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Blowing-Up Solutions of the Shallow Water Equations*

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Abstract In the paper, we study the question about global unsolvability of the Kawahara and Kaup-Kupershmidt shallow water equations in a bounded domain. For certain initial-boundary-value problems of the shallow-water equations, we establish the necessary conditions for the existence of global solutions. The proof of the results is based on the nonlinear capacity method. In closing, we provide some examples.

Keywords Kaup-Kupershmidt equation, Kawahara equation, shallow water, blow-up solution, mixed problem

MSC(2010) 35K55, 35R11.

1. Introduction

In the paper, we consider some shallow water equations as follows

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u + u\partial_x u = 0,\ (x,t) \in (a,b) \times (0,T), \tag{1.1}$$

$$\partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u - \partial_x u\partial_x^2 u = 0,\ (x,t) \in (a,b) \times (0,T), \tag{1.2}$$

with the initial data

$$u(x,0) = u_0(x), \tag{1.3}$$

where $-\infty < a < b < +\infty$, $\alpha \neq 0$, β, γ are real numbers.

The model (1.1) is also called the Kawahara equation [9]. It arises in the study of the water waves with surface tension, in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime (see [3, 10]). Equation (1.2) is often called the Kaup-Kupershmidt equation [8]. This model has arisen from the study of the capillary-gravity waves [2, 6, 7, 21].

If $\alpha = 0$, then the model (1.1) reduces to the well-known Korteweg-de Vries equation [4]

$$\partial_t u + \beta\partial_x^3 u + \gamma\partial_x u + u\partial_x u.$$

The local and global well-posedness of problems (1.1) - (1.3) for $\alpha = -1$, $\beta = 1$, $\gamma = 0$, with the boundary conditions

$$u(0,t) = 0,\ u(1,t) = 0,\ t \geq 0,$$

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$$\partial_x u(0,t) = 0,\ \partial_x u(1,t) = 0,\ t \geq 0,$$
$$\partial_x^2 u(1,t) = 0,\ t \geq 0,$$

were studied by Larkin et al. in [12–14]. Note that the local well-posedness of the equations (1.1)-(1.2) with the initial data (1.3) and several boundary conditions was investigated in [1, 5, 15, 18, 20, 22, 23].

The main aim of this paper is to obtain the blowing-up solutions of the above equations, more precisely, solutions that blow up in finite time for the large class of boundary conditions. Our approach is based on the Mitidieri-Pohozhaev nonlinear capacity method (see [16, 17], also [11, 19]), more precisely, on the choice of specific functions corresponding to the initial and boundary conditions.

A small difference in our approach is that we will study shallow-water equations without boundary conditions. We suppose the boundary conditions are such that there exists a sufficiently smooth function φ, for which the functional $B(u, \phi)$, containing u and φ and their k-derivatives, is lower bounds by a certain functional of φ. This allows us to study some classes of boundary conditions.

2. Blow-up of solution of the Kawahara equation

Let us consider the function $\varphi \in C^5([a,b])$ defined on the domain $a < x < b$ with arbitrary parameters $a, b \in \mathbb{R}$ and monotonically nondecreasing:

$$\varphi'(x) \geq 0 \ \text{ for } \ x \in [a,b], \tag{2.1}$$

and let φ satisfy the following properties:

$$\begin{cases} \theta_1 := \int\limits_a^b \frac{(\alpha\varphi^{(5)} + \beta\varphi''' + \gamma\varphi')^2}{\varphi'} dx < \infty; \\ \theta_2 := \int\limits_a^b \frac{\varphi^2}{\varphi'} dx < \infty. \end{cases} \tag{2.2}$$

Suppose that there is a classical solution $u(x,t) \in C^{1,5}_{t,x}(\mathbb{R} \times (0,T))$.

Multiplying Kawahara shallow water equation (1.1) by φ, we have

$$\begin{aligned} \int\limits_a^b \partial_t u(x,t)\varphi(x)dx = &- \alpha \int\limits_a^b \partial_x^5 u(x,t)\varphi(x)dx - \beta \int\limits_a^b \partial_x^3 u(x,t)\varphi(x)dx \\ &- \gamma \int\limits_a^b \partial_x u(x,t)\varphi(x)dx - \int\limits_a^b u(x,t)\partial_x u(x,t)\varphi(x)dx. \end{aligned}$$

Applying integration by parts, we arrive at

$$\begin{aligned} \partial_t \int\limits_a^b u(x,t)\varphi(x)dx = &\alpha \int\limits_a^b u(x,t)\varphi^{(5)}(x)dx + \beta \int\limits_a^b u(x,t)\varphi'''(x)dx \\ &+ \gamma \int\limits_a^b u(x,t)\varphi'(x)dx + \frac{1}{2}\int\limits_a^b u^2(x,t)\varphi'(x)dx \\ &+ \mathcal{B}(u(x,t),\varphi(x))\Big|_{x=a}^{x=b}, \end{aligned} \tag{2.3}$$

where

$$\mathcal{B}(u(x,t),\varphi(x)) = \alpha\left(\partial_x^4 u\varphi - \partial_x^3 u\varphi' + \partial_x^2 u\varphi'' - \partial_x u\varphi''' + u\varphi^{(4)}\right) + \beta\left(\partial_x^2 u\varphi - \partial_x u\varphi' + u\varphi''\right) + \gamma u\varphi + \frac{1}{2}u^2\varphi.$$

Then, using properties (2.1), we find

$$\int_a^b \left(2u(x,t)\left(\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)\right) + u^2(x,t)\varphi'(x)\right)dx$$
$$= \int_a^b \left(u(x,t) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\varphi'(x)}\right)^2 \varphi'(x)dx - \int_a^b \frac{\left(\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)\right)^2}{\varphi'(x)}dx.$$

We introduce a new functional:

$$F(t) = \int_a^b w(x,t)\varphi(x)dx,$$

where

$$w(x,t) = u(x,t) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\varphi'(x)}.$$

Using the Hölder inequality, we obtain the following estimate

$$\left(\int_a^b w(x,t)\varphi(x)dx\right)^2 \le \int_a^b w^2(x,t)\varphi'(x)dx \int_a^b \frac{\varphi^2(x)}{\varphi'(x)}dx.$$

Then, due to properties (2.2), expression (2.3) takes the form

$$F'(t) \ge \frac{\theta_2^{-1}}{2}F^2(t) + \Phi(t) - \frac{\theta_1}{2}, \tag{2.4}$$

with the initial condition

$$F(0) = \int_a^b \left(u_0(x) + \frac{\alpha\varphi^{(5)}(x) + \beta\varphi'''(x) + \gamma\varphi'(x)}{\varphi'(x)}\right)\varphi(x)dx,$$

where $\Phi(t) = \mathcal{B}(u(b,t),\varphi(b)) - \mathcal{B}(u(a,t),\varphi(a))$.

Then the following results are true.

Theorem 2.1. *Let* $u_0(x) \in L^1([a,b])$ *and* u *be the solution of the equation* (2.1) *such that* $u \in C_{t,x}^{1,5}(\mathbb{R}\times(0,T))$. *Let a function* φ *satisfy conditions* (2.1), (2.2) *and let*

$$2\Phi(t) \ge \sigma, \text{ for all } t > 0, \tag{2.5}$$

where σ *is a constant.*

(A) *If* $\sigma > \theta_1$ *and* $F(0) > 0$, *then*

$$F(t) \to +\infty \ \textit{for}\ t \to T_1^*,$$

where

$$T_1^* = \frac{2\sqrt{\theta_2}}{\sqrt{\sigma - \theta_1}} \left(\frac{\pi}{2} - \arctan \frac{F(0)}{2\sqrt{\theta_2(\sigma - \theta_1)}} \right).$$

(B) *If* $\sigma = \theta_1$ *and* $F(0) > 0$, *then*

$$F(t) \to +\infty \ \textit{for}\ t \to T_2^*,$$

where $T_2^* = \frac{4\theta_2}{F(0)}$.

(C) *If* $\sigma < \theta_1$ *and* $F(0) > 2\sqrt{\theta_2(\theta_1 - \sigma)}$, *then*

$$F(t) \to +\infty \ \textit{for}\ t \to T_3^*,$$

where

$$T_3^* = \frac{\sqrt{\theta_2}}{\sqrt{\theta_1 - \sigma}} \ln \frac{F(0) + 2\sqrt{\theta_2(\theta_1 - \sigma)}}{F(0) - 2\sqrt{\theta_2(\theta_1 - \sigma)}}.$$

Proof. Applying the theory of ordinary differential inequalities, we can prove Theorem 2.1. Indeed, let us consider the following differential equation

$$E'(t) = E^2(t) + \lambda. \tag{2.6}$$

(A) If $\lambda > 0$, then $E(t) \to +\infty$ at $t \to \frac{1}{\sqrt{\lambda}} \left(\frac{\pi}{2} - \arctan \frac{E(0)}{\sqrt{\lambda)}} \right)$;

(B) If $\lambda = 0$ and $E(0) > 0$, then $E(t) \to +\infty$ at $t \to \frac{1}{E(0)}$;

(C) If $\lambda < 0$ and $E(0) > \sqrt{\lambda}$, then $E(t) \to +\infty$ at $t \to \frac{1}{\sqrt{\lambda}} \ln \frac{E(0)+\sqrt{\lambda}}{F(0)-\sqrt{\lambda}}$.

Since the function $F(t)$ is an upper solution of equation (2.6), by comparison principle we have $F(t) \to +\infty$ at the same time with $E(t)$. □

Remark 2.1. If $\alpha = 0$, $\beta = 1$ and $\gamma = 0$ then Kawahara equation (1.1) coinsides with Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x u + u\partial_x u = 0.$$

Then our result in Theorem 2.1 implies the blow-up results for the Korteweg-de Vries equation obtained by Pohozhaev in [19].

Below, we give some examples for different classes of boundary conditions.

Example 2.1. Note that the nonlinear capacity method has great practical convenience. For example, let problem (1.1), (1.3) with $\beta = 0$, $\gamma > 0$, on $[0, 1]$ satisfy Dirichlet type boundary conditions

$$\begin{aligned} &u(0,t) = 0, \ u(1,t) = 0, \\ &\partial_x^2 u(0,t) = 0, \ \partial_x^2 u(1,t) = 0, \\ &\partial_x^4 u(0,t) + 4\partial_x^3 u(0,t) + 24\partial_x u(0,t) = 0, \ \ t \ge 0. \end{aligned}$$

Then, by taking $\varphi(x) = -(1-x)^4$, we obtain

$$\theta_1 := 4\gamma^2, \ \theta_2 := \frac{1}{24}$$

and

$$\Phi(t) = 0, \ \text{for all} \ \ t > 0.$$

Hence it follows from Theorem 2.1, that under condition

$$\int_0^1 u_0(x)(1-x)^4 dx < -\frac{\gamma}{5}$$

the solution of problem (1.1), (1.3) blows up in finite time

$$T^* = 2\sqrt{6}\gamma \ln \frac{\sqrt{6}F(0) + \gamma}{\sqrt{6}F(0) - \gamma}.$$

Example 2.2. Let $\alpha = 1$, $\beta = 1$, $\gamma > 0$, and $[a, b] = [0, 1]$. Consider the problem (1.1), (1.3) with nonlocal boundary conditions

$$\begin{aligned}
&u(0,t) = 0, \ u(1,t) = 0,\\
&\partial_x u(0,t) = 0, \ \partial_x u(1,t) = 0,\\
&\partial_x^4 u(1,t) - \partial_x^3 u(1,t) + \partial_x^2 u(1,t) + \partial_x^3 u(0,t) = f(t) \geq \frac{4}{3}, \ t \geq 0.
\end{aligned}$$

Letting $\varphi(x) = x$, we have

$$\theta_1 := \gamma^2, \ \theta_2 := \frac{1}{3},$$

and

$$\Phi(t) \geq \frac{4}{3}, \ \text{for all} \ \ t > 0.$$

Therefore, if

$$\int_0^1 u_0(x) x dx > -\frac{\gamma}{2},$$

then from Theorem 2.1 it follows that the solution of problem (1.1), (1.3) blows-up in finite

$$T^* = 2\left(\frac{\pi}{2} - \arctan \frac{F(0)}{2\gamma}\right).$$

3. Gradient blow-up of solution of the Kaup-Kupershmidt equation

In this section, we obtain a result on the ”soft blow-up” for the initial problem (1.2), (1.3) in the bounded domain. Suppose that there exists a smooth bounded classical solution. Differentiating equation (1.2) with respect to space variable, we obtain

$$\partial_{tx}^2 u + \alpha\partial_x^5 u + \beta\partial_x^4 u + \gamma\partial_x^2 u - \partial_x u\partial_x^3 u - \partial_x^2 u\partial_x^2 u = 0,\ a < x < b,\ t > 0. \tag{3.1}$$

We consider a function $\phi \in C^5([a,b])$ defined on the domain $a < x < b$ with arbitrary parameters $a, b \in \mathbb{R}$ and nonconvex:

$$\phi''(x) \geq 0 \ \text{ for } \ x \in [a,b], \tag{3.2}$$

and let function φ satisfy the following properties:

$$\begin{cases} \omega_1 := \int\limits_a^b \frac{(\alpha\phi^{(5)}+\beta\phi'''+\gamma\varphi')^2}{\phi''} dx < \infty; \\ \omega_2 := \int\limits_a^b \frac{\phi^2}{\phi''} dx < \infty. \end{cases} \tag{3.3}$$

Multiplying equation (3.1) by $\phi(x)$ and letting $\partial_x u = v$, we have

$$\begin{aligned} \frac{d}{dt}\int_a^b v(x,t)\phi(x)dx =& \alpha\int_a^b v(x,t)\phi^{(5)}(x)dx \\ &+ \beta\int_a^b v(x,t)\phi'''(x)dx + \gamma\int_a^b v(x,t)\phi'(x)dx \\ &+ \frac{1}{2}\int_a^b v^2(x,t)\phi''(x)dx \\ &+ \mathcal{M}(v(x,t),\phi(x))\Big|_{x=a}^{x=b}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{M}(v(x,t),\phi(x)) =& -\alpha\partial_x^4 v(x,t)\phi(x) + \alpha\partial_x^3 v(x,t)\phi'(x) \\ &- \alpha\partial_x^2 v(x,t)\phi''(x) + \alpha\partial_x v(x,t)\phi'''(x) \\ &- \alpha v(x,t)\phi^{(4)}(x) - \beta\partial_x^2 v(x,t)\phi(x) + \beta\partial_x v(x,t)\phi'(x) \\ &- \beta v(x,t)\phi''(x) - \gamma v(x,t)\phi(x) \\ &+ \frac{1}{2}\partial_x v^2(x,t)\phi(x) - \frac{1}{2}v^2(x,t)\phi'(x). \end{aligned}$$

We denote by $w(x,t)$ as follows

$$w(x,t) = v(x,t) + \frac{\alpha\phi^{(5)}(x) + \beta\phi'''(x) + \gamma\phi'(x)}{\phi''(x)}.$$

By using the Hölder inequality for the functional

$$H(t) = \int\limits_a^b w(x,t)\phi(x)dx,$$

we obtain the following estimate

$$\left(\int\limits_a^b w(x,t)\phi(x)dx\right)^2 \leq \int\limits_a^b w^2(x,t)\phi''(x)dx \int\limits_a^b \frac{\phi^2(x)}{\phi''(x)}dx.$$

We introduce the notation

$$\Phi(t) = \mathcal{M}(v(b,t), \phi(b)) - \mathcal{M}(v(a,t), \phi(a)).$$

Suppose that there exists a function $\phi(x)$ for which $\Phi(t)$ is independent of time. If there is no such a function, then $\Phi(t)$ must be considered separately, for example, by assuming that the constant independent of t is bounded above.

Consequently, by properties (3.2) and (3.3) for the function $H(t)$ we obtain the following ordinary differential inequality

$$H'(t) \geq \frac{\omega_2^{-1}}{2} H^2(t) - \omega^2, \tag{3.5}$$

where $\omega = \frac{\omega_1}{2} - \Phi(t)$.

Applying the theory of ordinary differential inequalities, we obtain the following result.

Theorem 3.1. *Let* $u_0(x) \in H^1([a,b])$ *and the solution* $u \in C^{1,5}_{t,x}(\mathbb{R} \times (0,T))$ *of the equation* (3.1) *be such that there exists a function* ϕ *satisfying conditions* (3.2), (3.3) *such that*

$$H(0) = \int\limits_a^b \left(u_0'(x) + \frac{\alpha\phi^{(5)}(x) + \beta\phi'''(x) + \gamma\phi'(x)}{\phi'(x)} \right) \phi(x)dx > \omega\sqrt{2\omega_2}.$$

Then the gradient solution of equation (1.2) *blows up in finite time and the following estimate holds:*

$$H(t) \geq \omega\sqrt{2\omega_2}\frac{1 + h_0 \exp\left(2h_0\sqrt{2\omega_2}^{-1}t\right)}{1 - h_0 \exp\left(2h_0\sqrt{2\omega_2}^{-1}t\right)}, \quad h_0 = \frac{\sqrt{2\omega_2}^{-1}H(0) - \omega}{\sqrt{2\omega_2}^{-1}H(0) + \omega},$$

and hence

$$\lim_{t \to T^*} H(t) = +\infty,$$

where $T^* = -\frac{\sqrt{2\omega_2}}{2\omega} \ln h_0$.

Theorem 3.1 can be proved in a similar way to Theorem 2.1.

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