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Subdifferential Frictional Contact Problem with Thermo-Electro-Visco-Elastic Locking Materials: Analysis and Approximation

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Abstract This paper investigates a frictional contact problem involving a thermo-electro-visco-elastic model for locking materials in contact with a rigid foundation. Friction is described by the subgradient of a locally Lipschitz function, while contact is governed by Signorini's unilateral condition. We formulate the problem as a system of three hemivariational inequalities and establish an existence and uniqueness theorem using a fixed-point argument and recent advances in hemivariational inequalities theory. Finally, we present a fully discrete finite element approximation of the model and derive error estimates for the approximate solution.

Keywords Thermo-electro-visco-elastic materials, locking piezoelectric, frictional contact problem, finite element method, error estimate

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1. Introduction

The investigation of piezoelectric materials has gained significant attention in recent years due to their extensive applications across various industries, including railways, automotive systems, civil engineering, and aeronautics. These materials possess unique properties, such as the ability to produce electrical charges under mechanical deformation and to undergo mechanical deformation when exposed to electric fields. Nonetheless, the interaction between a deformable piezoelectric structure and a conductive base presents intricate challenges. For example, friction-induced energy dissipation can heat the material, and through the pyroelectric effect, this heating affects certain piezoelectric systems by generating electrical charges or voltage. Analyzing these complex interactions requires a thorough study of coupled thermo-electro-mechanical phenomena, especially in contact problems with or without friction. Understanding these behaviors is essential for accurately modeling electro-elastic materials in real-world applications.

This study explores a contact problem involving a nonlinear thermo-visco-electro-elastic body with locking materials in contact with a rigid foundation. The model is formulated using hemivariational inequalities that account for visco-piezoelectric, thermal, subdifferential friction, and material degradation effects. For a comprehensive discussion on contact problems in piezoelectric and visco-piezoelectric materials,

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see [2, 17, 21, 24, 26–28, 32, 36, 38, 45, 46], while further insights into thermo-visco-piezoelectric materials are available in [1, 3, 7–9, 16, 18–20, 22, 29, 30, 39, 42–44, 47]. Furthermore, the concept of locking materials, first introduced by Prager [40, 41], describes materials that exhibit a significant increase in stiffness beyond a certain deformation threshold, effectively restricting further strain. This behavior is particularly observed in the study of contact problems, where interactions between a deformable body and a rigid foundation can lead to localized stress concentrations. Incorporating locking effects into our model allows for a more accurate representation of real-world materials, where excessive deformation must be restricted to prevent structural failure. In the context of thermo-electro-visco-elasticity, accounting for these effects is crucial, as they influence the overall mechanical response, frictional dissipation, and electro-thermal coupling in the system. For further discussions on the physical interpretation of locking materials, we refer the reader to [5, 21] and the references therein.

The analysis is performed over a time interval $[0, T]$, where $T > 0$, with time derivatives indicated by dots (e.g., $\dot{u} = \frac{\partial u}{\partial t}$). The focus is on thermo-electro-visco-elastic materials with locking properties, without explicitly expressing the dependence of various functions on the independent variables $x \in \Omega \cup \Gamma$. The governing laws of such materials are given as follows:

$$\sigma(t) \in \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) - \mathscr{P}^T E(\varphi(t)) - \mathscr{C}\theta(t) + \partial I_{L_1}\varepsilon(u(t)), \tag{1.1}$$

$$D(t) \in \mathscr{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathscr{G}\theta(t) + \partial I_{L_2} E(\varphi(t)), \tag{1.2}$$

$$\dot{\theta}(t) - div\,\mathscr{K}(\nabla\theta(t)) - h_0(t) \in \mathscr{M}\varepsilon(u)(t) - \mathscr{N}E(\varphi(t)) + \partial I_{L_3}\nabla\theta(t), \tag{1.3}$$

in which σ is the stress tensor, u is the displacement field, φ represents the electric potential field and θ is the temperature field. Moreover, $\partial I_{L_1} : \mathbb{S}^d \to 2^{\mathbb{S}^d}$, $\partial I_{L_2} : L^2(\Omega) \to 2^{L^2(\Omega)}$ and $\partial I_{L_3} : L^2(\Omega) \to 2^{L^2(\Omega)}$ stand for the subdifferentials of the indicator maps of the sets L_1, L_2 and L_3, defined by

$$I_{L_i}(q, \varepsilon) = \begin{cases} 0 & \text{if} \quad \varepsilon \in L_i\,, \\ +\infty & \text{if} \quad \varepsilon \notin L_i\,. \end{cases}$$

The subsets $L_1 \subset \mathbb{S}^d$, $L_1 \subset L^2(\Omega)$ and $L_2 \subset L^2(\Omega)$ define the locking constraints and characterize the material properties. These sets can take various forms, as explored in [5]. In this paper, we specifically focus on the case of perfectly locking materials, where the sets L_1, L_2, and L_3 are given by

$$L_1 = \{\varepsilon \in \mathbb{S}^d : Q_1(\varepsilon) \leq 0\}, \tag{1.4}$$

$$L_2 = \{\psi \in L^2(\Omega) : Q_2(\psi) \leq 0\}, \tag{1.5}$$

and

$$L_3 = \{\theta \in L^2(\Omega) : Q_3(\theta) \leq 0\}. \tag{1.6}$$

Here, the locking functions $Q_1 : \mathbb{S}^d \to \mathbb{R}$, $Q_2 : L^2(\Omega) \to \mathbb{R}$, and $Q_3 : L^2(\Omega) \to \mathbb{R}$ are convex, continuous, and the initial condition $Q_i(0) \leq 0$ for $i = 1, 2, 3$.

Mathematically, models describing thermo-electro-visco-elastic materials are relatively recent advancements, as seen in works such as [20, 43]. The first contribution of this paper is the extension of these models to thermo-electro-visco-elastic contact problems involving locking materials, framed within the theory of hemivariational inequalities. We establish both the existence and uniqueness of a solution for Signorini's contact problem with non-monotone boundary conditions defined by the Clarke subdifferential. The second contribution is the introduction of a numerical analysis of the hemivariational inequalities arising in locking thermo-electro-visco-elastic contact problems. Specifically, we present a fully discrete numerical scheme, where we apply the finite element method for spatial approximation and finite differences for time derivatives, leading to the derivation of error estimates for the approximate solutions. For further details on the numerical treatment of elastic and electro-elastic contact problems, we refer the reader to [5, 6, 44, 46] and references therein.

The paper is structured as follows. Section 2 is dedicated to studying the existence and uniqueness of the unique solution to a contact problem for locking materials. Additionally, we derive the variational formulation of this problem, expressed as a coupled system of three hemivariational inequalities and a parabolic equation. Finally, in Section 3, we examine a fully discrete approximation of the related model, and we derive error estimates and convergence results.

2. Problem description and weak formulation

In this section, we analyze a quasi-static contact problem for a nonlinear thermo-electro-visco-elastic body with locking materials. The problem is governed by unilateral constraints involving a multi-valued normal compliance function and a non-monotone, slip-dependent multi-valued friction condition. We first outline the physical context and then present its classical variational-hemivariational formulation as a system of three hemivariational inequalities. Finally, we investigate the existence and uniqueness of weak solutions to this system.

We consider a thermo-visco-piezoelectric-locking material's body occupying the domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$. The domain Ω is assumed to be open, bounded, and connected, with a Lipschitz boundary $\Gamma = \partial\Omega$. The body is subjected to body forces f_0, a volume free electric charge q_0, a surface electric charge q_b, and a heat source h_n. Mechanically and electrically, it is constrained on Γ, and to describe these constraints, we partition Γ into three open, measurable parts Γ_1, Γ_2, and Γ_3 such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma$ with $meas(\Gamma_1) > 0$. Furthermore, we decompose $\Gamma_1 \cup \Gamma_2$ into two open, measurable subsets Γ_a and Γ_b with $meas(\Gamma_a) > 0$.

Throughout this paper, the indices i, j, k range from 1 to d, the summation over repeated indices is implied, and an index following a comma represents the partial derivative with respect to the corresponding component of the variable. Let $\mathbb{S}^d$ denote the space of second-order symmetric tensors on $\mathbb{R}^d$. We use the notation " $\cdot$ " and $\|\cdot\|$ to represent the inner product and the associated Euclidean norm on both $\mathbb{R}^d$ and $\mathbb{S}^d$, defined as follows:

$$u \cdot v = u_i\, v_i \ , \quad \|v\| = (v \cdot v)^{1/2}, \qquad \forall\, u,\, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij}\, \tau_{ij} \ , \quad \|\tau\| = (\tau \cdot \tau)^{1/2}, \qquad \forall\, \sigma,\, \tau \in \mathbb{S}^d.$$

Let ν denote the outward unit normal to Γ. The normal and tangential components of the displacement vector $v \in \mathbb{R}^d$ and the stress tensor $\sigma \in \mathbb{S}^d$ on the boundary Γ are given by:

$$v_\nu = v \cdot \nu \ , \quad v_\tau = v - v_\nu \nu \quad \text{and} \quad \sigma_\nu = (\sigma\nu) \cdot \nu \ , \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

From the two orthogonality relations $v_\tau \cdot \nu = 0$ and $\sigma_\tau \cdot \nu = 0$, we derive the following useful result:

$$\sigma\nu \cdot v = \sigma_\nu\, v_\nu + \sigma_\tau \cdot v_\tau.$$

Thus, the classical formulation of our locking thermo-visco-piezoelectric contact problem, is given by the following:

Problem (P). *Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ and a temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}$ that satisfy the following conditions:*

$$\sigma(t) \in \mathscr{A}\varepsilon(\dot u(t)) + \mathscr{B}\varepsilon(u(t)) - \mathscr{P}^T E(\varphi(t)) - \mathscr{C}\theta(t) + \partial I_{L_1}\varepsilon(u(t)) \quad \text{in } \Omega \times [0,T], \tag{2.1}$$

$$D(t) \in \mathscr{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathscr{G}\theta(t) + \partial I_{L_2} E(\varphi(t)) \quad \text{in } \Omega \times [0,T], \tag{2.2}$$

$$\dot\theta(t) - div\mathscr{K}(\nabla\theta(t)) - h_0(t) \in \mathscr{M}\varepsilon(u)(t) - \mathscr{N}E(\varphi(t)) + \partial I_{L_3}\nabla\theta(t) \quad \text{in } \Omega \times [0,T], \tag{2.3}$$

$$Div\,\sigma(t) + f_0(t) = 0 \quad \text{in } \Omega \times [0,T], \tag{2.4}$$

$$div\, D(t) - q_0(t) = 0 \quad \text{in } \Omega \times [0,T], \tag{2.5}$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{2.6}$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a \times [0,T], \tag{2.7}$$

$$\theta(t) = 0 \quad \text{on } \Gamma_1 \times [0,T], \tag{2.8}$$

$$\sigma(t)\,\nu = f_2(t) \quad \text{on } \Gamma_2 \times [0,T], \tag{2.9}$$

$$D(t) \cdot \nu = q_b(t) \quad \text{on } \Gamma_b \times [0,T], \tag{2.10}$$

$$q(t) \cdot \nu = h_n(t) \quad \text{on } \Gamma_2 \times [0,T], \tag{2.11}$$

$$-\sigma_\nu(t) \in \partial j_\nu(\dot u_\nu(t)), \quad -\sigma_\tau(t) \in \partial j_\tau(\dot u_\tau(t)) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.12}$$

$$D(t) \cdot \nu \in h_e(u_\nu(t))\,\partial j_e(\varphi(t) - \varphi_0) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.13}$$

$$-\mathscr{K}(\nabla\theta(t)) \cdot \nu \in \partial j_\theta(\theta(t)) \quad \text{on } \Gamma_3 \times [0,T], \tag{2.14}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{2.15}$$

The conditions (2.1)-(2.3) define the constitutive laws for thermo-electro-visco-elastic materials with locking properties; further details can be found in [1,35,39,43]. Specifically, the tensors $\mathscr{A}$ and $\mathscr{B}$ represent the viscous and elastic components, respectively. The piezoelectric and thermal expansion tensors are denoted by $\mathscr{P}$ and $\mathscr{C}$, respectively, while β and $\mathscr{G}$ represent the electric permittivity and pyroelectric tensors, respectively. Additionally, $\mathscr{K}$ denotes the thermal conductivity tensor, $\mathscr{M}$ represents the thermo-mechanical coupling operator, and $\mathscr{N}$ stands for the thermo-electric coupling vector field. The linearized strain tensor is given by $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$, and the electric field by $E(\varphi) = -\nabla\varphi$. The transpose of the piezoelectric tensor is denoted by $\mathscr{P}^T = (e_{ijk})^T = (e_{kij})$.

The equilibrium equations for the stress and electric displacement fields are formulated in (2.4)-(2.5), where Div and div denote divergence operators for tensors and vector-valued functions, respectively. The boundary conditions describing mechanical, electrical, and thermal interactions are specified in (2.6)-(2.11). Furthermore, the conditions (2.12) address the normal stress and normal velocity under a non-monotone damped response and friction law, where j_ν and j_τ are locally Lipschitz functions, and ∂j_ν and ∂j_τ denote their Clarke generalized gradients. The regularized electrical contact condition on Γ_3 is given in (2.13), incorporating the foundation's electric potential φ_0 and the prescribed functions h_e and j_e. Finally, heat exchange between the surface Γ_3 and the foundation is governed by (2.14), while the initial conditions for displacement and temperature are outlined in (2.15).

Remark 2.1 ([13–15, 48])**.** This remark provides an overview of two fundamental notions in nonsmooth analysis: the convex subgradient and Clarke subgradient. These concepts are crucial in various applications, particularly in contact problems, where hemivariational and variational methods are often required to handle unilateral constraints and frictional effects. The convex subgradient is central to the study of convex functions, offering a natural extension of the gradient for nondifferentiable cases. Given a convex function $f : X \to \mathbb{R}$, its convex subdifferential at a point $x \in X$ is the set

$$\partial f(x) = \big\{\xi \in X^*\,;\quad f(y) \geq f(x) + \langle \xi, y - x\rangle,\quad \forall\, y \in X\big\}.$$

This set characterizes affine functions that provide global lower bounds for f, ensuring that f remains convex in a generalized sense. The convex subdifferential has a key role in variational inequalities and optimization problems, particularly in contact mechanics where energy functionals often exhibit convexity.

The Clarke subgradient, on the other hand, extends the notion of convex subgradients to nonsmooth, possibly nonconvex functions. Given a locally Lipschitz function $j : X \to \mathbb{R}$, its Clarke subdifferential at a point $u \in X$ is defined as

$$\partial j(u) = \big\{\xi \in X^*\,;\quad j^0(u; v) \geq \langle \xi, v\rangle,\quad \forall\, v \in X\big\},$$

where $j^0(u; v)$ is the Clarke directional derivative of j at $u \in X$ in the direction $v \in X$, given by

$$j^0(u; v) := \limsup_{\lambda \to 0^+,\, w \to u} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

Unlike the convex subgradient, the Clarke subgradient accounts for nonsmooth and nonconvex behaviors, making it a powerful tool in quasidifferentiability, differential inclusions, and nonsmooth mechanics. In contact problems, it is particularly useful for handling friction laws, where discontinuities in tangential forces require a more general subdifferential framework. These two notions provide complementary perspectives in nonsmooth analysis, enabling a rigorous formulation of variational principles and optimality conditions in contact mechanics and beyond.

To derive the variational-hemivariational formulation of Problem (**P**), we first introduce essential definitions and notations. We consider the following function spaces:

$$H = L^2(\Omega)^d, \quad Z = H^1(\Omega)^d, \quad Z_0 = L^2(\Omega),$$
$$\mathcal{H} = \big\{\tau = (\tau_{ij})\,;\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\}, \quad \mathcal{H}_1 = \big\{\sigma \in \mathcal{H}\,;\, \mathrm{Div}\,\sigma \in \mathcal{H}\big\}.$$

These are real Hilbert spaces, endowed with the following inner products and associated norms:

$$(u, v)_H = \int_\Omega u_i v_i\, dx, \quad (u, v)_Z = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_{\mathcal{H}}.$$

Based on the boundary conditions, we also define the following corresponding variational subspaces:

$$V = \big\{v \in H^1(\Omega)^d\,;\, v = 0 \ \text{ on } \ \Gamma_1\big\}, \quad W = \big\{\psi \in H^1(\Omega)\,;\, \psi = 0 \ \text{ on } \ \Gamma_a\big\},$$

$$Q = \big\{\theta \in H^1(\Omega)\,;\, \theta = 0 \ \text{ on } \ \Gamma_1\big\}.$$

These subspaces are Hilbert spaces for the following Euclidean structures:

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\,, \quad \|u\|_V = (u,u)_V^{1/2}, \quad \forall\, u,\, v \in V, \tag{2.16}$$

$$(\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_H\,, \quad \|\varphi\|_W = (\varphi,\varphi)_W^{1/2}, \quad \forall\, \varphi,\, \psi \in W, \tag{2.17}$$

and

$$(\theta,\eta)_Q = (\nabla\theta,\nabla\eta)_{\mathcal{H}}\,, \quad \|\theta\|_Q = (\theta,\theta)_Q^{1/2}, \quad \forall\, \theta,\, \eta \in Q. \tag{2.18}$$

Furthermore, we define the following locking constraint spaces:

$$V_1 = \big\{v \in V\,;\ \varepsilon(v(x)) \in L_1 \ \text{ a.e. } \ x \in \Omega\big\},$$

$$W_1 = \big\{\psi \in W\,;\ E(\psi(x)) \in L_2 \ \text{ a.e. } \ x \in \Omega\big\},$$

and

$$Q_1 = \big\{\eta \in Q\,;\ \nabla\eta(x) \in L_3 \ \text{ a.e. } \ x \in \Omega\big\}.$$

Since V is a closed subspace of Z and $meas(\Gamma_1) > 0$, Korn's inequality ensures the existence of a nonnegative constant c_k, depending only on Ω and Γ_1, such that

$$\|v\|_Z \le c_k\,\|\varepsilon(v)\|_{\mathcal{H}}, \quad \forall\, v \in V. \tag{2.19}$$

Thus, the two norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_V$ are equivalent on V, ensuring that $(V, \|\cdot\|_V)$ forms a real Hilbert space. Additionally, by the Sobolev trace theorem, there exists a nonnegative constant c_0, depending only on Ω, Γ_3, and Γ_1, such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0\,\|v\|_V, \quad \forall\, v \in V. \tag{2.20}$$

Moreover, since $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds, guaranteeing the existence of a nonnegative constant c_F that depends only on Ω and Γ_a, such that

$$\|\psi\|_{H^1(\Omega)} \le c_F\,\|\nabla\psi\|_{\mathcal{H}}, \quad \forall\, \psi \in W. \tag{2.21}$$

From the relations (2.17) and (2.21), the norms $\|\cdot\|_W$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on W, ensuring that $(W, \|\cdot\|_W)$ forms a real Hilbert space. Furthermore, by the Sobolev trace theorem, there exists a nonnegative constant c_1 that depends on Ω, Γ_a, and Γ_3 such that

$$\|\xi\|_{L^2(\Gamma_3)} \le c_1\,\|\xi\|_W, \quad \forall\, \xi \in W. \tag{2.22}$$

Similarly, since $meas(\Gamma_1) > 0$, the Friedrichs-Poincaré inequality ensures the existence of a nonnegative constant c_R that depends only on Ω and Γ_1, such that

$$\|\theta\|_{H^1(\Omega)} \le c_R\,\|\nabla\theta\|_{\mathcal{H}}, \quad \forall\, \theta \in Q. \tag{2.23}$$

Then, recalling (2.18) and (2.23), the norms $\|\cdot\|_Q$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on Q, guaranteeing that $(Q, \|\cdot\|_Q)$ forms a real Hilbert space. Additionally, the Sobolev trace theorem implies that there exists a constant $c_2 > 0$ depending on Ω, Γ_1, and Γ_3 such that

$$\|\eta\|_{L^2(\Gamma_3)} \leq c_2 \|\eta\|_Q, \quad \forall \eta \in Q. \tag{2.24}$$

Starting now from the constitutive law (2.1), we derive

$$\sigma(u(t)) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) - \mathscr{P}^T E(\varphi(t)) - \mathscr{C}\theta(t) + \zeta(u),$$
$$\text{where} \quad \zeta(u) \in \partial I_{L_1}\varepsilon(u(t)) \quad \text{in} \quad \Omega.$$

Consequently, for all u, $v \in V_1$, we infer the following inequalities:

$$\big\langle \zeta(u(t)), \varepsilon(v) - \varepsilon(u(t)) \big\rangle \leq I_{L_1}\big(\varepsilon(v)\big) - I_{L_1}\big(\varepsilon(u(t))\big) \leq 0.$$

Thus, from the above inequalities, we obtain

$$\begin{aligned} &(\sigma(u(t)), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}} \\ &\leq (\mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) - \mathscr{P}^T E(\varphi(t)) - \mathscr{C}\theta(t), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}}. \end{aligned} \tag{2.25}$$

Similarly, by using the constitutive law (2.2), we write

$$\begin{aligned} &D(t) = \mathscr{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathscr{G}\theta(t) + p(\varphi(t)), \\ &\text{where} \quad p(\varphi(t)) \in \partial I_{L_2} E(\varphi(t)) \quad \text{in} \quad \Omega. \end{aligned} \tag{2.26}$$

Thus, for all φ, $\psi \in W_1$, we deduce that

$$\big\langle p(\varphi(t)), E(\psi) - E(\varphi(t)) \big\rangle \leq I_{L_2}\big(E(\psi)\big) - I_{L_2}\big(E(\varphi(t))\big) \leq 0,$$

which implies that for all φ, $\psi \in W_1$, we have

$$\big(D(t), \nabla\varphi(t) - \nabla\psi(t)\big)_{L^2(\Omega)} \leq \big(\mathscr{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathscr{G}\theta(t), \nabla\varphi - \nabla\psi\big)_{L^2(\Omega)}. \tag{2.27}$$

Now, by utilizing the constitutive law (2.3), we write

$$\begin{aligned} &\dot{\theta}(t) = div\,\mathscr{K}(\nabla\theta(t))\mathscr{M}\varepsilon(u)(t) - \mathscr{N}E(\varphi(t)) + h_0(t) + g(\theta(t)), \\ &\text{where} \quad g(\theta(t)) \in \partial I_D \nabla\theta(t) \quad \text{in} \quad \Omega. \end{aligned} \tag{2.28}$$

Therefore, for all θ, $\eta \in Q_1$, we conclude that

$$\big\langle g(\eta), \nabla\theta - \nabla\eta \big\rangle \leq I_D(\nabla\theta) - I_D(\nabla\eta) \leq 0,$$

leading subsequently, to the following inequality:

$$\big(\dot{\theta}, \nabla\theta - \nabla\eta\big)_{L^2(\Omega)} \leq \big(\operatorname{div}\mathscr{K}(\nabla\theta) + \mathscr{M}\varepsilon(u) + \mathscr{N}E(\varphi) + h_0, \nabla\theta - \nabla\eta\big)_{L^2(\Omega)}. \tag{2.29}$$

At this step, the following assumptions are required to analyze the solvability of Problem (**P**).

($\mathcal{H}_1$) The viscous stress tensor $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(a) for all $\varepsilon \in \mathbb{S}^d$, the mapping $\mathscr{A}(\cdot, \varepsilon)$ is measurable on Ω;

(b) there exists $L_{\mathscr{A}} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$, we have

$$\|\mathscr{A}(x, \varepsilon_1) - \mathscr{A}(x, \varepsilon_2)\| \leq L_{\mathscr{A}} \|\varepsilon_1 - \varepsilon_2\|; \tag{2.30}$$

(c) there exists $\alpha_{\mathscr{A}} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$, we have

$$(\mathscr{A}(x, \varepsilon_1) - \mathscr{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq \alpha_{\mathscr{A}} \|\varepsilon_1 - \varepsilon_2\|^2; \tag{2.31}$$

(d) for all $x \in \Omega$, the condition $\mathscr{A}(x, 0) = 0$ holds.

($\mathcal{H}_2$) The elastic stress tensor $\mathscr{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(a) for all $\varepsilon \in \mathbb{S}^d$, the mapping $\mathscr{B}(\cdot, \varepsilon)$ is measurable on Ω;

(b) there exists $L_{\mathscr{B}} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$\|\mathscr{B}(x, \varepsilon_1) - \mathscr{B}(x, \varepsilon_2)\| \leq L_{\mathscr{B}} \|\varepsilon_1 - \varepsilon_2\|; \tag{2.32}$$

(c) for all $x \in \Omega$, the condition $\mathscr{B}(x, 0) = 0$ holds.

($\mathcal{H}_3$) The piezoelectric tensor $\mathscr{P} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ verifies

(a) $e_{ijk} \in L^\infty(\Omega)$;

(b) there exists $L_{\mathscr{P}} > 0$ such that for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$ and $x \in \Omega$, we have

$$\|\mathscr{P}(x, \varepsilon_1) - \mathscr{P}(x, p_2, \varepsilon_2)\| \leq L_{\mathscr{P}} \|\varepsilon_1 - \varepsilon_2\|; \tag{2.33}$$

(c) for all $x \in \Omega$, the condition $\mathscr{P}(x, 0) = 0$ holds.

($\mathcal{H}_4$) The electrical permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ verifies

(a) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$;

(b) there exists $L_\beta > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\beta(x, \xi_1) - \beta(x, \xi_2)\| \leq L_\beta \|\xi_1 - \xi_2\|; \tag{2.34}$$

(c) there exists $\alpha_\beta > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$(\beta(x, \xi_1) - \beta(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha_\beta \|\xi_1 - \xi_2\|^2; \tag{2.35}$$

(d) for all $x \in \Omega$, the condition $\beta(x, 0) = 0$ holds.

($\mathcal{H}_5$) The functions $j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$, $j_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ and j_e, $j_\theta : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfy

(**I**)(a) for all $r \in \mathbb{R}$, the mapping $j_\nu(\cdot, r)$ is measurable on Γ_3;

(b) for all $x \in \Gamma_3$, the mapping $j_\nu(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}$;

(c) there exist $c_{0\nu}$, $c_{1\nu} \geq 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$|\partial j_\nu(x, r)| \leq c_{0\nu} + c_{1\nu} |r|; \tag{2.36}$$

(d) there exists $\alpha_{j\nu} \geq 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$j_\nu^0(x, r_1; r_2 - r_1) + j_\nu^0(x, r_2; r_1 - r_2) \leq \alpha_{j\nu} |r_1 - r_2|^2. \tag{2.37}$$

(II)(a) for all $\xi \in \mathbb{R}^d$, the mapping $j_\tau(\cdot, \xi)$ is measurable on Γ_3;

(b) for all $x \in \Gamma_3$, the mapping $j_\tau(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^d$;

(c) there exist $c_{0\tau}$, $c_{1\tau} \geq 0$ such that for all $\xi \in \mathbb{R}^d$ and $x \in \Gamma_3$, we have

$$\|\partial j_\tau(x, \xi)\| \leq c_{0\tau} + c_{1\tau} \|\xi\|_{\mathbb{R}^d}; \tag{2.38}$$

(d) there exists $\alpha_{j\tau} \geq 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Gamma_3$, we have

$$j_\tau^0(x, \xi_1; \xi_2 - \xi_1) + j_\tau^0(x, \xi_2; \xi_1 - \xi_2) \leq \alpha_{j\tau} \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2. \tag{2.39}$$

(III)(a) for all $r \in \mathbb{R}$, the mapping $j_e(\cdot, r)$ is measurable on Γ_3;

(b) for all $x \in \Gamma_3$, the mapping $j_e(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}$;

(c)) there exist c_{0e}, $c_{1e} \geq 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$|\partial j_e(x, r)| \leq c_{0e} + c_{1e} |r|; \tag{2.40}$$

(d)) there exists $\alpha_{je} \geq 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$j_e^0(x, r_1; r_2 - r_1) + j_e^0(x, r_2; r_1 - r_2) \leq \alpha_{je} |r_1 - r_2|^2. \tag{2.41}$$

(IV)(a) for all $r \in \mathbb{R}$, the map $j_\theta(\cdot, r)$ is measurable on Γ_3;

(b) for all $x \in \Gamma_3$, the mapping $j_\theta(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}$;

(c) there exist $c_{0\theta}$, $c_{1\theta} \geq 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$|\partial j_\theta(x, r)| \leq c_{0\theta} + c_{1\theta} |r|; \tag{2.42}$$

(d) there exists $\alpha_{j\theta} \geq 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$j_\theta^0(x, r_1; r_2 - r_1) + j_\theta^0(x, r_2; r_1 - r_2) \leq \alpha_{j\theta} |r_1 - r_2|^2. \tag{2.43}$$

$(\mathcal{H}_6)$ The function $h_e : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions

(a) for all $r \in \mathbb{R}$, the mapping $h_e(\cdot, r)$ is measurable on Γ_3;

(b) for all $x \in \Gamma_3$, the mapping $h_e(x, \cdot)$ is continuous on $\mathbb{R}$;

(c) there exists $\overline{h}_e > 0$ such that for all $r \in \mathbb{R}$ and $x \in \Gamma_3$, we have

$$0 \leq h_e(x, r) \leq \overline{h}_e. \tag{2.44}$$

$(\mathcal{H}_7)$ The thermal expansion tensor $\mathscr{C} : \Omega \times \mathbb{R} \to \mathbb{S}^d$ verifies

(a) for all $r \in \mathbb{R}$, the mapping $\mathscr{C}(\cdot, r)$ is measurable on Ω;

(b) there exists $L_{\mathscr{C}} > 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Omega$, we have

$$\|\mathscr{C}(x, r_1) - \mathscr{C}(x, r_2)\| \leq L_{\mathscr{C}} |r_1 - r_2|; \tag{2.45}$$

(c) for all $x \in \Omega$, the condition $\mathscr{C}(x, 0) = 0$ holds.

$(\mathcal{H}_8)$ The thermo-mechanical coupling operator $\mathscr{M} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ verifies

(a) for all $\xi \in \mathbb{R}^d$, one has $\mathscr{M}(\cdot, \xi) \in L^\infty(\Omega)$;

(b) there exists $L_{\mathscr{M}} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathscr{M}(x,\xi_1) - \mathscr{M}(x,\xi_2)\| \leq L_{\mathscr{M}} \|\xi_1 - \xi_2\|. \tag{2.46}$$

($\mathcal{H}_9$) The thermo-electrical coupling vector field $\mathscr{N} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfies

(a) for all $\xi \in \mathbb{R}^d$, one has $\mathscr{N}(\cdot,\xi) \in L^\infty(\Omega)$;

(b) there exists $L_{\mathscr{N}} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathscr{N}(x,\xi_1) - \mathscr{N}(x,\xi_2)\| \leq L_{\mathscr{N}} \|\xi_1 - \xi_2\|; \tag{2.47}$$

(c) for all $x \in \Omega$, the condition $\mathscr{N}(x,0) = 0$ holds.

($\mathcal{H}_{10}$) The pyroelectric tensor $\mathscr{G} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^d$ verifies

(a) for all $r \in \mathbb{R}$, one has $\mathscr{G}(\cdot,r) \in L^\infty(\Omega)$;

(b) there exists $L_{\mathscr{G}} > 0$ such that for all r_1, $r_2 \in \mathbb{R}$ and $x \in \Omega$, we have

$$\|\mathscr{G}(x,r_1) - \mathscr{G}(x,r_2)\| \leq L_{\mathscr{G}} |r_1 - r_2|. \tag{2.48}$$

($\mathcal{H}_{11}$) The thermal conductivity tensor $\mathscr{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(a) for all $\xi \in \mathbb{R}^d$, the map $\mathscr{K}(\cdot,\xi)$ is measurable on Ω;

(b) there exists $L_{\mathscr{K}} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$\|\mathscr{K}(x,\xi_1) - \mathscr{K}(x,\xi_2)\| \leq L_{\mathscr{K}} \|\xi_1 - \xi_2\|; \tag{2.49}$$

(c) there exists $\alpha_{\mathscr{K}} > 0$ such that for all ξ_1, $\xi_2 \in \mathbb{R}^d$ and $x \in \Omega$, we have

$$(\mathscr{K}(x,\xi_1) - \mathscr{K}(x,\xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha_{\mathscr{K}} \|\xi_1 - \xi_2\|^2; \tag{2.50}$$

(d) for all $x \in \Omega$, the condition $\mathscr{K}(x,0) = 0$.

($\mathcal{H}_{12}$) The forces, tractions, electrical charges, thermal source densities and the initial data satisfy

(i) $f_0 \in L^2(\Omega)^d$, $f_2 \in L^2(\Gamma_2)^d$, $q_0, h_0 \in L^2(\Omega)$, $q_b \in L^2(\Gamma_b)$, $h_n \in L^2(\Gamma_2)$;

(ii) $\varphi_0 \in L^\infty(\Gamma_3)$, $u_0 \in V$, $\theta_0 \in Q$.

($\mathcal{H}_{13}$) The subsets $L_1 \subset \mathbb{S}^d$, $L_2 \subset L^2(\Omega)$ and $L_3 \subset L^2(\Omega)$ are nonempty closed convex such that

$$0_{\mathbb{S}^d} \in L_1 , \quad 0_{L^2(\Omega)} \in L_2 \cap L_3.$$

By applying Riesz's representation theorem, we define $f \in V$, $q \in W$, and $h \in Q$ as follows:

$$\langle f, v\rangle_V = \langle f_0, v\rangle_{L^2(\Omega)^d} + \langle f_2, v\rangle_{L^2(\Gamma_2)^d}, \quad \forall\, v \in V, \tag{2.51}$$

$$\langle q, \psi\rangle_W = \langle q_0, \psi\rangle_{L^2(\Omega)} - \langle q_b, \psi\rangle_{L^2(\Gamma_b)}, \quad \forall\, \psi \in W, \tag{2.52}$$

$$\langle h, \xi\rangle_Q = \langle h_0, \xi\rangle_{L^2(\Omega)} - \langle h_n, \xi\rangle_{L^2(\Gamma_2)}, \quad \forall\, \xi \in Q. \tag{2.53}$$

Then, using standard arguments, the variational formulation of Problem (**P**) is given by the following:

Problem (PV). *Find a displacement* $u : [0,T] \to V_1$*, an electric potential* $\varphi : [0,T] \to W_1$ *and a temperature* $\theta : [0,T] \to Q_1$ *such that:*

$$\begin{aligned}
&\big\langle \mathscr{A}\varepsilon(\dot u(t)), \varepsilon(v-\dot u(t))\big\rangle_{\mathcal{H}} + \big\langle \mathscr{B}\varepsilon(u(t)) + \mathscr{P}^T\nabla\varphi(t) - \mathscr{C}\theta(t), \varepsilon(v-\dot u(t))\big\rangle_{\mathcal{H}}\\
&\quad + \int_{\Gamma_3} \big(j^0_\nu(\dot u_\nu(t); v_\nu - \dot u_\nu(t)) + j^0_\tau(\dot u_\tau(t); v_\tau - \dot u_\tau(t))\big)\, da \qquad (2.54)\\
&\quad \geq \big\langle f(t), v - \dot u(t)\big\rangle_V, \quad \forall v \in V_1,
\end{aligned}$$

$$\begin{aligned}
&\big\langle \beta\nabla(\varphi(t)) - \mathscr{P}\varepsilon(u(t)) - \mathscr{G}\theta(t), \nabla(\psi - \varphi(t))\big\rangle_{\mathcal{H}}\\
&\quad + \int_{\Gamma_3} h_e(u(t))\, j^0_e(\varphi(t)-\varphi_0; \psi - \varphi(t))\, da \qquad (2.55)\\
&\quad \geq \big\langle q, \psi - \varphi(t)\big\rangle_W, \quad \forall \psi \in W_1,
\end{aligned}$$

$$\begin{aligned}
&\big\langle \dot\theta(t), \lambda - \theta(t)\big\rangle_{\mathcal{H}} + \big\langle \mathscr{K}\nabla\theta(t), \nabla(\lambda - \theta(t))\big\rangle_{\mathcal{H}}\\
&\quad - \big\langle \mathscr{M}\varepsilon(u(t)) - \mathscr{N}\nabla\varphi(t), \lambda - \theta(t)\big\rangle_{\mathcal{H}} + \int_{\Gamma_3} j^0_\theta(\theta(t); \lambda - \theta(t))\, da \qquad (2.56)\\
&\quad \geq \big\langle h(t), \lambda - \theta(t)\big\rangle_Q, \quad \forall \eta \in Q_1.
\end{aligned}$$

Building on these considerations, we establish the following existence and uniqueness result.

Theorem 2.1. *Assume that hypotheses* $(\mathcal{H}_1)$*-*$(\mathcal{H}_{13})$ *and the following smallness conditions hold.*

$$\alpha_{\mathscr{A}} > c_0^2\,(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{meas(\Gamma_3)}, \qquad (2.57)$$

$$\alpha_\beta > \overline{h}_e \alpha_{j_e} c_0^2\,\sqrt{meas(\Gamma_3)}, \qquad (2.58)$$

$$\alpha_{\mathscr{K}} - c_0^2\alpha_{j_\theta}\sqrt{meas(\Gamma_3)} > L_{\mathscr{M}}T/2. \qquad (2.59)$$

Then, Problem (**PV**) *has a unique solution.*

The proof of this theorem follows the same approach as in [20, Theorem 2.1], based on the following fixed-point result (see [39, Lemma 1] or [43, Lemma 2.1]).

Lemma 2.1. *Let* $\big(X, \|\cdot\|_X\big)$ *be a Banach space and* $\Lambda : L^2(0,T;X) \to L^2(0,T;X)$ *be an operator satisfying, for some nonnegative constant* $c > 0$*, the following condition:*

$$\big\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\big\|_X^2 \leq c\int_0^t \big\|\eta_1(s) - \eta_2(s)\big\|_X^2\, ds, \quad \forall\, \eta_1,\, \eta_2 \in L^2(0,T;X).$$

Then, Λ *admits a unique fixed point, i.e.; there exists a unique* $\eta^* \in L^2(0,T;X)$ *such that* $\Lambda\eta^* = \eta^*$*.*

It is also based on results from the theory of hemivariational inequalities, namely [15, Theorem 6.3.73], [31, Theorem 7], and [33, Lemma 9], [35, Theorem 8.6], [39, Lemma 11] for the existence and uniqueness analysis, as well as [34, Corollary 3.1] and [34, Corollary 3.2] for the estimates.

Remark 2.2. It is important to note that the solvability of Problem (**PV**) is local in time, as the smallness condition (2.59) imposes a restriction on the time interval T. Therefore, the existence and uniqueness of a solution are guaranteed only for a sufficiently small time interval. A global solution would require further analysis or additional assumptions beyond those presented in this work.

3. Numerical analysis of Problem (P)

In this section, we introduce a fully discrete approach for solving Problem (**PV**) and provide an error estimate for the approximate solution. We first use the finite difference method to approximate the function's derivatives. Specifically, we partition the interval $[0, T]$ uniformly as $0 = t_0 < t_1 < ... < t_N = T$, with a time step size $k = T/N$. Then, for any continuous function v, we define

$$v(t_n) = v_n, \quad w_n := \delta v_n = \frac{v_n - v_{n-1}}{k}.$$

We also apply the finite element method for spatial discretization. For that, let Ω be a polygonal domain, and consider a regular family of partitions $\{\mathcal{T}^h\}$ of $\overline{\Omega}$ into triangles, aligned with the boundary partition $\partial\Omega$ into $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1 \cup \Gamma_2 = \Gamma_a \cup \Gamma_b$. Here, $h > 0$ represents the discretization parameter, and c is a generic positive constant independent of the discretization parameters h and k. To approximate the spaces V, W, and Q, we introduce the following linear finite element spaces corresponding to $\mathcal{T}^h$:

$$V^h = \big\{v^h \in C(\overline{\Omega})\,;\ \ v^h_{|_T} \in \mathbb{P}_1(T)^d \ \text{ for } \ T \in \mathcal{T}^h, \ \text{and} \ \ v^h = 0 \ \text{ on } \ \Gamma_1\big\},$$

$$W^h = \big\{\psi^h \in C(\overline{\Omega})\,;\ \ \psi^h_{|_T} \in \mathbb{P}_1(T)^d \ \text{ for } \ T \in \mathcal{T}^h, \ \text{and} \ \ \psi^h = 0 \ \text{ on } \ \Gamma_1\big\},$$

$$Q^h = \big\{\theta^h \in C(\overline{\Omega})\,;\ \ \theta^h_{|_T} \in \mathbb{P}_1(T)^d \ \text{ for } \ T \in \mathcal{T}^h, \ \text{and} \ \ \theta^h = 0 \ \text{ on } \ \Gamma_1\big\}.$$

We also define the following subspaces approximating the locking constraints subsets:

$$V^h_1 = \big\{v^h \in V\,;\ \ \varepsilon(v^h(x)) \in L_1 \ \text{ a.e. } \ x \in \Omega\big\},$$

$$W^h_1 = \big\{\psi^h \in W\,;\ \ E(\psi^h(x)) \in L_2 \ \text{ a.e. } \ x \in \Omega\big\},$$

$$Q^h_1 = \big\{\theta^h \in Q\,;\ \ \nabla\theta^h(x) \in L_3 \ \text{ a.e. } \ x \in \Omega\big\}.$$

Furthermore, we introduce the following piecewise constant finite element space for the stress field:

$$\mathcal{H}^h = \big\{\tau^h \in \mathcal{H}\,;\ \ \tau^h_{|_T} \in \mathbb{R}^{d\times d} \ \text{ for } \ T \in \mathcal{T}^h\big\}.$$

Finally, let $u^{hk}_0 = u^h_0 \in V^h_1$ and $\theta^{hk}_0 = \theta^h_0 \in Q^h_1$ be appropriate approximation of the initiale conditions u_0 and θ_0, respectively, meaning that they satisfy:

$$\|u_0 - u^h_0\|_V \le c\,h\ , \quad \|\theta_0 - \theta^h_0\|_Q \le c\,h. \tag{3.1}$$

Thus, the discrete formulation for Problem (**PV**) is presented as follows.

Problem $(\mathcal{PV}_{\mathbf{hk}})$. *Find a displacement* $\{u^{hk}_n\}_{n=0}^N \subset V^h_1$*, an electric potential*

$\{\varphi_n^{hk}\}_{n=0}^N \subset W_1^h$ *and a temperature* $\{\theta_n^{hk}\}_{n=0}^N \subset Q_1^h$ *such that for all* $n \in \{1, ..N\}$, *one has*

$$\begin{aligned}&\big\langle \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\big\rangle_{\mathcal{H}} + \big\langle \mathscr{B}\varepsilon(u_n^{hk}) + \mathscr{P}^T\nabla\varphi_n^{hk} - \mathscr{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\big\rangle_{\mathcal{H}}\\&+\int_{\Gamma_3}\big(j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\tau}^h - w_{n\tau}^{hk})\big)\,da\\&\geq \big\langle f_n, v_n^h - w_n^{hk}\big\rangle, \quad \forall\, v_n^h \in V_1^h,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\big\langle \beta\nabla\varphi_n^{hk} - \mathscr{P}\varepsilon(u_n^{hk}) - \mathscr{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\big\rangle_{\mathcal{H}}\\&+\int_{\Gamma_3} h_e(u_{n\nu}^{hk})\, j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n^{hk})\,da \geq \big\langle q, \psi_n^h - \varphi_n^{hk}\big\rangle_W, \quad \forall\, w_n^h \in W_1^h,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\big\langle \delta\theta_n^{hk}, \lambda_n^h - \theta_n^{hk}\big\rangle_{\mathcal{H}} + \big\langle \mathscr{K}\nabla\theta_n^{hk}, \nabla(\lambda_n^h - \theta_n^{hk})\big\rangle_{\mathcal{H}}\\&-\big\langle \mathscr{M}\varepsilon(u_n^{hk}) + \mathscr{N}\nabla\varphi_n^{hk}, \lambda_n^h - \theta_n^{hk}\big\rangle_{\mathcal{H}} + \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n^{hk})_{\mathcal{H}}\,da\\&\geq \big\langle h_n, \lambda_n^h - \theta_n^{hk}\big\rangle_{\mathcal{H}}\big\rangle_Q, \quad \forall\, \lambda_n^h \in Q_1^h,\end{aligned} \tag{3.4}$$

$$u_0^{hk} = u_0^h \quad \text{and} \quad \theta_0^{hk} = \theta_0^h. \tag{3.5}$$

Here, the sequences $\{u_n^{hk}\}_{n=0}^N$ and $\{w_n^{hk}\}_{n=0}^N$ are related by the following relations:

$$w_n^{hk} = \delta u_n^{hk} \;\text{ and }\; u_n^{hk} = u_0^h + k\sum_{j=1}^n w_j^{hk} \quad (n = 1, ..., N). \tag{3.6}$$

Under the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_{13})$, and applying reasoning similar to that used for Problem (**PV**), we show that Problem $(\mathcal{PV}_{\mathbf{hk}})$ has a unique solution $(u_n^{hk}, \varphi_n^{hk}, \theta_n^{hk}) \in V^h \times W^h \times Q^h$. Next, the error estimates will be derived through the application of the well known Céa inequalities. Before stating the result in the theorem below, we strengthen hypothesis $(\mathcal{H}_5)$ by assuming that the function $j_\rho(x, \cdot)$, for $\rho \in \{\nu, \tau, e, \theta\}$, is uniformly (globally) Lipschitz continuous rather than merely locally Lipschitz. With this additional assumption, the corresponding Lipschitz constants can be considered independent of x.

Theorem 3.1. *Given that the assumptions of Theorem 2.1 hold, along with the condition* (3.1), *let* (u, φ, θ) *be the solution to Problem* (**PV**), *and* $(u_n^{hk}, \varphi_n^{hk}, \theta_n^{hk})$ *the solution to Problem* $(\mathcal{PV}_{\mathbf{hk}})$. *Then, for each* $n = 1, .., N$, *the following error estimate holds.*

$$\begin{aligned}&\max_{1\leq n\leq N}\Big\{\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2\Big\}\\&\leq c\max_{1\leq n\leq N}\Big\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)}\Big\}\\&+c\sum_{n=1}^N \|\theta_n - \lambda_n^h\|_Q^2 + \|\theta_n - \lambda_n^h\|_{L^2(\Gamma_3)} + c\sum_{n=1}^{N-1}\|(\theta_n - \lambda_n^h) - (\theta_{n+1} - \lambda_{n+1}^h)\|\\&+c\left(\|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2\right) + c\left(h^2 + k^2\right).\end{aligned} \tag{3.7}$$

Proof. We first write the following relation

$$\begin{aligned}&\langle \mathscr{A}\varepsilon(w_n) - \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}}\\&= \langle \mathscr{A}\varepsilon(w_n) - \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathscr{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}}\\&\quad + \langle \mathscr{A}\varepsilon(w_n), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n^{hk}) - \varepsilon(v_n^h)\rangle_{\mathcal{H}}.\end{aligned} \tag{3.8}$$

We further substitute $t = t_n$ and $v = w_n^{hk}$ into the inequality (2.54) to obtain

$$\begin{aligned}&\langle \mathscr{A}\varepsilon(w_n), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}}\\&\le \langle \mathscr{B}\varepsilon(u_n), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathscr{P}^T\nabla\varphi_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} - \langle \mathscr{C}\theta_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}}\\&\quad + \int_{\Gamma_3} \left(j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\tau^0(w_{n\tau}; w_{n\nu}^{hk} - w_{n\nu})\right) da + \left\langle f_n, w_n - w_n^{hk}\right\rangle_V.\end{aligned} \tag{3.9}$$

Then, by referencing the condition (3.2), the previous expression leads to

$$\begin{aligned}&\langle \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n^{hk} - v_n^h)\rangle_{\mathcal{H}}\\&\le \langle \mathscr{B}\varepsilon(u_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathscr{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} - \langle \mathscr{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}}\\&\quad + \int_{\Gamma_3} \left(j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\right) da + \left\langle f_n, w_n^{hk} - v_n^h\right\rangle_V.\end{aligned} \tag{3.10}$$

Next, we apply hypothesis (2.31) along with inequalities (3.8)-(3.10) to infer

$$\begin{aligned}&\alpha_{\mathscr{A}}\|w_n - w_n^{hk}\|_V^2\\&\le \langle \mathscr{A}\varepsilon(w_n) - \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathscr{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}}\\&\quad + \langle f_n, w_n - v_n^h\rangle_V + \langle \mathscr{B}\varepsilon(u_n), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathscr{B}\varepsilon(u_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}}\\&\quad + \langle \mathscr{P}^T\nabla\varphi_n, \varepsilon(w_n^{hk} - w_n)\rangle + \langle \mathscr{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} - \langle \mathscr{C}\theta_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}}\\&\quad - \langle \mathscr{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \int_{\Gamma_3} \left(j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\right) da\\&\quad + \int_{\Gamma_3} \left(j_\tau^0(w_{n\tau}; w_{n\nu}^{hk} - w_{n\nu}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\right) da.\end{aligned}$$

Then, we can deduce that

$$
\begin{aligned}
&\alpha_{\mathscr{A}}\|w_n - w_n^{hk}\|_V^2 \\
&\le \langle \mathscr{A}\varepsilon(w_n) - \mathscr{A}\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathscr{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
&\quad + \langle \mathscr{B}\varepsilon(u_n^{hk}), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle \mathscr{B}\varepsilon(u_n) - \mathscr{B}(\varepsilon(u_n^{hk}), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} \\
&\quad + \langle \mathscr{P}^T\nabla\varphi_n - \mathscr{P}^T\nabla\varphi_n^{hk}, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathscr{P}^T\nabla\varphi_n^{hk}, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
&\quad - \langle \mathscr{C}\theta_n - \mathscr{C}\theta_n^{hk}, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} - \langle \mathscr{C}\theta_n^{hk}, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle f_n, w_n - v_n^h\rangle_V \\
&\quad + \int_{\Gamma_3} \big(j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\big)\, da \\
&\quad + \int_{\Gamma_3} \big(j_\tau^0(w_{n\tau}; w_{n\nu}^{hk} - w_{n\nu}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\big)\, da \\
&\quad + \int_{\Gamma_3} \big(j_\nu^0(w_{n\nu}^{hk}; v_n^h - w_n)\, da + \int_{\Gamma_3} j_\tau^0(w_{n\tau}^{hk}; v_n^h - w_{n\nu})\big)\, da.
\end{aligned}
$$

Subsequently, we utilize (2.30), (2.32), (2.33), (2.37), (2.39), and (2.45) to establish

$$
\begin{aligned}
&\alpha_{\mathscr{A}}\,\|w_n - w_n^{hk}\|_V^2 \\
&\le L_{\mathscr{A}}\,\|w_n - w_n^{hk}\|_V\|w_n - v_n^h\|_V \\
&\quad + L_{\mathscr{P}}\,\|\varphi_n - \varphi_n^{hk}\|_W\big(\|w_n - w_n^{hk}\|_V + \|w_n - v_n^h\|_V\big) \\
&\quad + L_{\mathscr{B}}\,\|u_n - u_n^{hk}\|_V\big(\|w_n - w_n^{hk}\|_V + \|w_n - v_n^h\|_V\big) + S_1(u_n, \varphi_n, \theta_n) \\
&\quad + I_1(\varphi_n^{hk}, \varphi_n, \psi_n) + L_{\mathscr{M}}\,\|\theta_n - \theta_n^{hk}\|_Q\big(\|w_n - w_n^{hk}\|_V + \|w_n - v_n^h\|_V\big) \\
&\quad + c_0^2\,\sqrt{meas(\Gamma_3)}\,(\alpha_{j\nu} + \alpha_{j\tau})\,\|w_n - w_n^{hk}\|_V^2,
\end{aligned}
$$

where the terms S_1 and I_1 are defined as follows:

$$
\begin{aligned}
&S_1(u_n, \varphi_n, \theta_n) \\
&= \langle \mathscr{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle \mathscr{B}\varepsilon(u_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
&\quad + \langle \mathscr{P}^T\nabla\varphi_n, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} - \langle \mathscr{C}\theta_n, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle f_n, w_n - v_n^h\rangle_V,
\end{aligned}
\tag{3.11}
$$

and

$$
I_1(w_n^{hk}, w_n, v_n^h) = \int_{\Gamma_3} j_\nu^0(w_{n\nu}^{hk}; v_n^h - w_n)\, da + \int_{\Gamma_3} j_\tau^0(w_{n\tau}^{hk}; v_n^h - w_{n\nu})\, da. \tag{3.12}
$$

We recall that for almost every $x \in \Gamma_3$, the maps $j_\nu(x,\cdot)$ and $j_\tau(x,\cdot)$ are assumed to be uniformly Lipschitz continuous. Then, their Lipschitz constants $L_{j_\nu(x,\cdot)}$ and $L_{j_\tau(x,\cdot)}$ are independent of x. Consequently, there exists $c > 0$ independent of x, such that

$$
j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}) \le c\,\|w_n - v_n^h\|_{L^2(\Gamma_3)},
$$

and

$$j_\tau^0(w_{n\tau}^{hk}; v_{n\tau}^h - w_{n\tau}) \le c\,\|w_n - v_n^h\|_{L^2(\Gamma_3)}.$$

Therefore, we have

$$I_1(w_n^{hk}, w_n, v_n^h) \le c\,\|w_n - v_n^h\|_{L^2(\Gamma_3)}. \tag{3.13}$$

Next, by multiplying the equation (2.4) with an arbitrary element $v \in V_1$, we obtain

$$\int_\Omega \sigma\nu \cdot \varepsilon(v)\,da - \int_\Omega f \cdot v\,da = \int_{\Gamma_3} \sigma \cdot v\,da \quad \text{and} \quad \sigma\nu \in L^2(\Gamma_3;\mathbb{R}^d).$$

Therefore, it can be concluded that the following inequalities hold.

$$\begin{aligned} S_1(u_n,\varphi_n,\theta_n) &= \int_{\Gamma_3} \sigma\nu \cdot (v_n^h - w_n)\,da \\ &\le c\,\|\sigma\|\,\big\|w_n - v_n^h\big\|_{L^2(\Gamma_3)} \\ &\le c\,\big\|w_n - v_n^h\big\|_{L^2(\Gamma_3)}. \end{aligned} \tag{3.14}$$

Moreover, by employing the Cauchy inequality for $\epsilon > 0$, we can derive

$$\begin{aligned} &\big(\alpha_{\mathscr{A}} - c_0^2\sqrt{mes(\Gamma_3)}(\alpha_{j\nu} + \alpha_{j\tau}) - 5\epsilon\big)\,\|w_n - w_n^{hk}\|_V^2 \\ &\le c\left\{\|w_n - v_n^h\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right\}. \end{aligned} \tag{3.15}$$

Hence, using similar reasoning as in [9], we conclude

$$\big\|u_n - u_n^{hk}\big\|_V^2 \le c\big(h^2 + k^2\big) + ck\sum_{i=1}^n \|w_i - w_i^{hk}\|_V^2. \tag{3.16}$$

Subsequently, by combining the inequalities (3.15) and (3.16), we infer that

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 \\ &\le c\left\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right\} \\ &\quad + c\left(h^2 + k^2\right) + ck\sum_{i=1}^n \|w_i - w_i^{hk}\|_V^2. \end{aligned} \tag{3.17}$$

Next, by applying the Gronwall inequality to (3.17), we derive

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 \\ &\le c\left\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right\} \\ &\quad + c\left(h^2 + k^2\right). \end{aligned} \tag{3.18}$$

We now combine the relations (3.16) and (3.18) to obtain a constant $c > 0$ such that

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 \\ &\le c\left\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right\} \\ &\quad + c\left(h^2 + k^2\right) + ck\sum_{i=1}^n \left(\|w_i - w_i^{hk}\|_V^2 + \|u_i - u_i^{hk}\|_V^2\right). \end{aligned} \tag{3.19}$$

For simplicity of notation, let us assume

$$e_n = \|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2, \tag{3.20}$$

$$\begin{aligned} g_n &= \|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 \\ &\quad + \|w_n - v_n^h\|_{L^2(\Gamma_3} + h^2 + k^2. \end{aligned} \tag{3.21}$$

Therefore, there exists a nonnegative constant c such that

$$e_n \le c\, g_n + c \sum_{j=1}^{n} e_j. \tag{3.22}$$

Therefore, upon applying the Gronwall inequality once more, we obtain

$$\begin{aligned} &\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 \\ &\le c\left\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right\} \\ &\quad + c\left(h^2 + k^2\right). \end{aligned} \tag{3.23}$$

Furthermore, based on the assumption (2.35), it follows that

$$\begin{aligned} \alpha_\beta\|\varphi_n - \varphi_n^{hk}\|_W^2 &\le \langle\beta\nabla\varphi_n - \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n - \psi_n^h)\rangle_{\mathcal{H}} + \langle\beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\ &\quad + \langle\beta\nabla\varphi_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle\beta\nabla\varphi_n^{hk}, \nabla(\varphi_n^{hk} - \psi_n^h)\rangle_{\mathcal{H}}. \end{aligned} \tag{3.24}$$

By choosing $t = t_n$ and $\psi = \varphi_n^{hk}$ in the relation (2.55), we find

$$\begin{aligned} &\langle\beta\nabla\varphi_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} \\ &\le \langle\mathscr{P}\varepsilon(u_n), \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle\mathscr{G}\theta_n, \nabla(\varphi_n - \varphi_n^{hk})\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} h_e(u_{n\nu})\, j_e^0(\varphi_n - \varphi_0; \varphi_n^{hk} - \varphi_n)\, da + \left\langle q_n, \varphi_n - \varphi_n^{hk}\right\rangle_W. \end{aligned} \tag{3.25}$$

Referring next to the inequality (3.3), we derive

$$\begin{aligned} &\langle\beta\nabla\varphi_n^{hk}, \nabla(\varphi_n^{hk} - \psi_n^h)\rangle_{\mathcal{H}} \\ &\le \langle\mathscr{P}\varepsilon(u_n^{hk}), \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} + \langle\mathscr{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} h_e(u_{n\nu}^{hk})\, j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n^{hk})\, da + \left\langle q_n, \varphi_n^{hk} - \psi_n^h\right\rangle_W. \end{aligned} \tag{3.26}$$

Now, by combining the inequalities (3.24) to (3.26), we can infer that

$$\begin{aligned} &\alpha_\beta\|\varphi_n - \varphi_n^{hk}\|_W^2 \\ &\le \langle\beta\nabla\varphi_n - \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n - \psi_n^h)\rangle_{\mathcal{H}} + \langle\beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\ &- \langle\mathscr{P}\varepsilon(u_n), \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} - \langle\mathscr{P}\varepsilon(u_n^{hk}), \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} - \langle\mathscr{G}\theta_n, \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} \end{aligned}$$

$$
\begin{aligned}
&+ \langle \mathscr{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n^{hk})\rangle_{\mathcal{H}} + \int_{\Gamma_3} \overline{h}_e j_e^0(\varphi_n - \varphi_0; \varphi_n^{hk} - \varphi_n)\, da \\
&+ \int_{\Gamma_3} \overline{h}_e j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n^{hk})\, da + \big\langle q_n, \varphi_n - \psi_n^h \big\rangle_W \\
&\le \langle \beta\nabla\varphi_n - \beta\nabla\varphi_n^{hk}, \nabla(\varphi_n - \psi_n^h)\rangle_{\mathcal{H}} + \langle \beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\
&+ \langle \mathscr{P}\varepsilon(u_n) - \mathscr{P}\varepsilon(u_n^{hk}), \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} + \langle \mathscr{P}\varepsilon(u_n^{hk}), \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\
&+ \langle \mathscr{G}\theta_n - \mathscr{G}\theta_n^{hk}, \nabla(\varphi_n^{hk} - \varphi_n)\rangle_{\mathcal{H}} + \langle \mathscr{G}\theta_n^{hk}, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} + \langle q_n, \varphi_n - \psi_n^h\rangle_W \\
&+ \int_{\Gamma_3} \overline{h}_e \big(j_e^0(\varphi_n - \varphi_0; \varphi_n^{hk} - \varphi_n) + j_e^0(\varphi_n^{hk} - \varphi_0; \varphi_n - \varphi_n^{hk}) \\
&+ j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n)\big)\, da.
\end{aligned}
$$

We now apply inequalities (2.33), (2.34), (2.41), and (2.48) to deduce

$$
\begin{aligned}
&\alpha_\beta \|\varphi_n - \varphi_n^{hk}\|_W^2 \\
&\le L_\beta \|\varphi_n - \varphi_n^{hk}\|_W \|\varphi_n - \psi_n^h\|_W + L_{\mathscr{P}} \|u_n - u_n^{hk}\|_V \|\varphi_n - \varphi_n^{hk}\|_W \\
&\quad + L_{\mathscr{P}} \|u_n - u_n^{hk}\|_V \|\psi_n^h - \varphi_n\|_W + L_{\mathscr{G}} \|\theta_n - \theta_n^{hk}\|_Q \|\varphi_n - \varphi_n^{hk}\|_W \\
&\quad + L_{\mathscr{G}} \|\theta_n - \theta_n^{hk}\|_Q \|\psi_n^h - \varphi_n\|_W + \overline{h}_e \alpha_{j_e} c_0^2 \, \|\varphi_n - \varphi_n^{hk}\|_W^2 \\
&\quad + S_2(u_n, \varphi_n, \theta_n) + I_2(\varphi_n^{hk}, \varphi_n, \psi_n),
\end{aligned}
$$

where the quantities S_2 and I_2 are given by

$$
\begin{aligned}
S_2(u_n, \varphi_n, \theta_n) = {}& \langle \beta\nabla\varphi_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} - \langle \mathscr{P}\varepsilon(u_n), \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} \\
& - \langle \mathscr{G}\theta_n, \nabla(\psi_n^h - \varphi_n)\rangle_{\mathcal{H}} + \langle q_n, \varphi_n - \psi_n^h\rangle_W,
\end{aligned} \tag{3.27}
$$

$$
I_2(\varphi_n^{hk}, \varphi_n, \psi_n) = \int_{\Gamma_3} \overline{h}_e \, j_e^0(\varphi_n^{hk} - \varphi_0; \psi_n^h - \varphi_n)\, da. \tag{3.28}
$$

On the other hand, multiplying (2.5) by an arbitrary element $\psi_n^h \in W_1$, yields

$$
\begin{aligned}
&\int_\Omega D \cdot \nu \, (\psi_n^h - \varphi_n)\, dx - \int_\Omega q_0 \, (\psi_n^h - \varphi_n)\, dx \\
&\qquad + \int_{\Gamma_b} (\psi_n^h - \varphi_n)\, d\Gamma = \int_{\Gamma_3} D \cdot \nu \, (\psi_n^h - \varphi_n)\, d\Gamma.
\end{aligned}
$$

Remembering the condition $D \cdot \nu \in L^2(0, T; L^2(\Gamma_3))$, we find that

$$
S_2(u_n, \varphi_n, \theta_n) \le c \, \big\| \varphi_n - \psi_n^h \big\|_{L^2(\Gamma_3)}. \tag{3.29}
$$

The mapping $j_e(x, \cdot)$ is assumed to be uniformly Lipschitz continuous, then its Lipschitz constant, denoted by c, is independent of x. Consequently, we can write

$$
I_2(\varphi_n^{hk}, \varphi_n, \psi_n) \le c \, \big\| \varphi_n - \psi_n^h \big\|_{L^2(\Gamma_3)}. \tag{3.30}
$$

Thus, by applying the Cauchy inequality, we derive the following estimate

$$
\begin{aligned}
&\big\| \varphi_n - \varphi_n^{hk} \big\|_W^2 \\
&\le c \left\{ \|\varphi_n - \psi_n^h\|_W^2 + \|u_n - u_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \right\}.
\end{aligned} \tag{3.31}
$$

Furthermore, by employing the condition (2.50), we infer that

$$\alpha_{\mathscr{K}}\|\theta_n-\theta_n^{hk}\|_Q^2 \le \langle \mathscr{K}\nabla\theta_n-\mathscr{K}\nabla\theta_n^{hk},\nabla(\theta_n-\lambda_n^h\rangle_{\mathcal{H}}+\langle \mathscr{K}\nabla\theta_n,\nabla(\lambda_n^h-\theta_n)\rangle_{\mathcal{H}} + \langle \mathscr{K}\nabla\theta_n,\nabla(\theta_n-\theta_n^{hk}\rangle_{\mathcal{H}}+\langle \mathscr{K}\nabla\theta_n^{hk},\nabla(\theta_n^{hk}-\lambda_n^h)\rangle_{\mathcal{H}}. \tag{3.32}$$

Next, by substituting $t=t_n$ and $\lambda=\theta_n^{hk}$ into the inequality (2.56), we obtain

$$\begin{aligned}&\langle \mathscr{K}\nabla\theta_n,\nabla(\theta_n-\theta_n^{hk}\rangle_{\mathcal{H}}\\ &\le \langle\dot\theta_n,\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}-\langle \mathscr{M}\varepsilon(u_n),\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}+\langle \mathscr{N}\nabla\varphi_n,\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}\\ &\quad+\int_{\Gamma_3} j_\theta^0(\theta_n;\theta_n^{hk}-\theta_n)\,da+\langle h_n,\theta_n-\theta^{hk}\rangle_Q.\end{aligned} \tag{3.33}$$

Moreover, it follows from the inequality (3.4) that

$$\begin{aligned}&\langle \mathscr{K}\nabla\theta_n^{hk},\nabla(\theta_n^{hk}-\lambda_n^h)\rangle_{\mathcal{H}}\\ &\le\langle\delta\theta_n^{hk},\lambda_n^h-\theta_n^{hk}\rangle_{\mathcal{H}}-\langle \mathscr{M}\varepsilon(u_n^{hk}),\lambda_n^h-\theta_n^{hk}\rangle_{\mathcal{H}}\\ &\quad+\langle \mathscr{N}\nabla\varphi_n^{hk},\lambda_n^h-\theta_n^{hk}\rangle_{\mathcal{H}}+\int_{\Gamma_3} j_\theta^0(\theta_n^{hk};\lambda_n^h-\theta_n^{hk})\,da+\langle h_n,\theta_n^{hk}-\lambda_n^h\rangle_Q.\end{aligned} \tag{3.34}$$

Thus, by combining the preceding inequalities, we deduce

$$\begin{aligned}&\alpha_{\mathscr{K}}\|\theta_n-\theta_n^{hk}\|_Q^2\\ &\le\langle \mathscr{K}\nabla\theta_n-\mathscr{K}\nabla\theta_n^{hk},\nabla(\theta_n-\lambda_n^h\rangle_{\mathcal{H}}+\langle \mathscr{K}\nabla\theta_n,\nabla(\lambda_n^h-\theta_n)\rangle_{\mathcal{H}}\\ &\quad+\langle\delta\theta_n-\delta\theta_n^{hk},\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}+\langle\delta\theta_n^{hk},\lambda_n^h-\theta_n\rangle_{\mathcal{H}}\\ &\quad-\langle \mathscr{M}\varepsilon(u_n)-\mathscr{M}\varepsilon(u_n^{hk}),\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}+\langle \mathscr{M}\varepsilon(u_n^{hk}),\lambda_n^h-\theta_n\rangle_{\mathcal{H}}\\ &\quad-\langle \mathscr{N}\nabla\varphi_n-\mathscr{N}\nabla\varphi_n^{hk},\theta_n^{hk}-\theta_n\rangle_{\mathcal{H}}+\langle \mathscr{N}\nabla\varphi_n,\lambda_n^h-\theta_n^{hk}\rangle_{\mathcal{H}}\\ &\quad+\langle h_n,\theta_n-\lambda_n^h\rangle_Q+\int_{\Gamma_3} j_\theta^0(\theta_n^{hk};\lambda_n^h-\theta_n)\,da\\ &\quad+\int_{\Gamma_3}\big(j_\theta^0(\theta_n;\theta_n^{hk}-\theta_n)+j_\theta^0(\theta_n^{hk};\theta_n-\theta_n^{hk})\big)\,da\end{aligned}$$

Therefore, we can obtain the following estimate

$$\begin{aligned}&\alpha_{\mathscr{K}}\big\|\theta_n-\theta_n^{hk}\big\|_Q^2+\big\langle\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\big\rangle\\ &\le c\left\{\|\theta_n-\lambda_n^h\|_Q^2+\|u_n-u_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_Q^2\right\}\\ &\quad+\big\langle\delta\theta_n^{hk}-\delta\theta_n,\lambda_n^h-\theta_n\big\rangle_{\mathcal{H}}+S_3(u_n,\varphi_n,\theta_n)+I_3(\theta_n^{hk},\theta_n,\lambda_n^h),\end{aligned} \tag{3.35}$$

where the terms S_3 and I_3 are given as follows:

$$\begin{aligned}S_3(u_n,\varphi_n,\theta_n)&=\langle\dot\theta_n,\lambda_n^h-\theta_n\rangle_{\mathcal{H}}+\langle \mathscr{K}\nabla\theta_n,\nabla(\lambda_n^h-\theta_n)\rangle_{\mathcal{H}}\\ &\quad-\langle \mathscr{M}\varepsilon(u_n),\lambda_n^h-\theta_n\rangle_{\mathcal{H}}+\langle \mathscr{N}\nabla\varphi_n,\lambda_n^h-\theta_n\rangle_{\mathcal{H}}\\ &\quad+\big\langle h_n,\theta_n-\lambda_n^h\big\rangle_{\mathcal{H}},\end{aligned} \tag{3.36}$$

$$I_3(\theta_n^{hk}, \theta, \lambda_n^h) = \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n)\, da. \tag{3.37}$$

Analogously to the relation (3.14), we can obtain

$$\int_\Omega \mathscr{K} \nabla\theta \cdot \nu(\lambda_n^h - \theta_n)\, dx - \int_\Omega h_0(\lambda_n^h - \theta_n)\, dx \\ + \int_{\Gamma_2} (\lambda_n^h - \theta_n)\, d\Gamma = \int_{\Gamma_3} \mathscr{K} \nabla\theta \cdot \nu.\nu(\lambda_n^h - \theta_n))\, d\Gamma, \quad \forall\, \lambda_n^h \in Q_3^h.$$

Then, since $\mathscr{K} \nabla\theta \cdot \nu \in L^2(0,T; L^2(\Gamma_3))$, it yields that

$$S_3(u_n, \varphi_n, \theta_n) \le c \left\| \theta_n - \lambda_n^h \right\|_{L^2(\Gamma_3)}. \tag{3.38}$$

We recall that $j_\theta(x, \cdot)$ is assumed to be uniformly Lipschitz continuous a.e. on $x \in \Gamma_3$, so that its Lipschitz constant $c > 0$ is independent of x. Consequently, the following estimate holds.

$$I_3(\theta_n^{hk}, \theta, \lambda_n^h) \le c \left\| \theta_n - \lambda_n^h \right\|_{L^2(\Gamma_3)}. \tag{3.39}$$

We then combine the inequalities (3.35), (3.38), and (3.39) to find

$$\begin{aligned} &\alpha_{\mathscr{K}} \|\theta_n - \theta_n^{hk}\|_Q^2 + \langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk} \rangle_{L(\Omega)} \\ &\le c \left\{ \|\theta_n - \lambda_n^h\|_Q^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_Q^2 + \|\theta_n - \lambda_n^h\|_{L^2(\Gamma_3)} \right\} \\ &\quad + \langle \delta\theta_n^{hk} - \delta\theta_n, \lambda_n^h - \theta_n \rangle_{\mathcal{H}}. \end{aligned} \tag{3.40}$$

Next, the identity $2\langle a - b, a\rangle = \|a - b\|^2 + \|a\|^2 - \|b\|^2$ for $a = \theta_n - \theta_n^{hk}$ and $b = \theta_{n-1} - \theta_{n-1}^{hk}$ implies

$$\frac{1}{2k}\left(\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_Q^2 \right) \le \left\langle \delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk} \right\rangle_{L^2(\Omega)}. \tag{3.41}$$

Then, by combining the inequalities (3.40) and (3.41), the following upper bound yields:

$$\begin{aligned} &\alpha_{\mathscr{K}} \left\|\theta_n - \theta_n^{hk}\right\|_Q^2 + \frac{1}{2k}\left(\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_Q^2 \right) \\ &\le c \left\{ \left\|\theta_n - \lambda_n^h\right\|_Q^2 + \left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\varphi_n - \varphi_n^{hk}\right\|_Q^2 + \left\|\theta_n - \lambda_n^h\right\|_{L^2(\Gamma_3)} \right\} \\ &\quad + \left\langle \delta\theta_n^{hk} - \delta\theta_n, \lambda_n^h - \theta_n \right\rangle_{\mathcal{H}}. \end{aligned} \tag{3.42}$$

Replacing n with j in the above relation and summing from $j = 1$ to n, we obtain

$$\begin{aligned} &\|\theta_n - \theta_n^{hk}\|_Q^2 + 2k\, \alpha_{\mathscr{K}} \sum_{j=1}^n \|\theta_j - \theta_j^{hk}\|_Q^2 \\ &\le ck \sum_{j=1}^n \left\{ \|\theta_j - \lambda_j^h\|_Q^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \right\} \\ &\quad + 2k \sum_{j=1}^n \left\langle \delta\theta_j^{hk} - \delta\theta_j, \lambda_j^h - \theta_j \right\rangle_{\mathcal{H}} + \left\|\theta_0 - \theta_0^h\right\|_Q^2. \end{aligned} \tag{3.43}$$

Following the same approach as in [4], we can derive the following upper bound

$$
\begin{aligned}
&2k \sum_{j=1}^{n} \left\langle \delta\theta_j^{hk} - \delta\theta_j, \lambda_j^h - \theta_j \right\rangle_{\mathcal{H}} \\
&\leq c \left\{ \|\theta_n - \theta_n^{hk}\|_Q^2 + \|\theta_n - \lambda_n^h\|_Q^2 + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 \right\} \\
&\quad + \frac{k}{2} \sum_{j=1}^{n-1} \left\|\theta_j - \theta_j^{hk}\right\|_Q^2 + \frac{2}{k} \sum_{j=1}^{n-1} \left\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\right\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{3.44}
$$

For convenience, we consider the following notations

$$
e_n = \|\theta_n - \theta_n^{hk}\|_Q^2 + 2k\alpha_{\mathscr{K}} \sum_{j=1}^{n} \|\theta_j - \theta_j^{hk}\|_Q^2, \tag{3.45}
$$

$$
\begin{aligned}
g_n = k &\sum_{j=1}^{n} \left\{ \|\theta_j - \lambda_j^h\|_Q^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \right\} \\
&+ \frac{1}{k} \sum_{j=1}^{n-1} \left\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\right\|_{L^2(\Omega)}^2 \\
&+ \left\|\theta_0 - \theta_0^h\right\|_Q^2 + \left\|\theta_1 - \lambda_1^h\right\|_Q^2 + \left\|\theta_n - \lambda_n^h\right\|_Q^2.
\end{aligned}
\tag{3.46}
$$

Thus, the preceding upper bounds imply to the existence of a constant $c > 0$ such that

$$
e_n \leq c\, g_n + c \sum_{j=1}^{n} e_j. \tag{3.47}
$$

We then apply the Gronwall inequality to establish that

$$
\begin{aligned}
&\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 + \sum_{j=1}^{n} \left\|\theta_j - \theta_j^{hk}\right\|_Q^2 \\
&\leq ck \sum_{j=1}^{n} \left\{ \|\theta_j - \lambda_j^h\|_Q^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)} \right\} \\
&\quad + c \sum_{j=1}^{n-1} \left\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_Q^2 + \left\|\theta_1 - \lambda_1^h\right\|_Q^2.
\end{aligned}
\tag{3.48}
$$

Based on the previous estimates (3.16), (3.18), and (3.48), we conclude

$$\begin{aligned}
&\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 \\
&\le c\Big\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\
&\quad + \sum_{j=1}^{n}(\|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}) + \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\
&\quad + \sum_{j=1}^{n}(\|w_j - w_j^{hk}\|_V^2 + \|u_j - u_j^{hk}\|_V^2 + \|\varphi_j - \varphi_j^{hk}\|_Q^2 + \|\theta_j - \theta_j^{hk}\|_Q^2) \\
&\quad + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2\Big\} + c\left(h^2 + k^2\right).
\end{aligned} \tag{3.49}$$

Let us then define the following two quantities:

$$e_n = \|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2, \tag{3.50}$$

and

$$\begin{aligned}
g_n &= \|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\
&\quad + \sum_{j=1}^{n}(\|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}) + \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\
&\quad + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 + h^2 + k^2.
\end{aligned} \tag{3.51}$$

So, by using these notations, the inequality (3.49) can be rewritten as follows:

$$e_n \le c\, g_n + c\sum_{j=}^{n} e_j. \tag{3.52}$$

Thus, applying the Gronwall inequality to the preceding result, we infer that

$$\begin{aligned}
&\|w_n - w_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\theta_n - \theta_n^{hk}\|_Q^2 \\
&\le c\Big\{\|w_n - v_n^h\|_V^2 + \|\varphi_n - \psi_n^h\|_W^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \|\varphi_n - \psi_n^h\|_{L^2(\Gamma_3)} \\
&\quad + \sum_{j=1}^{n}(\|\theta_j - \lambda_j^h\|_Q^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}) + \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 \\
&\quad + \|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2\Big\} + c\big(h^2 + k^2\big).
\end{aligned} \tag{3.53}$$

Finally, by combining (3.23) and (3.53), we derive the estimate (3.7), which validates Theorem 3.1.

□

Using the finite element approximation (see [1, 4–6, 11, 23, 25]), the following error estimate holds.

Corollary 3.1. *Under the conditions of Theorem 2.1, along with the following regularities*

$$u \in L^2([0,T];V_1) \cap C^1([0,T];H^2(\Omega))\,,\ \ w \in C([0,T];H^2(\Omega)) \cap L^2([0,T];H^2(\Gamma_3)),$$

$$\varphi \in C([0,T];H^2(\Omega))\,,\ \theta \in C([0,T];H^2(\Omega)) \cap L^2(0,T;H^2(\Gamma_3))\,,\ \dot{\theta} \in\ L^2(0,T;H^2(\Omega)),$$

the following first order error estimate is obtained.

$$\begin{aligned}&\max_{1\le n\le N} \|w_n - w_n^{hk}\|_V + \max_{1\le n\le N} \|u_n - u_n^{hk}\|_V \\ &+ \max_{1\le n\le N} \|\varphi_n - \varphi_n^{hk}\|_W + \max_{1\le n\le N} \|\theta_n - \theta_n^{hk}\|_Q \le c\,\big(h+k\big)\end{aligned} \tag{3.54}$$

for some constant c independent of the parameters h and k.

We now outline the key steps in the proof of this corollary. First, let $\Pi_V^h w_n$, $\Pi_V^h u_n$, $\Pi_W^h \varphi_n$ and $\Pi_Q^h \theta_n$ denote the finite element interpolants of w_n, u_n, φ_n and θ_n, respectively, where Π_S^h represents the standard interpolation operator over a given set S. Next, under appropriate regularity conditions, we derive the corresponding finite element interpolation error estimates (see [12, p. 133] for details). We then utilize the same techniques as in [10, p. 126], [1, Theorem 3], and [11] to obtain the desired error estimate.

4. Conclusion

This paper analyzed a thermo-electro-visco-elastic contact problem with locking materials, formulating it as a coupled system of hemivariational inequalities and a parabolic equation. We proved the existence and uniqueness of the solution, addressing complex interactions involving non-monotone boundary conditions and locking constraints. Additionally, a fully discrete scheme was developed using finite elements for space and finite differences for time, with error estimates and convergence results established. These findings extend the theoretical understanding and numerical modeling of locking materials in contact problems. Future work could explore non-convex constraints and optimal control applications for such systems.

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