently, we introduce the concept of the variable exponent fractional Sobolev space to provide a suitable variational framework for addressing our problem. We define $W^{\kappa(\cdot,\cdot),p(\cdot,\cdot)}(\Omega)$ as the variable exponent fractional Sobolev space according to the following characterization:

$\mathbb{W} := W^{\kappa(\cdot,\cdot),p(\cdot,\cdot)}(\Omega)$

$$= \left\{ u \in L^{\widetilde{p}(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy < \infty, \text{ for some } \zeta > 0 \right\}.$$

Equip $\mathbb{W}$ with the norm

$$||u||_{\mathbb{W}} = [u]_{\mathbb{W}} + ||u||_{\widetilde{p}(\cdot)},$$

where

$$[u]_{\mathbb{W}} = \inf\left\{ \zeta > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \le 1 \right\}.$$

Then $(\mathbb{W}, ||u||_{\mathbb{W}})$ is a reflexive Banach space. For any $x \in \overline{\Omega}$, we set

$$p^{\star}_{\kappa}(x) := \frac{N\widetilde{p}(x)}{N - \widetilde{\kappa}(x)\widetilde{p}(x)}.$$

Lemma 2.3 ([26, 27]). *Assume that (1.1) and (1.2) hold. Then for any $\xi \in C(\overline{\Omega}, (1,\infty))$ such that $\xi(x) < p^{\star}_{\kappa}(x)$ for all $x \in \overline{\Omega}$. Then the embedding $\mathbb{W} \hookrightarrow L^{\xi(\cdot)}(\Omega)$ is continuous. Moreover, this embedding is compact.*

Because of the homogeneous Dirichlet boundary condition on $\mathbb{R}^N\backslash\Omega$, it becomes necessary to incorporate this constraint into the weak formulation of (P_μ). To address this, we introduce a new function space:

$$\begin{aligned} \mathbb{X} &:= X^{\kappa(\cdot,\cdot),p(\cdot,\cdot)}(\Omega) \\ &= \Big\{ u : \mathbb{R}^N \to \mathbb{R},\ u|_{\Omega} \in L^{\widetilde{p}(\cdot)}(\Omega), \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+\kappa(x,y)p(x,y)}}\, dx\, dy < \infty, \\ &\quad \text{for some } \zeta > 0 \Big\}, \end{aligned}$$

where $\mathcal{Q} = (\mathbb{R}^N \times \mathbb{R}^N)\backslash(\Omega^c \times \Omega^c)$. Endow $\mathbb{X}$ with the norm

$$||u||_{\mathbb{X}} = [u]_{\mathbb{X}} + ||u||_{\widetilde{p}(\cdot)},$$

where

$$[u]_{\mathbb{X}} = \inf\left\{ \zeta > 0 : \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \le 1 \right\}.$$

In the same way $(\mathbb{X}, ||\cdot||_{\mathbb{X}})$ is a separable reflexive Banach space.

Since the variable exponents p, $\widetilde{p}$ and ξ are continuous, we can extend p to $\mathbb{R}^N \times \mathbb{R}^N$ and $\widetilde{p}, \xi$ to $\mathbb{R}^N$ continuously under the conditions given in Lemma 2.3. Let $\mathbb{X}_0$ be the linear space:

$$\mathbb{X}_0 = \{u \in \mathbb{X} : u = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\}$$

equipped with the norm

$$||u||_{\mathbb{X}_0} = [u]_{\mathbb{X}} = \inf\left\{ \zeta > 0 : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{\zeta^{p(x,y)}|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \le 1 \right\}.$$

Obviously, $(\mathbb{X}_0, ||\cdot||_{\mathbb{X}_0})$ is a reflexive Banach space. Set

$$\rho_0(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \text{ for all } u \in \mathbb{X}_0.$$

Lemma 2.4 ([26]). *For all $u, u_n \in \mathbb{X}_0$, the following properties hold true:*

(1) $||u||_{\mathbb{X}_0} > 1 \Rightarrow ||u||_{\mathbb{X}_0}^{p^-} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^+}$;

(2) $||u||_{\mathbb{X}_0} \leq 1 \Rightarrow ||u||_{\mathbb{X}_0}^{p^+} \leq \rho_0(u) \leq ||u||_{\mathbb{X}_0}^{p^-}$;

(3) $||u_n - u||_{\mathbb{X}_0} \to 0 \Leftrightarrow \rho_0(u_n - u) \to 0$.

Lemma 2.5. *[8, Theorem 3.6] Assume that (1.1) and (1.2) hold. Then for any $\xi \in C(\overline{\Omega}, (1, \infty))$ such that $\xi(x) < p_\kappa^\star(x)$ for all $x \in \overline{\Omega}$. Then the embedding $\mathbb{X}_0 \hookrightarrow L^{\xi(\cdot)}(\Omega)$ is continuous. Moreover, this embedding is compact.*

Remark 2.1. Since $1 < \widetilde{p}(x) < p_s^\star(x)$ for all $x \in \overline{\Omega}$, by Lemma 2.5, the norms $|| \cdot ||_{\mathbb{X}_0}$ and $|| \cdot ||_{\mathbb{X}}$ are equivalent in $\mathbb{X}_0$.

Let $\mathcal{L}$ denote the operator associated with $\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)}$ defined as

$$\begin{aligned} \mathcal{L} : \mathbb{X}_0 &\to \mathbb{X}_0^* \\ u &\mapsto \mathcal{L}(u) : \mathbb{X}_0 \ni \phi \mapsto \langle \mathcal{L}(u), \phi \rangle \in \mathbb{R} \end{aligned}$$

such that

$$\langle \mathcal{L}(u), \phi \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+\kappa(x,y)p(x,y)}} dxdy,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between $\mathbb{X}_0$ and its dual space $\mathbb{X}_0^*$.

The proof of the following lemma can be shown similarly to the proofs of Lemma 4.2 in [6] and Proposition 2.6 in [28].

Lemma 2.6. *Assume that (1.1) and (1.2) hold. Then, the following assertions hold:*

($\mathcal{L}_1$) *$\mathcal{L}$ is a bounded and strictly monotone operator on $\mathbb{X}_0$;*

($\mathcal{L}_2$) *$\mathcal{L}$ is a mapping of type (S_+), that is, if $\liminf_{k\to\infty} \langle \mathcal{L}(u_k) - \mathcal{L}(u), u_k - u \rangle \leq 0$, and $u_k \rightharpoonup u$ in $\mathbb{X}_0$, then $u_k \to u$ in $\mathbb{X}_0$;*

($\mathcal{L}_3$) *$\mathcal{L}$ is a homeomorphism on $\mathbb{X}_0$.*

For the proof of Theorem 3.1, we will apply the following lemma from monotone operator theory.

Lemma 2.7. *[25, Theorems 25.F] Let $(\mathbb{E}, || \cdot ||_{\mathbb{E}})$ be a reflexive Banach space and let the functional $I \in C^1(\mathbb{E}, \mathbb{R})$ satisfy the following two properties:*

(*i*) *I is weakly coercive on $\mathbb{E}$ (i.e., $I(v) \to \infty$ as $||v||_{\mathbb{E}} \to \infty$ on $\mathbb{E}$);*

(*ii*) *I' is strictly monotone on $\mathbb{E}$ (i.e., $\langle I'(z) - I'(w), z - w \rangle > 0$ for all $z, w \in \mathbb{E}$ with $z \neq w$).*

Then, there exists a unique $\tilde{z} \in \mathbb{E}$ such that $\langle I'(\tilde{z}), w \rangle = 0$ for all $w \in \mathbb{E}$ and $I(\tilde{z}) = \inf_{z\in\mathbb{E}} I(z)$.

3. Main result

Prior to presenting our principal theorem, we shall initially formulate certain assumptions concerning the data associated with problem (P_μ).

(F_1) $f(x,t)$ is non-increasing in the second variable for each fixed $x \in \Omega$;

(F_2) There exist $g \in L^{\widehat{\widetilde{p}}(\cdot)}(\Omega)$, $\gamma \in C(\overline{\Omega}, [0,\infty))$ and $\sigma > 0$ such that $p^- > \gamma^+$ and

$$|f(x,t)| \le |g(x)| + \sigma |t|^{\gamma(x)-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $L^{\widehat{\widetilde{p}}(\cdot)}(\Omega)$ is the conjugate space of $L^{\widetilde{p}(\cdot)}(\Omega)$ with $\widehat{\widetilde{p}}(x) = \dfrac{\widetilde{p}(x)}{\widetilde{p}(x)-1}$;

(F_3) There exist $\overline{\alpha} > 0$, $x_0 \in \Omega$ and $s_2 > s_1 > 0$ such that

$$\begin{cases} B_{s_1}(x_0) \subset B_{s_2}(x_0) \subset \Omega, \\ f(x,\overline{\alpha}) \ge 0 \text{ for all } x \in B_{s_2}(x_0) \backslash B_{s_1}(x_0), \\ \limsup\limits_{\alpha \to 0^+} \dfrac{\inf_{x \in B_{s_1}(x_0)} F(x,\alpha)}{\alpha^{p^-}} = \infty, \end{cases}$$

where $B_s(x_0)$ is the open $s-$ ball in $\mathbb{R}^N$ centered at x_0, and $F(x,\alpha) := \displaystyle\int_0^\alpha f(x,\tau)d\tau$.

Next, we give the sense in which we will take a solution to problem (P_μ).

Definition 3.1. We say that $u \in \mathbb{X}_0$ is a weak solution of problem (P_μ) if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy + \int_\Omega a(x)|u|^{\widetilde{p}(x)-2} u\phi dx$$
$$= \mu \int_\Omega f(x,u)\phi dx \text{ for all } \phi \in \mathbb{X}_0.$$

It is a widely recognized fact that the weak solution of (P_μ) corresponds to a critical point of the energy functional, which is defined on the space $\mathbb{X}_0$ as follows:

$$\begin{aligned} \mathcal{I}_\mu(u) &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy + \int_\Omega a(x) \frac{|u|^{\widetilde{p}(x)}}{\widetilde{p}(x)} dx - \mu \int_\Omega F(x,u)dx \\ &= \Psi(u) - \mu\Phi(u). \end{aligned} \tag{3.1}$$

By a standard argument we show that $\mathcal{I}_\mu \in C^1(\mathbb{X}_0, \mathbb{R})$, and its derivative is given by

$$\begin{aligned} \langle \mathcal{I}_\mu'(u), \phi \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \\ &\quad + \int_\Omega a(x)|u|^{\widetilde{p}(x)-2} u\phi dx - \mu \int_\Omega f(x,u)\phi dx \\ &= \langle \Psi'(u), \phi \rangle - \mu \langle \Phi'(u), \phi \rangle \text{ for all } u, \phi \in \mathbb{X}_0. \end{aligned} \tag{3.2}$$

The main result can be stated as follows.

Theorem 3.1. *Assume that the assumptions (F_1), (F_2) and (F_3) hold. Then for each $\mu > 0$, problem (P_μ) possesses a unique nontrivial weak solution $u_\mu \in \mathbb{X}_0$ verifying:*

(i) $\mathcal{I}_\mu(u_\mu) = \inf_{u\in\mathbb{X}_0} \mathcal{I}_\mu(u) < 0$;

(ii) *The mapping $(0,\infty) \ni \mu \mapsto \mathcal{I}_\mu(u_\mu)$ is strictly decreasing;*

(iii) *For each nonempty bounded set $\Lambda \subset (0,\infty)$, there exists $C_{\mu,\Lambda} > 0$ such that*

$$||u_\mu||_{\mathbb{X}_0} \leq C_{\mu,\Lambda} \text{ for all } \mu \in \Lambda.$$

Moreover, $||u_\mu||_{\mathbb{X}_0} \to 0$ as $\mu \to 0^+$.

4. Proof of the main result

Proof of Theorem 3.1. To apply Lemma 2.4, we need the following two lemmas.

Lemma 4.1. *For any $\mu > 0$, the energy functional $\mathcal{I}_\mu : \mathbb{X}_0 \to \mathbb{R}$ is weakly coercive.*

Proof. Let $u \in \mathbb{X}_0$ such that $||u||_{\mathbb{X}_0} > 1$. By (F_2), (3.1) and Lemmas 2.1, 2.4 and 2.5, it follows that

$$\begin{aligned}
\mathcal{I}_\mu(u) &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy + \int_\Omega a(x)\frac{|u|^{\widetilde{p}(x)}}{\widetilde{p}(x)}dx - \mu\int_\Omega F(x,u)dx \\
&\geq \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy - \mu\int_\Omega |F(x,u)|dx \\
&\geq \frac{1}{p^+}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy - \mu\int_\Omega |g(x)||u|dx - \mu\sigma\int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}dx \\
&\geq \frac{1}{p^+}||u||_{\mathbb{X}_0}^{p^-} - 2\mu||g||_{\widehat{\widetilde{p}}(\cdot)}||u||_{\widetilde{p}(\cdot)} - \frac{\mu\sigma}{\gamma^-}||u||_{\gamma(\cdot)}^{\gamma^+} \\
&\geq \frac{1}{p^+}||u||_{\mathbb{X}_0}^{p^-} - 2\mu C_1||g||_{\widehat{\widetilde{p}}(\cdot)}||u||_{\mathbb{X}_0} - \frac{\mu\sigma}{\gamma^-}C_2^{\gamma^+}||u||_{\mathbb{X}_0}^{\gamma^+}.
\end{aligned}$$

This implies that $\mathcal{I}_\mu(u) \to \infty$ as $||u||_{\mathbb{X}_0} \to \infty$ on $\mathbb{X}_0$, since $p^- > \gamma^+$ and $p^- > 1$. Consequently, $\mathcal{I}_\mu$ is weakly coercive. The proof is finished. □

Lemma 4.2. *For any $\mu > 0$, the mapping $\mathcal{I}_\mu' : \mathbb{X}_0 \to \mathbb{X}_0^*$ is strictly monotone.*

Proof. Let $u, v \in \mathbb{X}_0$ with $u \neq v$. Set $\mathcal{U}(x,y) = u(x) - u(y)$ and $\mathcal{V}(x,y) = v(x) - v(y)$. From (F_1) and (3.2) we have

$$\begin{aligned}
&\langle \mathcal{I}_\mu'(u) - \mathcal{I}_\mu'(v), u - v\rangle \\
&= \int_{\mathbb{R}^{2N}} \left(\frac{|\mathcal{U}(x,y)|^{p(x,y)-2}\mathcal{U}(x,y)}{|x-y|^{N+\kappa(x,y)p(x,y)}} - \frac{|\mathcal{V}(x,y)|^{p(x,y)-2}\mathcal{V}(x,y)}{|x-y|^{N+\kappa(x,y)p(x,y)}}\right)(\mathcal{U}(x,y)-\mathcal{V}(x,y))\,dxdy \\
&\quad + \int_\Omega a(x)\left(|u|^{\widetilde{p}(x)-2}u - |v|^{\widetilde{p}(x)-2}v\right)(u-v)dx - \mu\int_\Omega (f(x,u)-f(x,v))(u-v)dx \\
&\geq \int_{\mathbb{R}^{2N}} \left(\frac{|\mathcal{U}(x,y)|^{p(x,y)-2}\mathcal{U}(x,y)}{|x-y|^{N+\kappa(x,y)p(x,y)}} - \frac{|\mathcal{V}(x,y)|^{p(x,y)-2}\mathcal{V}(x,y)}{|x-y|^{N+\kappa(x,y)p(x,y)}}\right)(\mathcal{U}(x,y)-\mathcal{V}(x,y))\,dxdy \\
&\quad + \int_\Omega a(x)\left(|u|^{\widetilde{p}(x)-2}u - |v|^{\widetilde{p}(x)-2}v\right)(u-v)dx. \qquad (4.1)
\end{aligned}$$

We refer to the following inequalities, credited to J. Simon [3]:

$$\langle |\zeta|^{r-2}\zeta - |\eta|^{r-2}\eta, \zeta-\eta\rangle \geq \begin{cases} \dfrac{(r-1)|\zeta-\eta|^2}{(|\zeta|+|\eta|)^{2-r}} & \text{if } 1<r<2, \\ \\ \dfrac{1}{2^r}|\zeta-\eta|^r & \text{if } r\geq 2, \end{cases} \tag{4.2}$$

for any $\zeta, \eta \in \mathbb{R}^d$ with $d \in \mathbb{N}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^d$.

Now, from (4.1), (4.2) and Lemma 2.6($\mathcal{L}_1$), we see that

$$\langle \mathcal{I}_\mu^{'}(u) - \mathcal{I}_\mu^{'}(v), u-v\rangle > 0 \text{ for all } u, v \in \mathbb{X}_0 \text{ with } u \neq v.$$

Consequently, $\mathcal{I}_\mu^{'}$ is a strictly monotone operator. □

Thus, from Lemmas 4.1 and 4.2, and by applying [25, Theorems 25.F], we conclude that problem (P_μ) has a unique weak solution $u_\mu \in \mathbb{X}_0$.

Now, we show that $u_\mu \not\equiv 0$ and (i) holds, for this we define the following function:

$$\psi : \mathbb{R}^N \to \mathbb{R}$$

$$x \mapsto \psi(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \backslash B_{s_2}(x_0), \\ \dfrac{\operatorname{dist}(x,x_0)-s_2}{s_1-s_2} & \text{if } x \in B_{s_2}(x_0)\backslash B_{s_1}(x_0), \\ 1 & \text{if } x \in B_{s_1}(x_0), \end{cases}$$

where $\operatorname{dist}(x,x_0)$ is the Euclidean distance on $\mathbb{R}^N$. Then $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^N$. Moreover, $\psi \in \mathbb{X}_0$. In fact, clearly, $\psi|_\Omega \in L^{\widetilde{p}(\cdot)}(\Omega)$. Observe that

$$|x-y|^{N+\kappa(x,y)p(x,y)} \geq \begin{cases} |x-y|^{N+\kappa^+p^+} & \text{if } |x-y|\geq 1, \\ \\ |x-y|^{N+\kappa^-p^-} & \text{if } |x-y|<1. \end{cases}$$

Then

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}dxdy + \iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^-p^-}}dxdy. \end{aligned} \tag{4.3}$$

Put:

$$\begin{cases} \Omega_1 = B_{s_1}(x_0) \times \left(\mathbb{R}^N \setminus B_{s_2}(x_0)\right), \\ \Omega_2 = B_{s_1}(x_0) \times \left(B_{s_2}(x_0) \setminus B_{s_1}(x_0)\right), \\ \Omega_3 = \left(B_{s_2}(x_0) \setminus B_{s_1}(x_0)\right) \times \left(B_{s_2}(x_0) \setminus B_{s_1}(x_0)\right), \\ \Omega_4 = \left(B_{s_2}(x_0) \setminus B_{s_1}(x_0)\right) \times \left(\mathbb{R}^N \setminus B_{s_2}(x_0)\right). \end{cases}$$

Then, we can write

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}\,dxdy = {} & 2\iint_{\Omega_1} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}\,dxdy \\ & + 2\iint_{\Omega_2} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}\,dxdy \\ & + 2\iint_{\Omega_3} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}\,dxdy \\ & + 2\iint_{\Omega_4} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}\,dxdy \\ =: {} & 2I_1^+ + 2I_2^+ + 2I_3^+ + 2I_4^+. \end{aligned} \tag{4.4}$$

Before estimating the integrals I_1^+, I_2^+, I_3^+ and I_4^+ by direct calculations, we first recall that if $f : B_s(0) \ni x \mapsto f(x)$ is a continuous and radial function (i.e., there exists a function $\widetilde{f} : [0,s] \ni x \mapsto \widetilde{f}(|x|)$), then

$$\int_{B_s(0)} f(x)dx = \omega_{N-1}\int_0^s \widetilde{f}(r)r^{N-1}dr,$$

where ω_{N-1} is $(N-1)$-dimensional measure of the unit sphere and $r = |x| = \left(\sum_{i=1}^N x_i^2\right)^{\frac{1}{2}}$. Note that if $s \to \infty$, it is now straightforward to see that if f is a continuous and improperly Riemann integrable function on $\mathbb{R}^N$ which is radial, then

$$\int_{\mathbb{R}^N} f(x)dx = \omega_{N-1}\int_0^\infty \widetilde{f}(r)r^{N-1}dr.$$

From the definition of ψ and the preceding reminders, we deduce that

$$\begin{aligned} I_1^+ &= \iint_{\Omega_1} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}dxdy \\ &= \int_{B_{s_1}(x_0)} \left(\int_{\mathbb{R}^N\setminus B_{s_2}(x_0)} \frac{1}{|x-y|^{N+\kappa^+p^+}}dy\right)dx \\ &= \int_{B_{s_1}(x_0)} \left(\int_{\mathbb{R}^N\setminus B_{s_2}(x-x_0)} \frac{1}{|z|^{N+\kappa^+p^+}}dz\right)dx \\ &= \omega_{N-1}\int_{s_2}^\infty \frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds \int_{B_{s_1}(x_0)} dx \\ &= \varpi_N\omega_{N-1}s_1^N\int_{s_2}^\infty \frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds \\ &= \frac{\varpi_N\omega_{N-1}}{\kappa^+p^+}s_1^N s_2^{-\kappa^+p^+}, \end{aligned} \tag{4.5}$$

where ϖ_N is N-dimensional measure of the unit ball,

$$I_2^+ = \iint_{\Omega_2} \frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}dxdy$$

$$
\begin{aligned}
&=\iint_{\Omega_2} \frac{1}{|x-y|^{N+\kappa^+p^+}}\left|\frac{s_2-\operatorname{dist}(y,x_0)}{s_1-s_2}\right|^{p(x,y)} dxdy\\
&\le \int_{B_{s_1}(x_0)}\left(\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)} \frac{1}{|x-y|^{N+\kappa^+p^+}}dy\right)dx\\
&= \int_{B_{s_1}(x_0)}\left(\int_{B_{s_2}(x-x_0)\setminus B_{s_1}(x-x_0)} \frac{1}{|z|^{N+\kappa^+p^+}}dz\right)dx\\
&=\omega_{N-1}\int_{s_1}^{s_2}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\int_{B_{s_1}(x_0)}dx\\
&=\varpi_N\omega_{N-1}s_1^N\int_{s_1}^{s_2}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\\
&\le\frac{\varpi_N\omega_{N-1}}{\kappa^+p^+}s_1^{N-\kappa^+p^+},
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
I_3^+ &=\iint_{\Omega_3}\frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}dxdy}dxdy\\
&=\iint_{\Omega_3}\frac{1}{|x-y|^{N+\kappa^+p^+}}\left|\frac{\operatorname{dist}(y,x_0)-\operatorname{dist}(x,x_0)}{s_1-s_2}\right|^{p(x,y)}dxdy\\
&\le\iint_{\Omega_3}\frac{1}{|x-y|^{N+\kappa^+p^+}}dxdy\\
&=\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}\left(\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}\frac{1}{|x-y|^{N+\kappa^+p^+}}dy\right)dx\\
&=\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}\left(\int_{B_{s_2}(x-x_0)\setminus B_{s_1}(x-x_0)}\frac{1}{|z|^{N+\kappa^+p^+}}dz\right)dx\\
&=\omega_{N-1}\int_{s_1}^{s_2}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}dx\\
&=\varpi_N\omega_{N-1}(s_2^N-s_1^N)\int_{s_1}^{s_2}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\\
&\le\frac{\varpi_N\omega_{N-1}}{\kappa^+p^+}s_2^N s_1^{-\kappa^+p^+}
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
I_4^+ &=\iint_{\Omega_4}\frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}dxdy\\
&=\iint_{\Omega_4}\frac{1}{|x-y|^{N+\kappa^+p^+}}dxdy\\
&=\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}\left(\int_{\mathbb{R}^N\setminus B_{s_2}(x_0)}\frac{1}{|x-y|^{N+\kappa^+p^+}}dy\right)dx\\
&=\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}\left(\int_{\mathbb{R}^N\setminus B_{s_2}(x-x_0)}\frac{1}{|z|^{N+\kappa^+p^+}}dz\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&=\omega_{N-1}\int_{s_2}^{\infty}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)}dx\\
&=\varpi_N\omega_{N-1}(s_2^N-s_1^N)\int_{s_2}^{\infty}\frac{s^{N-1}}{s^{N+\kappa^+p^+}}ds\\
&\leq\frac{\varpi_N\omega_{N-1}}{\kappa^+p^+}s_2^{N-\kappa^+p^+}. \qquad (4.8)
\end{aligned}
$$

By (4.4)-(4.8), there is a constant $C_1 := C(N, p^+, \kappa^+, s_1, s_2) > 0$ such that

$$\iint_{\mathbb{R}^{2N}}\frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^+p^+}}dxdy\leq C_1.$$

Analogously, there is a constant $C_2 := C(N, p^-, \kappa^-, s_1, s_2) > 0$ such that

$$\iint_{\mathbb{R}^{2N}}\frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa^-p^-}}dxdy\leq C_2.$$

Therefore, it follows from (4.3) that

$$\iint_{\mathbb{R}^{2N}}\frac{|\psi(x)-\psi(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy\leq C_1+C_2<\infty,$$

which yields that $\psi\in\mathbb{X}_0$. By (F_3) there exists a sequence of $\{\alpha_n\}\subset(0,\overline{\alpha}]$ such that

$$\alpha_n\to 0 \quad\text{and}\quad \alpha_n^{-p^-}\inf_{x\in B_{s_1}(x_0)}F(x,\alpha_n)\to\infty \text{ as } n\to\infty. \qquad (4.9)$$

From (4.9), there exists $\delta > 0$ and for n large enough, one has,

$$0<\alpha_n<\min\left\{1,||\psi||_{\widetilde{p}(\cdot)}^{-1}\right\}\quad\text{and}\quad\inf_{x\in B_{s_1}(x_0)}F(x,\alpha_n)\geq\delta\alpha_n^{p^-}.$$

Then we have

$$||\alpha_n\psi||_{\widetilde{p}(\cdot)}<1 \text{ and } \int_{B_{s_1}(x_0)}F(x,\alpha_n)dx\geq\int_{B_{s_1}(x_0)}\delta\alpha_n^{p^-}dx=\delta\alpha_n^{p^-}\varpi_N s_1^N. \qquad (4.10)$$

Furthermore, by virtue of Lemma 2.4(2), we deduce the following

$$
\begin{aligned}
\Psi(\alpha_n\psi(x))&=\int_{\mathbb{R}^{2N}}\frac{|\alpha_n\psi(x)-\alpha_n\psi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy+\int_\Omega a(x)\frac{|\alpha_n\psi(x)|^{\widetilde{p}(x)}}{\widetilde{p}(x)}dx\\
&\leq\frac{\alpha_n^{p^-}}{p^-}\left(||\psi||_{\mathbb{X}_0}^{p^-}+||a||_1\right). \qquad (4.11)
\end{aligned}
$$

For any $x\in B_{s_2}(x_0)\setminus B_{s_1}(x_0)$, by the assumption (F_1), we obtain

$$
\begin{aligned}
F(x,\alpha_n\psi(x))&=\int_0^{\alpha_n\psi(x)}f(x,\tau)d\tau\geq\alpha_n\psi(x)f(x,\alpha_n\psi(x))\\
&\geq\alpha_n\psi(x)f(x,\overline{\alpha})\\
&\geq 0, \qquad (4.12)
\end{aligned}
$$

since $f(x,\overline{\alpha}) \geq 0$ and $0 \leq \alpha_n\psi(x) \leq \overline{\alpha}$. For any fixed $\Theta > 0$, choose the above constant δ large enough so that

$$\Theta < \frac{\delta\alpha_n^{p^-}\varpi_N s_1^N}{||\psi||_{\mathbb{X}_0}^{p^-} + ||a||_1}. \tag{4.13}$$

From (4.11)-(4.13), and for any n large enough, we derive that

$$\begin{aligned}\frac{\Phi(\alpha_n\psi)}{\Psi(\alpha_n\psi)} &= \frac{\displaystyle\int_{B_{s_1}(x_0)} F(x,\alpha_n\psi)dx + \int_{B_{s_2}(x_0)\setminus B_{s_1}(x_0)} F(x,\alpha_n\psi)dx}{\Psi(\alpha_n\psi)} \\ &\geq \frac{\displaystyle\int_{B_{s_1}(x_0)} F(x,\alpha_n)dx}{\Psi(\alpha_n\psi)} \\ &\geq \frac{\delta\alpha_n^{p^-}\varpi_N s_1^N}{||\psi||_{\mathbb{X}_0}^{p^-} + ||a||_1} > \Theta.\end{aligned}$$

Then $\dfrac{\Phi(\alpha_n\psi)}{\Psi(\alpha_n\psi)} \to \infty$ as $n \to \infty$, for Θ large enough. Hence, for n large enough, we have $\mathcal{I}_\mu(\alpha_n\psi) = \Psi(\alpha_n\psi) - \mu\Phi(\alpha_n\psi) < 0$. Then, from the fact that $\alpha_n\psi \in \mathbb{X}_0$ and u_μ is a global minimum of $\mathcal{I}_\mu$ in $\mathbb{X}_0$, we have $\mathcal{I}_\mu(u_\mu) < 0$, which gives that $u_\mu \not\equiv 0$ and (i) is satisfied.

Next, we prove (ii). For all $u \in \mathbb{X}_0$ and $\mu > 0$, in view of (3.1), we have $\mathcal{I}_\mu(u) = \mu\mathcal{J}_\mu(u)$, where

$$\mathcal{J}_\mu(u) = \frac{\Psi(u)}{\mu} - \Phi(u). \tag{4.14}$$

Let $0 < \mu_1 < \mu_2$ be fixed. Then from (i) and for $k = 1, 2$, we get that

$$\mu_k\mathcal{J}_{\mu_k}(u_{\mu_k}) = \mathcal{I}_{\mu_k}(u_{\mu_k}) = \inf_{u\in\mathbb{X}_0} \mathcal{I}_{\mu_k}(u) = \mu_k \inf_{u\in\mathbb{X}_0} \mathcal{J}_{\mu_k}(u) < 0.$$

Thus, $\mathcal{J}_{\mu_k}(u_{\mu_k}) = \inf_{u\in\mathbb{X}_0} \mathcal{J}_{\mu_k}(u)$. Moreover, in view of (4.14), we have $\mathcal{J}_{\mu_2}(u_{\mu_2}) \leq \mathcal{J}_{\mu_1}(u_{\mu_1})$. Therefore,

$$\mathcal{I}_{\mu_2}(u_{\mu_2}) = \mu_2\mathcal{J}_{\mu_2}(u_{\mu_2}) \leq \mu_2\mathcal{J}_{\mu_1}(u_{\mu_1}) < \mu_1\mathcal{J}_{\mu_1}(u_{\mu_1}) = \mathcal{I}_{\mu_1}(u_{\mu_1}).$$

Hence, (ii) holds.

Finally, we show that (iii). Arguing by contradiction, we assume that there exists $\{\mu_n\} \subset \Lambda$ such that $||u_{\mu_n}||_{\mathbb{X}_0} \to \infty$ as $n \to \infty$. Then, by the assumption (F_2), Lemma 2.4(1) and Lemma 2.3, for n large enough, we have

$$\begin{aligned}||u_{\mu_n}||_{\mathbb{X}_0}^{p^-} &\leq \int_{\mathbb{R}^{2N}} \frac{|u_{\mu_n}(x) - u_{\mu_n}(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}}dxdy \\ &\leq \langle\Psi'(u_{\mu_n}), u_{\mu_n}\rangle = \mu_n\langle\Phi'(u_{\mu_n}), u_{\mu_n}\rangle \\ &\leq \mu_n \int_\Omega |u_{\mu_n} f(x,u_{\mu_n})|dx \\ &\leq \overline{\mu}\int_\Omega g(x)|u_{\mu_n}|dx + \overline{\mu}\sigma\int_\Omega |u_{\mu_n}|^{\gamma(x)}dx\end{aligned}$$

$$\leq 2\overline{\mu}||g||_{\widehat{p}(\cdot)}||u_{\mu_n}||_{\widetilde{p}(\cdot)} + \overline{\mu}\sigma||u_{\mu_n}||_{\gamma(\cdot)}^{\gamma^+}$$
$$\leq 2\overline{\mu}C_3||g||_{\widehat{p}(\cdot)}||u_{\mu_n}||_{\mathbb{X}_0} + \overline{\mu}\sigma C_4||u_{\mu_n}||_{\mathbb{X}_0}^{\gamma^+}, \tag{4.15}$$

where $\overline{\mu} := \sup_{n\in\mathbb{N}} \mu_n < \infty$. We derive a contradiction since $p^- > \gamma^+$, and therefore there exists $C_{\mu,\Lambda} > 0$ such that $||u_\mu||_{\mathbb{X}_0} \leq C_{\mu,\Lambda}$ for all $\mu \in \Lambda$. In particular, there exists $C_5 > 1$ such that

$$||u_\mu||_{\mathbb{X}_0} \leq C_5 \text{ for all } \mu \in (0,1).$$

Then, similar to (4.15), we entail

$$\begin{aligned} ||u_\mu||_{\mathbb{X}_0}^{p^+} &\leq \int_{\mathbb{R}^{2N}} \frac{|u_\mu(x) - u_\mu(y)|^{p(x,y)}}{|x-y|^{N+\kappa(x,y)p(x,y)}} dxdy \\ &\leq 2\mu||g||_{\widehat{p}(\cdot)}||u_\mu||_{\widetilde{p}(\cdot)} + \mu\sigma||u_\mu||_{\gamma(\cdot)}^{\gamma^+} \\ &\leq 2\mu C_3||g||_{\widehat{p}(\cdot)}||u_\mu||_{\mathbb{X}_0} + \mu\sigma C_4||u_\mu||_{\mathbb{X}_0}^{\gamma^+} \\ &\leq 2\mu C_3C_5||g||_{\widehat{p}(\cdot)} + \mu\sigma C_4C_5^{\gamma^+} \to 0 \quad \text{as } \mu \to 0. \end{aligned}$$

Hence it yields that $||u_\mu||_{\mathbb{X}_0} \to 0$ as $\mu \to 0^+$. This ends the proof of Theorem 3.1.

Acknowledgements

The authors thank the anonymous referees for invaluable comments and insightful suggestions which improved the presentation of this manuscript.

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