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Spreading Speed for Some Cooperative Systems with Nonlocal Diffusion and Free Boundaries, Part 3: Rate of Shifting

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Abstract This is the last part of our work series on a class of cooperative reaction-diffusion systems with free boundaries in one space dimension, where the diffusion terms are nonlocal, given by integral operators involving suitable kernel functions, and some of the equations in the system do not have a diffusion term. Such a system covers various models arising from population biology and epidemiology, including in particular a West Nile virus model [10] and some epidemic models [22, 38], where a "spreading-vanishing" dichotomy is known to govern the long time dynamical behaviour, but the spreading rate was not well understood. In this work series, we develop a systematic approach to determine the spreading profile of the system. In Part 1 [11], we obtained threshold conditions on the kernel functions which decide exactly when the spreading has finite speed c_0, or infinite speed (accelerated spreading), and for the case of finite speed, we determined its value c_0 via semi-wave solutions. In this paper, for some typical classes of kernel functions, we obtain more precise descriptions of the spreading for the finite speed case by revealing the exact rate of shifting of the spreading front from c_0t; the infinite speed case is studied separately in Part 2 [14].

Keywords Free boundary, nonlocal diffusion system, spreading rate

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1. Introduction

We continue our efforts to determine the precise long-time behaviour of cooperative systems with nonlocal diffusion and free boundaries of the following form:

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$$\begin{cases} \partial_t u_i = d_i \mathcal{L}_i[u_i](t,x) + f_i(u_1, u_2, \cdots, u_m), & t > 0,\ x \in (g(t), h(t)),\ 1 \le i \le m_0, \\ \partial_t u_i = f_i(u_1, u_2, \cdots, u_m), & t > 0,\ x \in (g(t), h(t)),\ m_0 < i \le m, \\ u_i(t, g(t)) = u_i(t, h(t)) = 0, & t > 0,\ 1 \le i \le m, \\ g'(t) = -\displaystyle\sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_i(x-y) u_i(t,x) \mathrm{d}y \mathrm{d}x, & t > 0, \\ h'(t) = \displaystyle\sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y) u_i(t,x) \mathrm{d}y \mathrm{d}x, & t > 0, \\ u_i(0,x) = u_{i0}(x), & x \in [-h_0, h_0],\ 1 \le i \le m, \end{cases} \tag{1.1}$$

where $1 \le m_0 \le m$, and for $i \in \{1, ..., m_0\}$,

$$\mathcal{L}_i[v](t,x) := \int_{g(t)}^{h(t)} J_i(x-y) v(t,y) \mathrm{d}y - v(t,x),$$

$$d_i > 0 \text{ and } \mu_i \ge 0 \text{ are constants, with } \sum_{i=1}^{m_0} \mu_i > 0.$$

The initial functions satisfy for $1 \le i \le m$,

$$u_{i0} \in C([-h_0, h_0]), \quad u_{i0}(-h_0) = u_{i0}(h_0) = 0,\ u_{i0}(x) > 0 \ \text{ in } (-h_0, h_0). \tag{1.2}$$

The kernel functions satisfy, for $J \in \{J_i : 1 \le i \le m_0\}$,

(**J**): $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is nonnegative, even, $J(0) > 0$, $\displaystyle\int_{\mathbb{R}} J(x) \mathrm{d}x = 1$.

As in Part 1 [11], we will write $F = (f_1, ..., f_m) \in [C^1(\mathbb{R}^m_+)]^m$ with

$$\mathbb{R}^m_+ := \{x = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \ge 0 \text{ for } i = 1, ..., m\},$$

and use the following notations for vectors in $\mathbb{R}^m$:

(i) For $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, we simply write $(x_1, \cdots, x_m)$ as (x_i). For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$\begin{array}{lll} x \succeq (\preceq)\ y & \text{means} & x_i \ge (\le)\ y_i \text{ for } 1 \le i \le m, \\ x \succ (\prec)\ y & \text{means} & x \succeq (\preceq)\ y \text{ but } x \ne y, \\ x \succ\!\!\succ (\prec\!\!\prec)\ y & \text{means} & x_i > (<)\ y_i \text{ for } 1 \le i \le m. \end{array}$$

(ii) If $x \preceq y$, then $[x, y] := \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$.

(iii) Hadamard product: For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \circ y = (x_i y_i) \in \mathbb{R}^m.$$

(iv) Any $x \in \mathbb{R}^m$ is viewed as a row vector, namely a $1 \times m$ matrix, whose transpose is denoted by x^T.

Our basic assumptions on F are:

($\mathbf{f_1}$) (i) $F(u) = \mathbf{0}$ has only two roots in $\mathbb{R}^m_+$: $\mathbf{0} = (0, 0, \cdots, 0)$ and $\mathbf{u}^* = (u_1^*, u_2^*, \cdots, u_m^*) \succ\succ \mathbf{0}$.

(ii) $\partial_j f_i(u) \geq 0$ for $i \neq j$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$, where either $\hat{\mathbf{u}} = \infty$ meaning $[\mathbf{0}, \hat{\mathbf{u}}] = \mathbb{R}^m_+$, or $\mathbf{u}^* \prec\prec \hat{\mathbf{u}} \in \mathbb{R}^m$; which implies that (1.1) is a cooperative system in $[\mathbf{0}, \hat{\mathbf{u}}]$.

(iii) The matrix $\nabla F(\mathbf{0})$ is irreducible with positive principal eigenvalue, where $\nabla F(\mathbf{0}) = (a_{ij})_{m\times m}$ with $a_{ij} = \partial_j f_i(\mathbf{0})$.

(iv) If $m_0 < m$ then $\partial_j f_i(u) > 0$ for $1 \leq j \leq m_0 < i \leq m$ and $u \in [\mathbf{0}, \mathbf{u}^*]$.

($\mathbf{f_2}$) $F(ku) \geq kF(u)$ for any $0 \leq k \leq 1$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$.

($\mathbf{f_3}$) The matrix $\nabla F(\mathbf{u}^*)$ is invertible, $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \preceq \mathbf{0}$ and for each $i \in \{1, ..., m\}$, either

(i) $\displaystyle\sum_{j=1}^m \partial_j f_i(\mathbf{u}^*) u_j^* < 0$, or

(ii) $\displaystyle\sum_{j=1}^m \partial_j f_i(\mathbf{u}^*) u_j^* = 0$ and $f_i(u)$ is linear in $[\mathbf{u}^* - \epsilon_0 \mathbf{1}, \mathbf{u}^*]$ for some small $\epsilon_0 > 0$, where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^m$.

($\mathbf{f_4}$) The set $[\mathbf{0}, \hat{\mathbf{u}}]$ is invariant for

$$U_t = D \circ \int_{\mathbb{R}} \mathbf{J}(x-y) \circ U(t,y)\mathrm{d}y - D \circ U + F(U) \text{ for } t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

and the equilibrium $\mathbf{u}^*$ attracts all the nontrivial solutions in $[\mathbf{0}, \hat{\mathbf{u}}]$; namely, $U(t,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $t > 0, x \in \mathbb{R}$ if $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $x \in \mathbb{R}$, and $\lim_{t\to\infty} U(t,\cdot) = \mathbf{u}^*$ in $L^\infty_{loc}(\mathbb{R})$ if additionally $U(0,x) \not\equiv \mathbf{0}$.

In (1.3) we have used the convention that $d_i = 0$ and $J_i \equiv 0$ for $m_0 < i \leq m$, and

$$D = (d_i),\ \mathbf{J}(x) = (J_i(x)).$$

This convention will be used throughout the paper.

The above assumptions on F indicate that the system is cooperative in $[\mathbf{0}, \hat{\mathbf{u}}]$, and of monostable type, with $\mathbf{u}^*$ the unique stable equilibrium of (1.3), which is also the global attractor of all the nontrivial nonnegative solutions of (1.3) in $[\mathbf{0}, \hat{\mathbf{u}}]$.

Problems (1.1) and (1.3) arise frequently in population and epidemic models. For example, if $m_0 = m = 2$, (1.1) contains the West Nile virus model in [10] and the epidemic models in [22] as special cases, and with $(m_0, m) = (1, 2)$, it covers the epidemic model in [38]. In these special cases, it is known that the long-time dynamical behaviour of the solution to (1.1) exhibits a spreading-vanishing dichotomy.

Similar to the special cases mentioned in the last paragraph, it can be shown that (1.1) with initial data satisfying (1.2) and $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ has a unique positive solution $(U(t,x), g(t), h(t))$ defined for all $t > 0$. We say spreading happens if, as $t \to \infty$,

$$(g(t), h(t)) \to (-\infty, \infty) \text{ and } U(t,\cdot) \to \mathbf{u}^* \text{ component-wise in } L^\infty_{loc}(\mathbb{R}),$$

and we say vanishing happens if

$$(g(t), h(t)) \to (g_\infty, h_\infty) \text{ is a finite interval, and } \max_{x\in[g(t),h(t)]} |U(t,x)| \to 0.$$

1.1. Main results of Part 1

Since this paper is built upon the results in Part 1 [11], let us now recall the main results obtained there. When spreading happens for (1.1), we proved in Part 1 that the spreading speed is finite if and only if the following additional condition is satisfied by the kernel functions:

$$(\mathbf{J_1}): \qquad \int_0^{\infty} xJ(x)\mathrm{d}x < \infty \text{ for } J \in \{J_i : 1 \le i \le m_0,\ \mu_i > 0\}.$$

If $(\mathbf{J_1})$ is not satisfied, then the spreading speed is infinite, namely accelerated spreading happens, and this case was further investigated in Part 2 of this series [14]. Let us note that if for some $i \in \{1, ..., m_0\}$, $\mu_i = 0$, then no restriction on J_i is imposed by $(\mathbf{J_1})$. For convenience of later discussions, we will introduce the following notations:

$$A_+ := \{i : 1 \le i \le m_0,\ \mu_i > 0\}, \quad A_0 := \{i : 1 \le i \le m_0,\ \mu_i = 0\}.$$

The proof of these conclusions rely on a complete understanding of the associated semi-wave problem to (1.1), which consists of the following two equations (1.4) and (1.5) with unknowns $(c, \Phi(x))$:

$$\begin{cases} D \circ \displaystyle\int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Phi(y)\mathrm{d}y - D \circ \Phi + c\Phi'(x) + F(\Phi(x)) = 0 \text{ for } x < 0, \\ \Phi(-\infty) = \mathbf{u}^*, \quad \Phi(0) = \mathbf{0}, \end{cases} \tag{1.4}$$

and

$$c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_0^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x, \tag{1.5}$$

where we recall $D = (d_i)$, $\mathbf{J} = (J_i)$, $\Phi = (\phi_i)$ and "$\circ$" is the Hadamard product.

If (c, Φ) solves (1.4), we say that Φ is a semi-wave solution to (1.3) with speed c. This is not to be confused with the semi-wave to (1.1), for which the extra equation (1.5) should be satisfied, yielding a semi-wave solution of (1.3) with the desired speed $c = c_0$, which determines the spreading speed of (1.1).

We are interested in semi-waves which are monotone and with positive speed. The following condition on the kernel functions will be used:

$$(\mathbf{J_2}): \qquad \int_0^{\infty} e^{\lambda x}J(x)\mathrm{d}x < \infty \text{ for some } \lambda > 0 \text{ and every } J \in \{J_i : 1 \le i \le m_0\}.$$

Theorem A. *Suppose the kernel functions satisfy* $(\mathbf{J})$ *and* F *satisfies* $(\mathbf{f_1})$-$(\mathbf{f_4})$. *Then there exists* $C_* \in (0, +\infty]$ *such that*

(i) *for* $0 < c < C_*$, (1.4) *has a unique monotone solution* $\Phi^c = (\phi_i^c)$, *and*

$$\lim_{c \nearrow C_*} \Phi^c(x) = \mathbf{0} \text{ locally uniformly in } (-\infty, 0];$$

(ii) $C_* \ne \infty$ *if and only if* $(\mathbf{J_2})$ *holds;*

(iii) *the system* (1.4)-(1.5) *has a solution pair* (c, Φ) *with* $\Phi(x)$ *monotone if and only if* $(\mathbf{J_1})$ *holds, and when* $(\mathbf{J_1})$ *holds, there exists a unique* $c_0 \in (0, C_*)$ *such that* $(c, \Phi) = (c_0, \Phi^{c_0})$ *solves* (1.4) *and* (1.5).

The spreading speed of (1.1) is determined by the following result:

Theorem B. *Suppose the conditions in Theorem A are satisfied,* (U, g, h) *is the solution of* (1.1) *with* $U(0, x) \in [\mathbf{0}, \hat{\mathbf{u}}]$*, and spreading happens. Then the following conclusions hold for the spreading speed:*

(i) *If* $(\mathbf{J_1})$ *is satisfied, then the spreading speed is finite, and is determined by*

$$-\lim_{t\to\infty}\frac{g(t)}{t} = \lim_{t\to\infty}\frac{h(t)}{t} = c_0 \text{ with } c_0 \text{ given in Theorem A (iii).}$$

(ii) *If* $(\mathbf{J_1})$ *is not satisfied, then accelerated spreading happens, namely*

$$-\lim_{t\to\infty}\frac{g(t)}{t} = \lim_{t\to\infty}\frac{h(t)}{t} = \infty.$$

Note that for $(\mathbf{J_1})$ to be violated, it only requires a single J_i with $i \in A_+$ to satisfy $\int_0^\infty xJ_i(x)dx = \infty$.

1.2. Main results of this paper

The main purpose of this paper is to sharpen the conclusion in Theorem B (i) for some typical classes of kernel functions satisfying $(\mathbf{J_1})$. For the single species model (namely (1.1) with $m = m_0 = 1$), it was shown in [12] that the shift $c_0t - h(t)$ may stay bounded or go to infinity as $t \to \infty$, depending on the behaviour of the kernel function near infinity. We want to extend such estimate for a single equation in [12] to a rather general system of the form (1.1), where diffusion need not appear in every equation of the system.

Following [12], for $\alpha > 1$ and a continuous nonnegative even kernel function J, we introduce the condition

$$(\mathbf{J}^\alpha): \qquad \int_0^\infty x^{\alpha-1}J(x)dx < \infty.$$

Clearly $(\mathbf{J}^{\alpha_2})$ implies $(\mathbf{J}^{\alpha_1})$ if $\alpha_2 > \alpha_1 > 1$. Let us note that $(\mathbf{J_1})$ holds if and only if every J_i with $i \in A_+$ satisfies $(\mathbf{J^2})$. If $(\mathbf{J_2})$ holds, then $(\mathbf{J}^\alpha)$ is satisfied for all $\alpha > 1$ by every J_i $(i = 1, ..., m_0)$.

Theorem 1.1. *In Theorem B, suppose additionally that every* J_i *with* $i \in A_+$ *satisfies* $(\mathbf{J^3})$*, every* J_i *with* $i \in A_0$ *satisfies* $(\mathbf{J}^\alpha)$ *for some* $\alpha \in (2, 3)$*,* F *is* C^2 *and* $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \prec\!\!\prec \mathbf{0}$*. Then there exist positive constants* θ*,* C *and* t_0 *such that, for all* $t > t_0$ *and* $x \in [g(t), h(t)]$*,*

$$\begin{cases} |h(t) - c_0t| + |g(t) + c_0t| \le C, \\ U(t, x) \succeq [1 - \epsilon(t)]\big[\Phi^{c_0}(x - c_0t + C) + \Phi^{c_0}(-x - c_0t + C) - \mathbf{u}^*\big], \\ U(t, x) \preceq [1 + \epsilon(t)] \min\big\{\Phi^{c_0}(x - c_0t - C),\ \Phi^{c_0}(-x - c_0t - C)\big\}, \end{cases}$$

where $\epsilon(t) := (t + \theta)^{-\alpha}$*, and* (c_0, Φ^{c_0}) *is the unique pair solving* (1.4) *and* (1.5) *obtained in Theorem A* (iii)*, with* $\Phi^{c_0}(x)$ *extended by* $\mathbf{0}$ *for* $x > 0$.

Further estimates on $g(t)$ and $h(t)$ can be obtained for some slightly more restrictive classes of kernel functions. We will write

$$\eta(t) \approx \xi(t) \quad \text{if} \quad C_1\xi(t) \le \eta(t) \le C_2\xi(t)$$

for some positive constants $C_1 \le C_2$ and all t in a certain concerned range.

Our next result is about the situation that the kernel functions $\{J_i : i \in A_+\}$ have a dominating one J_{i^*}, by which we mean

$$J_i(x) \le C J_{i^*}(x) \text{ for some } C > 0 \text{ and all } i \in A_+, x \in \mathbb{R}.$$

For $\gamma > 0$ and a continuous nonnegative even kernel function J, we introduce the condition

$(\mathbf{J}^\gamma_\infty)$: $\quad J(x) \approx |x|^{-\gamma} \quad \text{for } |x| \gg 1.$

Note that for a kernel function satisfying $(\mathbf{J}^\gamma_\infty)$, it satisfies the condition in $(\mathbf{J})$ only if $\gamma > 1$, and it satisfies the condition in $(\mathbf{J_1})$ only if $\gamma > 2$.

For a kernel function satisfying $(\mathbf{J}^\gamma_\infty)$, clearly $(\mathbf{J}^\alpha)$ holds if and only if $\gamma > \alpha$. Therefore the case $\gamma > 3$ is covered by Theorem 1.1. When $\gamma \in (1, 2]$, $(\mathbf{J_1})$ does not hold and accelerated spreading may happen; the precise rate of acceleration for this case has been determined in [14]. The following theorem is concerned with the remaining case $\gamma \in (2, 3]$, which indicates that the result in Theorem 1.1 is sharp.

Theorem 1.2. *In Theorem B, suppose additionally the kernel functions $\{J_i : i \in A_+\}$ have a dominating one J_{i^*} which satisfies $(\mathbf{J}^\gamma_\infty)$ for some $\gamma \in (2, 3]$, every J_i with $i \in A_0$ satisfies $(\mathbf{J}^\alpha)$ for some $\alpha \ge \gamma - 1$, F is C^2 and*

$$F(v) - v[\nabla F(v)]^T \ggcurly \mathbf{0} \quad \text{for} \quad \mathbf{0} \llcurly v \preceq \mathbf{u}^*. \tag{1.6}$$

Then for $t \gg 1$,

$$c_0 t + g(t),\ c_0 t - h(t) \approx \begin{cases} \ln t & \text{if } \gamma = 3, \\ t^{3-\gamma} & \text{if } \gamma \in (2, 3). \end{cases}$$

Note that $(\mathbf{f_2})$ implies

$$F(v) - v[\nabla F(v)]^T \succeq \mathbf{0} \quad \text{for } v \in [\mathbf{0}, \mathbf{u}^*].$$

Therefore (1.6) is a strengthened version of $(\mathbf{f_2})$. If we take $v = \mathbf{u}^*$ in (1.6), then it yields $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \llcurly \mathbf{0}$. When $m = 1$, (1.6) reduces to $F(v) > F'(v)v$ for $0 < v \le \hat{u}$, which is satisfied, for example, by $F(v) = av - bv^p$ with $a, b > 0$ and $p > 1$.

The proofs of Theorems 1.1 and 1.2 rely on the following estimates for the semi-wave solutions of (1.3), which are also of independent interests.

Theorem 1.3. *Suppose that F satisfies $(\mathbf{f_1})$-$(\mathbf{f_4})$ and the kernel functions satisfy $(\mathbf{J})$, and $\Phi(x) = (\phi_i(x))$ is a monotone solution of (1.4) for some $c > 0$. Then the following conclusions hold:*

(i) *If $(\mathbf{J}^\alpha)$ is satisfied by every J_i $(i = 1, ..., m_0)$ for some $\alpha > 1$, then for every $i \in \{1, ..., m\}$,*

$$\int_{-\infty}^{-1} \big[u_i^* - \phi_i(x)\big] |x|^{\alpha-2} dx < \infty,$$

which implies, by the monotonicity of $\phi_i(x)$,

$$0 < u_i^* - \phi_i(x) \le C|x|^{1-\alpha} \text{ for some } C > 0 \text{ and all } x < -1.$$

(ii) *If* $(\mathbf{J}^\alpha)$ *is not satisfied by some* J_i *for some* $\alpha > 1$*, then*

$$\sum_{i=1}^{m} \int_{-\infty}^{-1} \big[u_i^* - \phi_i(x)\big]|x|^{\alpha-2}dx = \infty.$$

1.3. Applications to epidemic models

Let us now see how the results above can be applied to the models in [10] and [22].

The West Nile virus model in [10] is given by

$$\begin{cases} H_t = d_1\mathcal{L}_1[H](t,x) + a_1(e_1 - H)V - b_1 H, & x \in (g(t), h(t)),\ t > 0,\\ V_t = d_2\mathcal{L}_2[V](t,x) + a_1(e_2 - V)H - b_2 V, & x \in (g(t), h(t)),\ t > 0,\\ H(t,x) = V(t,x) = 0, & t > 0,\ x \in \{g(t), h(t)\},\\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x-y)V(t,x)\mathrm{d}y\mathrm{d}x, & t > 0,\\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x-y)V(t,x)\mathrm{d}y\mathrm{d}x, & t > 0,\\ -g(0) = h(0) = h_0,\ H(0,x) = u_1^0(x),\ V(0,x) = u_2^0(x), & x \in [-h_0, h_0]. \end{cases} \tag{1.7}$$

where a_i, e_i and b_i $(i = 1, 2)$ are positive constants satisfying $a_1a_2e_1e_2 > b_1b_2$ (which is necessary for spreading to happen). We thus have

$$F(u) = F_1(u) := \Big(a_1(e_1 - u_1)u_2 - b_1u_1, a_2(e_2 - u_2)u_1 - b_2u_2\Big),$$

$$\mathbf{u}^* = \left(\frac{a_1a_2e_1e_2 - b_1b_2}{a_1a_2e_2 + a_2b_1}, \frac{a_1a_2e_1e_2 - b_1b_2}{a_1a_2e_1 + a_1b_2}\right).$$

It is straightforward to check that conditions $(\mathbf{f_1})$-$(\mathbf{f_3})$ are satisfied by F_1 with $\hat{\mathbf{u}} = (e_1, e_2)$. Condition $(\mathbf{f_4})$ was shown to hold in [10]. It is easy to see that F_1 is C^2 and

$$F_1(u) - u[\nabla F_1(u)]^T = (a_1u_1u_2, a_2u_1u_2).$$

Therefore (1.6) holds. Thus all our results apply to (1.7).

The epidemic model in [22] is given by

$$\begin{cases} u_t = d_1\mathcal{L}_1[u] - au + H(v), & t > 0,\ x \in (g(t), h(t)),\\ v_t = d_2\mathcal{L}_1[v] - bv + G(u), & t > 0,\ x \in (g(t), h(t)),\\ u(t,x) = v(t,x) = 0, & t > 0,\ x = g(t) \text{ or } x = h(t),\\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} [J_1(x-y)u(t,x) + \rho J_2(x-y)v(t,x)]dydx, & t > 0,\\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} [J_1(x-y)u(t,x) + \rho J_2(x-y)v(t,x)]dydx, & t > 0,\\ -g(0) = h(0) = h_0,\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in [-h_0, h_0], \end{cases} \tag{1.8}$$

where a, b, d_1, d_2, μ, ρ and h_0 are positive constants, and the functions G and H satisfy

(i) $G, H \in C^2([0,\infty))$, $G(0) = H(0) = 0$, $G'(z), H'(z) > 0$, $G''(z), H''(z) < 0$ for $z \geq 0$;

(ii) there exists $\bar{z} > 0$ such that $G(H(\bar{z})/a) < b\bar{z}$.

In this example,

$$F(u) = F_2(u) := (H(u_2) - au_1, G(u_1) - bu_2), \quad \mathbf{u}^* = (K_1, K_2)$$

where $(K_1, K_2) \gg \mathbf{0}$ are uniquely determined by

$$K_1 = \frac{H(K_2)}{a}, \quad K_2 = \frac{G(H(K_2)/a)}{b}.$$

One easily checks that F_2 satisfies $(\mathbf{f_1}) - (\mathbf{f_3})$ with $\hat{\mathbf{u}} = (K_1, K_2)$. In [22], it was proved that $(\mathbf{f_4})$ also holds. Clearly F_2 is C^2 and

$$F_2(u) - u[\nabla F_2(u)]^T = (H(u_2) - u_2 H'(u_2), G(u_1) - u_1 G'(u_1)).$$

Hence (1.6) holds. Therefore all our results apply to (1.8). The above analysis indicates that our results in this paper also apply to (1.8) in the degenerate case $d_2 = \rho = 0$, which is a slight variation of the model in [38].

1.4. Related works

Some variations of the model (1.8) have been studied recently in [6,13,28,29], where the reaction term in the equation for v also contains a nonlocal term. The results obtained in this paper do not apply to these problems due to the variations, and vice versa.

Several local (random) diffusion versions of (1.1) or its variations have been extensively studied in the past decade, starting from the work [7]. In these local diffusion free boundary problems the spreading speed is always finite, and is determined by the associated semi-waves; see, for example, [1,7–9,15,19,21,30,31,37] as a small sample. Note, however, that for systems of equations with free boundaries, no sharp estimate for the shift is available except [31], where for the West Nile virus model, convergence of the shift was proved.

A striking difference of nonlocal diffusion models of the form (1.1) to their local diffusion counterparts is that accelerated spreading may occur. For the scalar case of (1.3), namely for the Fisher-KPP equation with nonlocal diffusion, it follows from the theory in [32] that accelerated spreading occurs exactly when the kernel function does not satisfy $(\mathbf{J_2})$ described above. On the other hand, when $(\mathbf{J_2})$ is satisfied by the kernel function (thin-tailed kernel) then there exists some $c_* > 0$ such that the associated traveling wave problem has a monotone traveling wave with speed c if and only if $c \geq c_*$, and c_* is the asymptotic spreading speed determined by the scalar nonlocal Fisher-KPP equation (1.3); see, for example, [20,24–26,32,36]. Related works on accelerated spreading can be found in [2,3,16–18,23,34,35] and the references therein. It is well known that the local diffusion version of (1.3) with compactly supported initial functions can only spread with finite speed, which is determined by the associated traveling waves [20,32,33,39].

As already mentioned in Part 1 [11], there are two fundamental differences between the free boundary model (1.1) and the corresponding model (1.3) where no free boundary appears. Firstly (1.1) provides the exact location of the spreading front, which is the free boundary, while the location of the front is not prescribed in (1.3) – one usually uses suitable level sets of the solution to describe the front behaviour. Secondly, the long time dynamical behaviour of (1.1) is often governed by a spreading-vanishing dichotomy [4,10,38], but (1.3) predicts successful spreading all the time. Let us also note that since $(\mathbf{J_2})$ implies $(\mathbf{J_1})$ but not the other way round, (1.3) is more readily than (1.1) to give rise to accelerated spreading.

1.5. Organisation of the paper

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.3, which will be used in Section 3 for the proof of Theorem 1.1. Sections 4 is devoted to the proof of Theorems 1.2, where the dominating kernel function behaves like $|x|^{-\gamma}$ near infinity, and we determine the exact growth rates of $c_0t - h(t)$ and $c_0t + g(t)$ for γ in the range $(2, 3]$. Note that when $\gamma > 3$, the spreading behaviour is already covered by Theorem 1.1 proved in Section 3.

2. Asymptotic behaviour of semi-wave solutions

The purpose of this section is to prove the following two theorems, which imply Theorem 1.3.

Theorem 2.1. *Suppose that F satisfies $(\mathbf{f_1})$-$(\mathbf{f_4})$ and the kernel functions J_i $(i = 1, ..., m_0)$ satisfy $(\mathbf{J})$ and $(\mathbf{J}^\alpha)$ for some $\alpha > 1$. If $\Phi(x) = (\phi_i(x))$ is a monotone solution of (1.4) for some $c > 0$, then for every $i \in \{1, ..., m\}$,*

$$\int_{-\infty}^{-1} [u_i^* - \phi_i(x)]|x|^{\alpha-2}dx < \infty,$$

which implies, by the monotonicity of $\Phi(x)$,

$$0 < u_i^* - \phi_i(x) \le C|x|^{1-\alpha} \text{ for some } C > 0 \text{ and all } x < 0,\ \ i \in \{1, ..., m\}.$$

The next result shows that this estimate is sharp.

Theorem 2.2. *Suppose that F satisfies $(\mathbf{f_1})$-$(\mathbf{f_4})$ and the kernel functions satisfy $(\mathbf{J})$. If $(\mathbf{J}^\alpha)$ is not satisfied for some $\alpha > 1$ by some J_i, and $\Phi(x) = (\phi_i(x))$ is a monotone solution of (1.4) for some $c > 0$, then*

$$\sum_{i=1}^{m} \int_{-\infty}^{-1} [u_i^* - \phi_i(x)]|x|^{\alpha-2}dx = \infty. \tag{2.1}$$

The proof of Theorem 2.1 is based on the following two lemmas, with the first taken from [11].

Lemma 2.1 (Lemma 2.1, [11]). *If $(\mathbf{f_1})$ holds, then there exist $\lambda_1 > 0$, small $\epsilon > 0$, and vectors $\Theta = (\theta_i) \gg \mathbf{0}$, $\tilde{\Theta} = (\tilde{\theta}_i) \gg \mathbf{0}$ such that*

$$\Theta \nabla F(\mathbf{0})^T = \lambda_1 \Theta, \quad \tilde{\Theta} \nabla F(\mathbf{0}) = \lambda_1 \tilde{\Theta}, \tag{2.2}$$

and

$$\begin{cases} f_i(\epsilon\Theta) \ge \epsilon\hat{\sigma} \sum_{j=1}^m \theta_j \text{ for } i = 1, ..., m, \\ \sum_{i=1}^m \tilde{\theta}_i f_i(X) \ge \sum_{i=1}^m b_i x_i \text{ for } X = (x_i) \in [\mathbf{0}, \epsilon\mathbf{1}], \end{cases} \tag{2.3}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$ *and*

$$\hat{\sigma} := \frac{\lambda_1}{2} \min_{1 \le i \le m} \theta_i \Big/ \sum_{j=1}^{m} \theta_i, \; b_i := \frac{\lambda_1 \tilde{\theta}_i}{2} > 0.$$

Proof. This is Lemma 2.1 in [11]. We reproduce its short proof below for convenience of later use.

Let λ_1 be the principal eigenvalue of $\nabla F(\mathbf{0})$. By the Perron-Frobenius theorem, there exist positive eigenvectors Θ and $\tilde{\Theta}$ such that the identities in (2.2) hold.

Moreover, in view of $F \in [C^1(\mathbb{R}^m_+)]^m$, for small $\epsilon > 0$ and $X = (x_i) \in [\mathbf{0}, \epsilon \mathbf{1}]$,

$$F(\epsilon\Theta) = \epsilon\Theta\big[\nabla F(\mathbf{0})^T + o(1)\mathbf{I}_m\big] = \epsilon[\lambda_1 + o(1)]\Theta,$$
$$\sum_{i=1}^{m} \tilde{\theta}_i f_i(X) = \tilde{\Theta}\big[\nabla F(\mathbf{0}) + o_\epsilon(X)\big]X^T = \tilde{\Theta}[\lambda_1 \mathbf{I}_m + o_\epsilon(X)]X^T,$$

with $|o_\epsilon(X)| \to 0$ as $\epsilon \to 0$ uniformly in $X \in [\mathbf{0}, \epsilon\mathbf{1}]$. Hence (2.3) holds provided that $\epsilon > 0$ is small enough. □

Denote

$$\Psi(x) = (\psi_i(x)) := \mathbf{u}^* - \Phi(x) \text{ and } G(u) = (g_i(u)) := -F(\mathbf{u}^* - u).$$

Then Ψ satisfies

$$\begin{cases} \mathbf{0} = D \circ \displaystyle\int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Psi(y)\mathrm{d}y - D \circ \Psi + D \circ \mathbf{u}^* \circ \int_{0}^{\infty} \mathbf{J}(x-y)\mathrm{d}y \\ \qquad\qquad\qquad\qquad + c\Psi'(x) + G(\Psi(x)) \quad \text{for } -\infty < x < 0, \\ \Psi(-\infty) = \mathbf{0}, \quad \Psi(0) = \mathbf{u}^*. \end{cases} \tag{2.4}$$

Since $\mathbf{u}^*$ is stable and $\nabla F(\mathbf{u}^*) = \nabla G(\mathbf{0})$ is invertible, the eigenvalues of $\nabla F(\mathbf{u}^*)$ are all negative. Therefore we can use the same reasoning as in the proof of Lemma 2.1 to find two vectors $\widetilde{A} = (\tilde{a}_i) \succ\succ \mathbf{0}$ and $\widetilde{B} = (\tilde{b}_i) \prec\prec \mathbf{0}$ such that, for $U = (u_i) \in [\mathbf{0}, \epsilon\mathbf{1}]$ with $\epsilon > 0$ sufficiently small,

$$\sum_{i=1}^{m} \tilde{a}_i g_i(U) \le \sum_{i=1}^{m} \tilde{b}_i u_i \le -\hat{b} \sum_{j=1}^{m} \tilde{a}_i u_i,$$

for some $\hat{b} > 0$.

Since $\Psi(-\infty) = \mathbf{0}$ and $\Psi(x) = (\psi_i(x)) \succ\succ \mathbf{0}$ for $x < 0$, we have $0 < \psi_i(x) < \epsilon$ for large negative x (denoted by $x \ll -1$), and so

$$\sum_{i=1}^{m} \tilde{a}_i g_i(\Psi(x)) \le -\hat{b}\widetilde{\psi}(x) \quad \text{for } x \ll -1, \text{ with} \tag{2.5}$$

$$\widetilde{\psi}(x) := \sum_{j=1}^{m} \tilde{a}_j \psi_j(x). \tag{2.6}$$

Lemma 2.2. *Suppose* $(\mathbf{J})$ *and* $(\mathbf{f_1})$-$(\mathbf{f_4})$ *are satisfied. If* $(\mathbf{J}^\alpha)$ *is satisfied by every* J_i $(i = 1, ..., m_0)$ *for some* $\alpha \ge 1$*, then*

$$\int_{-\infty}^{0} \widetilde{\psi}(x)\mathrm{d}x < \infty.$$

Proof. A simple calculation gives

$$\begin{aligned}&D\circ\int_{-\infty}^{0}\mathbf{J}(x-y)\circ\Psi(y)\mathrm{d}y-D\circ\Psi+D\circ\mathbf{u}^{*}\circ\int_{0}^{\infty}\mathbf{J}(x-y)\mathrm{d}y\\=&-D\circ\int_{-\infty}^{0}\mathbf{J}(x-y)\circ\Phi(y)\mathrm{d}y+D\circ\Phi.\end{aligned}$$

Integrating the equation satisfied by $\widetilde{\psi}$ over the interval (x,y) with $x<y\ll -1$, and making use of (2.5), we obtain

$$\begin{aligned}&c(\widetilde{\psi}(y)-\widetilde{\psi}(x))+\sum_{i=1}^{m}\int_{x}^{y}\tilde{a}_i d_i\left[\int_{-\infty}^{0}J_i(z-w)\psi_i(w)\mathrm{d}w-\psi_i(z)\right]\mathrm{d}z\\&+\sum_{i=1}^{m}\int_{x}^{y}\tilde{a}_i d_i u_i^{*}\int_{0}^{\infty}J_i(z-w)\mathrm{d}w\mathrm{d}z\\=&c(\widetilde{\psi}(y)-\widetilde{\psi}(x))-\sum_{i=1}^{m}\int_{x}^{y}\tilde{a}_i d_i\left[\int_{-\infty}^{0}J_i(z-w)\phi_i(w)\mathrm{d}w-\phi_i(z)\right]\mathrm{d}z\\=&-\int_{x}^{y}\sum_{i=1}^{m}\tilde{a}_i g_i(\Psi(z))\mathrm{d}z\geq\hat{b}\int_{x}^{y}\widetilde{\psi}(z)\mathrm{d}z.\end{aligned}$$

We extend Φ to $\mathbb{R}$ by define $\phi_i(x)=0$ for $x>0$. Then the new function Φ is differentiable on $\mathbb{R}$ except at $x=0$. Due to $(\mathbf{J}^{\alpha})$, we have, for $i\in\{1,...,m_0\}$,

$$\begin{aligned}&\left|\int_{x}^{y}\left(\int_{-\infty}^{0}J_i(z-w)\phi_i(w)\mathrm{d}w-\phi_i(z)\right)\mathrm{d}z\right|\\=&\left|\int_{x}^{y}\left(\int_{\mathbb{R}}J_i(z-w)\phi_i(w)\mathrm{d}w-\phi_i(z)\right)\mathrm{d}z\right|\\=&\left|\int_{x}^{y}\int_{\mathbb{R}}J_i(w)(\phi_i(z+w)-\phi(z))\mathrm{d}w\mathrm{d}z\right|=\left|\int_{x}^{y}\int_{\mathbb{R}}J_i(w)\int_{0}^{1}w\phi_i'(z+sw)\mathrm{d}s\mathrm{d}w\mathrm{d}z\right|\\=&\left|\int_{\mathbb{R}}wJ_i(w)\int_{0}^{1}[\phi_i(y+sw)-\phi_i(x+sw)]\mathrm{d}s\mathrm{d}w\right|\leq a_i^{*}\int_{\mathbb{R}}|y|J_i(y)\mathrm{d}y=:M_i<\infty.\end{aligned}$$

Thus, for $x<y\ll -1$,

$$\hat{b}\int_{x}^{y}\widetilde{\psi}(z)\mathrm{d}z\leq c(\widetilde{\psi}(y)-\widetilde{\psi}(x))+\sum_{i=1}^{m}\tilde{a}_i d_i M_i\leq\sum_{i=1}^{m}\tilde{a}_i(cu_i^{*}+d_iM_i),$$

which implies $\int_{-\infty}^{0}\widetilde{\psi}(z)\mathrm{d}z<\infty$. □

Proof of Theorem 2.1.

Case 1. $\alpha\geq 2$.

With $\widetilde{\psi}=\sum_{i=1}^{m}\tilde{a}_i\psi_i$ given by (2.6), it suffices to show

$$\int_{-\infty}^{0}\widetilde{\psi}(x)|x|^{\alpha-2}\mathrm{d}x<\infty.$$

By Lemma 2.2 we have

$$\int_{-\infty}^{0}\tilde{\psi}(x)dx<\infty\text{ and hence }\int_{-\infty}^{0}\psi_i(x)dx<\infty\text{ for }i\in\{1,...,m\}.$$

So there is nothing to prove if $\alpha = 2$, and we only need to consider the case $\alpha > 2$.

Suppose $\alpha > 2$ and

$$\int_{-\infty}^{0} |x|^{\gamma} \tilde{\psi}(x) dx < \infty \text{ for some } \gamma \geq 0. \tag{2.7}$$

Then by Lemma 2.3 in [12], for any β satisfying $0 < \beta \leq \min\{\gamma + 1, \alpha - 1\}$, and $i \in \{1, ..., m_0\}$,

$$\int_{-M}^{0} \left[\int_{-\infty}^{0} J_i(x-y) \psi_i(y) \mathrm{d}y - \psi_i(x) \right] |x|^{\beta} \mathrm{d}x \leq C \tag{2.8}$$

for some $C > 0$ and all $M > 0$.

Moreover, if we fix $M_0 > 1$ so that (2.5) holds for $x \leq -M_0$, then for $M > M_0$ and β as above, we have

$$\begin{aligned} &\hat{b} \int_{-M}^{-M_0} \tilde{\psi}(x)|x|^{\beta} dx \leq -\sum_{i=1}^{m} \int_{-M}^{-M_0} \tilde{a}_i g_i(\Psi(x))|x|^{\beta} dx \\ &= c \int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta} dx + \sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x) \right] |x|^{\beta} \mathrm{d}x \\ &\quad + \sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J_i(x-y) \mathrm{d}y \mathrm{d}x. \end{aligned}$$

By (2.8),

$$\begin{aligned} &\sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x) \right] |x|^{\beta} \mathrm{d}x \\ &\leq C \sum_{i=1}^{m_0} \tilde{a}_i d_i - \sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M_0}^{0} \left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x) \right] |x|^{\beta} \mathrm{d}x \\ &:= C_1 < \infty \text{ for all } M > M_0. \end{aligned}$$

Moreover, if we assume additionally that $\beta \leq \alpha - 2$, then we have, for $i \in \{1, ..., m_0\}$,

$$\begin{aligned} &\int_{-M}^{-M_0} \int_{0}^{\infty} |x|^{\beta} J_i(x-y) \mathrm{d}y \mathrm{d}x \\ &\leq \int_{0}^{M} \int_{0}^{\infty} x^{\beta} J_i(x+y) dy dx = \int_{0}^{M} \int_{x}^{\infty} x^{\beta} J_i(y) dy dx \\ &\leq \int_{0}^{\infty} \int_{x}^{\infty} x^{\beta} J_i(y) dy dx = \frac{1}{\beta+1} \int_{0}^{\infty} y^{\beta+1} J_i(y) dy := C_2 < \infty. \end{aligned}$$

Therefore, for $\beta \in (0, \min\{\gamma + 1, \alpha - 2\}]$ and $M > M_0$,

$$\begin{aligned} &\hat{b} \int_{-M}^{-M_0} \tilde{\psi}(x)|x|^{\beta} dx \leq \ c \int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta} dx + C_1 + \sum_{i=1}^{m} \tilde{a}_i d_i u_i^* C_2 \\ &\leq c \int_{1}^{M} x^{\beta} \tilde{\psi}'(-x) dx + C_3 \leq c \int_{1}^{M} x^{\gamma+1} \tilde{\psi}'(-x) dx + C_3 \end{aligned}$$

$$\leq c\tilde{\psi}(-1) + c\int_1^M (\gamma+1)x^\gamma \tilde{\psi}(-x)dx + C_3 := C_4 < \infty \text{ by (2.7).}$$

It follows that

$$\int_{-\infty}^0 \tilde{\psi}(x)|x|^\beta dx < \infty. \tag{2.9}$$

Thus we have proved that (2.7) implies (2.9) for every $\beta \in (0, \min\{\gamma+1, \alpha-2\}]$.

If we write $\alpha - 2 = n + \theta$ with $n \geq 0$ an integer and $\theta \in (0,1]$. Then by the above conclusion and an induction argument we see that (2.9) holds with $\beta = n$. Thus (2.7) holds for $\gamma = n$. So applying the above conclusion once more we see that (2.9) holds for every $\beta \in (0, \min\{n+1, \alpha-2\}] = (0, \alpha-2]$, as desired.

Case 2. $\alpha \in (1,2)$.

Let $\beta = \alpha - 2$. As in Case 1, for $M > M_0$,

$$\begin{aligned}
&\hat{b}\int_{-M}^{-M_0} \tilde{\psi}(x)|x|^\beta dx \\
&\leq c\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^\beta dx + \sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[\int_{-\infty}^0 J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right] |x|^\beta \mathrm{d}x \\
&\quad + \sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x \\
&\leq c\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^\beta dx + \widetilde{C}_1 + \sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x,
\end{aligned}$$

where $\widetilde{C}_1 > 0$ is obtained by observing $\beta \leq \alpha - 1$ and making use of Lemma 2.4 in [12]. By $(\mathbf{J}^\alpha)$ and $\beta + 2 = \alpha$,

$$\begin{aligned}
\int_{-M}^{-M_0}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x &\leq \int_0^\infty\int_x^\infty x^\beta J_i(y)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\alpha}\int_0^\infty y^{\alpha-1}J_i(y)\mathrm{d}y := \widetilde{C}_2 < \infty.
\end{aligned}$$

Due to $\beta < 0$, we have

$$\begin{aligned}
&\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^\beta dx = \int_{M_0}^M \tilde{\psi}'(-x)x^\beta dx \\
&= \tilde{\psi}(-M_0)M_0^\beta - \tilde{\psi}(-M)M^\beta + \beta\int_{M_0}^M \tilde{\psi}(-x)x^{\beta-1}dx \leq \tilde{\psi}(-M_0)M_0^\beta := \widetilde{C}_3 < \infty.
\end{aligned}$$

Hence

$$\hat{b}\int_{-M}^{-M_0} \tilde{\psi}(x)|x|^\beta dx \leq \widetilde{C}_1 + \widetilde{C}_2\sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* + c\widetilde{C}_3 < \infty$$

for all $M > M_0$, which implies

$$\int_{-\infty}^{-1} \tilde{\psi}(x)|x|^{\alpha-2}dx < \infty.$$

The proof is completed. □

Proof of Theorem 2.2.

We have

$$|g_i(\Psi(x))| \le L\sum_{j=1}^m \psi_j(x) := L\hat\psi(x) \text{ for some } L>0 \text{ and all } x<0,\ i\in\{1,...,m\}.$$

Now for $M>1$ and $\beta=\alpha-2$,

$$\begin{aligned}
L\int_{-M}^{-1}\hat\psi(x)|x|^\beta dx &\ge -\sum_{i=1}^m\int_{-M}^{-1} g_i(\Psi(x))|x|^\beta dx\\
&= c\int_{-M}^{-1}\hat\psi'(x)|x|^\beta dx + \sum_{i=1}^{m_0} d_i\int_{-M}^{-1}\Big[\int_{-\infty}^0 J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\Big]|x|^\beta \mathrm{d}x\\
&\quad + \sum_{i=1}^{m_0} d_i u_i^*\int_{-M}^{-1}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&\ge -\sum_{i=1}^{m_0} d_i\int_{-M}^{-1}\psi_i(x)|x|^\beta\mathrm{d}x + \sum_{i=1}^{m_0} d_i u_i^*\int_{-M}^{-1}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x
\end{aligned}$$

Therefore, with $\widetilde L := L+\sum_{i=1}^{m_0} d_i$, we have

$$\begin{aligned}
\widetilde L\int_{-M}^{-1}\hat\psi(x)|x|^\beta dx &\ge \sum_{i=1}^{m_0} d_i u_i^*\int_{-M}^{-1}\int_0^\infty |x|^\beta J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&= \sum_{i=1}^{m_0} d_i u_i^*\int_1^M\int_x^\infty x^\beta J_i(y)dydx\\
&= \sum_{i=1}^{m_0} d_i u_i^*\Big[\int_1^M\int_1^\infty - \int_1^M\int_1^x\Big]x^\beta J_i(y)dydx\\
&= \sum_{i=1}^{m_0} \frac{d_i u_i^*}{\beta+1}\Big[\int_1^\infty (M^{\beta+1}-1)J_i(y)dy + \int_1^M (y^{\beta+1}-M^{\beta+1})J_i(y)dy\Big]\\
&\ge \sum_{i=1}^{m_0} \frac{d_i u_i^*}{\beta+1}\Big[\int_1^M y^{\beta+1}J_i(y)dy - \int_1^\infty J_i(y)dy\Big] \to\infty \text{ as } M\to\infty,
\end{aligned}$$

since $\beta+2=\alpha$. Therefore (2.1) holds, as we wanted. □

3. Bounds for $c_0t-h(t)$, $c_0t+g(t)$ and $U(t,x)$ for kernels of type $(\mathbf{J}^\alpha)$

Let us first observe that it suffices to estimate $h(t)-c_0t$, since that for $g(t)+c_0t$ follows by considering (1.1) with initial function $u_0(-x)$.

Theorem 1.1 will follow easily from Lemmas 3.1, 3.3 below and their proofs, where more general and stronger conclusions are proved.

3.1. Bound from below

Lemma 3.1. *In Theorem B, if additionally* $(\mathbf{J_1})$ *holds and the kernel functions* J_i $(i=1,...,m_0)$ *satisfy* $(\mathbf{J}^\alpha)$ *for some* $\alpha>1$, *F is* C^2 *and* $\mathbf{u}^*\nabla F(\mathbf{u}^*)\lll\mathbf{0}$, *then* *there exists $C>0$ such that for $t\geq 0$,*

$$h(t)-c_0t\geq -C\left[1+\int_0^t(1+x)^{1-\alpha}dx+\int_0^{\frac{c_0}{2}t}x^2\hat{J}(x)dx+t\int_{\frac{c_0}{2}t}^{\infty}x\hat{J}(x)dx\right],$$

where $c_0>0$ is given in Theorem A and $\hat{J}(x):=\sum_{i=1}^{m_0}\mu_iJ_i(x)$.

To prove Lemma 3.1, we will need the following result.

Lemma 3.2. *Suppose that $F=(f_i)\in C^2(\mathbb{R}^m,\mathbb{R}^m)$, $\mathbf{u}^*\succ\!\succ 0$ and*

$$F(\mathbf{u}^*)=\mathbf{0},\ \ \mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T\prec\!\prec\mathbf{0}.$$

Then there exists $\delta_0>0$ small such that for $0<\epsilon\ll 1$ and $u,v\in[(1-\delta_0]\mathbf{u}^*,\mathbf{u}^*]$ satisfying

$$(u_i^*-u_i)(u_j^*-v_j)\leq C\delta_0\epsilon \text{ for some } C>0 \text{ and all } i,j\in\{1,...,m\},$$

we have

$$(1-\epsilon)[F(u)+F(v)]-F((1-\epsilon)(u+v-\mathbf{u}^*))\preceq\frac{\epsilon}{2}\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T.$$

Proof. Define

$$G(u,v)=(g_i(u,v)):=(1-\epsilon)[F(u)+F(v)]-F((1-\epsilon)(u+v-\mathbf{u}^*)),\quad u,v\in\mathbb{R}^m.$$

For $u,\ v\in[(1-\delta_0)\mathbf{u}^*,\mathbf{u}^*]$ and each $i\in\{1,...,m\}$, we may apply the mean value theorem to the function

$$\xi_i(t):=g_i(\mathbf{u}^*+t(u-\mathbf{u}^*),\mathbf{u}^*+t(v-\mathbf{u}^*)$$

to obtain

$$\xi_i(1)=\xi_i(0)+\xi_i'(\zeta_i) \text{ for some } \zeta_i\in[0,1].$$

Denote

$$\tilde{u}=\tilde{u}^i:=\mathbf{u}^*+\zeta_i(u-\mathbf{u}^*),\quad \tilde{v}=\tilde{v}^i:=\mathbf{u}^*+\zeta_i(v-\mathbf{u}^*).$$

Then the above identity is equivalent to

$$\begin{aligned}g_i(u,v)=&g_i(\mathbf{u}^*,\mathbf{u}^*)+\nabla_u\,g_i(\tilde{u},\tilde{v})\cdot(u-\mathbf{u}^*)+\nabla_v\,g_i(\tilde{u},\tilde{v})\cdot(v-\mathbf{u}^*)\\ =&-f_i((1-\epsilon)\mathbf{u}^*)+(1-\epsilon)\nabla f_i(\tilde{u})\cdot(u-\mathbf{u}^*)+(1-\epsilon)\nabla f_i(\tilde{v})\cdot(v-\mathbf{u}^*)\\ &-(1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde{u}+\tilde{v}-\mathbf{u}^*)\big)\cdot(u-\mathbf{u}^*)\\ &-(1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde{u}+\tilde{v}-\mathbf{u}^*)\big)\cdot(v-\mathbf{u}^*).\end{aligned}$$

Let us note that $\tilde{u}\in[u,\mathbf{u}^*]$ and $\tilde{v}\in[v,\mathbf{u}^*]$. Since $F\in C^2$, there is C_1 such that

$$|\partial_{jk}f_i(u)|\leq C_1\quad\text{for}\quad u\in[\mathbf{0},\mathbf{u}^*],\ i,j,k\in\{1,...,m\}.$$

A simple calculation gives

$$\begin{aligned}&(1-\epsilon)\nabla f_i(\tilde{u})(u-\mathbf{u}^*)-(1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde{u}+\tilde{v}-\mathbf{u}^*)\big)\cdot(u-\mathbf{u}^*)\\ =&(1-\epsilon)\big[\nabla f_i(\tilde{u})-\nabla f_i\big((1-\epsilon)(\tilde{u}+\tilde{v}-\mathbf{u}^*)\big)\big]\cdot(u-\mathbf{u}^*)\leq(1-\epsilon)b_1\sum_{j=1}^m(u_j^*-u_j),\end{aligned}$$

where

$$\begin{aligned}b_1 &:= C_1|\tilde{u} - (1-\epsilon)(\tilde{u} + \tilde{v} - \mathbf{u}^*)| \\ &= C_1|\epsilon\tilde{u} - (1-\epsilon)(\tilde{v} - \mathbf{u}^*)| \le C_1 \sum_{j=1}^{m} [\epsilon\tilde{u}_j + (1-\epsilon)(u_j^* - \tilde{v}_j)] \\ &\le C_2\epsilon + C_1 \sum_{j=1}^{m} (u_j^* - v_j) \text{ with } C_2 := C_1 \textstyle\sum_{j=1}^{m} u_j^*.\end{aligned}$$

Similarly,

$$\begin{aligned}&(1-\epsilon)\nabla f_i(\tilde{u}) \cdot (v - \mathbf{u}^*) - (1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde{u} + \tilde{v} - \mathbf{u}^*)\big) \cdot (v - \mathbf{u}^*) \\ &\le (1-\epsilon) b_2 \sum_{j=1}^{m} (u_j^* - v_j),\end{aligned}$$

where

$$b_2 := C_1|\epsilon\tilde{u} - (1-\epsilon)(\tilde{u} - \mathbf{u}^*)| \le \ C_2\epsilon + C_1 \sum_{j=1}^{m} (u_j^* - u_j).$$

Thus

$$\begin{aligned}g_i(u,v) \le& - f_i((1-\epsilon)\mathbf{u}^*) + (1-\epsilon) b_1 \sum_{j=1}^{m} (u_j^* - v_j) + (1-\epsilon) b_2 \sum_{j=1}^{m} (u_j^* - u_j) \\ \le& - f_i((1-\epsilon)\mathbf{u}^*) + \Big[C_2\epsilon + C_1 \sum_{j=1}^{m} (u_j^* - v_j)\Big] \sum_{k=1}^{m} (u_k^* - u_k) \\ &+ \Big[C_2\epsilon + C_1 \sum_{j=1}^{m} (u_j^* - u_j)\Big] \sum_{k=1}^{m} (u_k^* - v_k) \\ =& - f_i((1-\epsilon)\mathbf{u}^*) + C_2\epsilon \sum_{k=1}^{m} \Big[(u_k^* - u_k) + (u_k^* - v_k)\Big] \\ &+ C_1 \sum_{j,k=1}^{m} (u_j^* - v_j)(u_k^* - u_k) + C_1 \sum_{j,k=1}^{m} (u_j^* - u_j)(u_k^* - v_k) \\ =& \ \epsilon \nabla f_i(\mathbf{u}^*) \cdot \mathbf{u}^* + o(\epsilon) + C_2\epsilon \sum_{k=1}^{m} \Big[(u_k^* - u_k) + (u_k^* - v_k)\Big] \\ &+ 2C_1 \sum_{j,k=1}^{m} (u_j^* - v_j)(u_k^* - u_k),\end{aligned}$$

where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.

If $u, v \in [(1-\delta_0)\mathbf{u}^*, \mathbf{u}^*]$, then

$$P = (p_i) := \mathbf{u}^* - u, \quad Q = (q_i) := \mathbf{u}^* - v \in [\mathbf{0}, \delta_0 \mathbf{u}^*], \tag{3.1}$$

and hence

$$g_i(u,v) = g_i(\mathbf{u}^* - P, \mathbf{u}^* - Q)$$

$$\le \epsilon \nabla f_i(\mathbf{u}^*)\cdot \mathbf{u}^* + o(\epsilon) + C_2\epsilon \sum_{k=1}^{m}(p_k+q_k) + 2C_1 \sum_{j,k=1}^{m} p_j q_k$$
$$\le \epsilon\Big[\mathbf{u}^*\cdot \nabla f_i(\mathbf{u}^*) + o(1) + 2(C_2+C_1)\delta_0\Big]$$
$$\le \frac{\epsilon}{2}\mathbf{u}^*\cdot \nabla f_i(\mathbf{u}^*) \quad \text{for} \quad i\in\{1,...,m\},\ 0<\epsilon\ll 1$$

provided that $\delta_0 > 0$ is sufficiently small. □

Proof of Lemma 3.1. Let (c_0, Φ^{c_0}) be the unique solution pair of (1.4)-(1.5) in Theorem A. To simplify notations we write $\Phi^{c_0}(x) = \Phi(x) = (\phi_i(x))$. By Theorem 2.1 and $(\mathbf{J}^\alpha)$, there is $C > 0$ such that for $x < -1,\ i \in \{1, ..., m\}$,

$$\sum_{j=1}^{m_0}\int_0^\infty J_j(y)|y|^{\alpha-1}\mathrm{d}y \le C, \quad 0 < u_i^* - \phi_i(x) \le \frac{C}{x^{\alpha-1}}. \tag{3.2}$$

Define

$$\begin{cases} \underline{h}(t) := c_0 t + \delta(t), \quad t\ge 0,\\ \underline{U}(t,x) := (1-\epsilon(t))[\Phi(x-\underline{h}(t)) + \Phi(-x-\underline{h}(t)) - \mathbf{u}^*], \quad t\ge 0, x\in[-\underline{h}(t), \underline{h}(t)],\end{cases}$$

where $\epsilon(t) := (t+\theta)^{1-\alpha}$ and

$$\delta(t) := K_1 - K_2\int_0^t \epsilon(\tau)\mathrm{d}\tau - 2\sum_{i=1}^{m_0}\mu_i u_i^* \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)} \int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau,$$

with θ, K_1 and K_2 large positive constants to be determined.

For any $M > 0$ and $i \in A_+$,

$$\int_{-\infty}^{-M}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x = \int_M^\infty \int_x^\infty J_i(y)\mathrm{d}y\mathrm{d}x$$
$$= \int_M^\infty\int_M^y J_i(y)\mathrm{d}x\mathrm{d}y = \int_M^\infty (y-M)J_i(y)\mathrm{d}y \le \int_M^\infty yJ_i(y)\mathrm{d}y.$$

Hence, due to $\int_0^\infty yJ_i(y)\mathrm{d}y < \infty$ for $i\in A_+$, we have

$$2\sum_{i=1}^{m_0}\mu_i u_i^*\int_0^t\int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau$$
$$\le 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_0^t\int_{-\infty}^{-\frac{c_0}{2}\theta}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau \le \Big[2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{\frac{c_0}{2}\theta}^\infty yJ_i(y)\mathrm{d}y\Big]t \le \frac{c_0}{4}t$$

provided that $\theta > 0$ is large enough, say $\theta \ge \theta_0$.

For any given small $\epsilon_0 > 0$, due to $\Phi(-\infty) = \mathbf{u}^*$ there is $K_0 = K_0(\epsilon_0) > 0$ such that

$$(1-\epsilon_0)\mathbf{u}^* \preceq \Phi(-K_0),$$

which implies that

$$\Phi(x-\underline{h}(t)), \Phi(-x-\underline{h}(t)) \in [(1-\epsilon_0)\mathbf{u}^*, \mathbf{u}^*] \text{ for } x\in[-\underline{h}(t)+K_0, \underline{h}(t)-K_0], \tag{3.3}$$

where we have assumed $\underline{h}(0) = K_1 > K_0$.

Clearly

$$K_2 \int_0^t (\tau+\theta)^{1-\alpha} \mathrm{d}\tau \le K_2 \theta^{1-\alpha} t \le \frac{c_0}{4} t$$

provided $\theta \ge (4K_2/c_0)^{1/(\alpha-1)}$. Therefore

$$\underline{h}(t) \ge \frac{c_0}{2} t + K_1 \ge \frac{c_0}{2}(t+\theta) > K_0 \text{ for all } t \ge 0 \text{ provided that} \tag{3.4}$$

$$K_1 \ge \frac{c_0}{2}\theta \text{ and } \theta \ge \max\left\{(4K_2/c_0)^{1/(\alpha-1)}, \theta_0, 2K_0/c_0\right\}. \tag{3.5}$$

Define

$$\epsilon_1 := \inf_{1\le i\le m} \inf_{x\in[-K_0,0]} |\phi_i'(x)| > 0.$$

Then

$$\begin{cases} \Phi'(x-\underline{h}(t)) < -\epsilon_1 \mathbf{1} & \text{for } x \in [\underline{h}(t)-K_0, \underline{h}(t)], \\ \Phi'(-x-\underline{h}(t)) < -\epsilon_1 \mathbf{1} & \text{for } x \in [-\underline{h}(t), -\underline{h}(t)+K_0]. \end{cases} \tag{3.6}$$

Claim 1: With $\underline{U} = (\underline{u}_i)$, and suitably chosen $\theta,\ K_1,\ K_2$, we have

$$\underline{h}'(t) \le \sum_{i=1}^m \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y, \quad t>0 \tag{3.7}$$

and

$$-\underline{h}'(t) \ge -\sum_{i=1}^m \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y, \quad t>0.$$

Due to $\underline{U}(t,x) = \underline{U}(t,-x)$ and $\mathbf{J}(x) = \mathbf{J}(-x)$, we just need to verify (3.7). We calculate

$$\begin{aligned} &\sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x \\ =&(1-\epsilon)\sum_{i=1}^{m_0} \mu_i \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x \\ &+(1-\epsilon)\sum_{i=1}^{m_0} \mu_i \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J_i(x-y)[\phi_i(-x-2\underline{h}(t)) - u_i^*]\mathrm{d}y\mathrm{d}x \\ =&(1-\epsilon)c_0 - (1-\epsilon)\sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x \\ &-(1-\epsilon)\sum_{i=1}^{m_0} \mu_i \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J_i(x-y)[u_i^* - \phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x. \end{aligned}$$

From (3.4), for $t \ge 0$,

$$(1-\epsilon)\sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x$$

$$+ (1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-2\underline{h}(t)}^{-\underline{h}(t)}\int_0^\infty J_i(x-y)[u_i^*-\phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x$$
$$\leq 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{-\infty}^{-\underline{h}(t)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x \leq 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x.$$

And by (3.2), we have, for $t>0$,

$$\begin{aligned}
&(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{0}\int_0^\infty J_i(x-y)[u_i^*-\phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x\\
&\leq \sum_{i=1}^{m_0}\mu_i[u_i^*-\phi_i(-\underline{h}(t))]\int_{-\underline{h}(t)}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&\leq \sum_{i=1}^{m_0}\mu_i\frac{C}{h(t)^{\alpha-1}}\int_{-\infty}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&= \sum_{i=1}^{m_0}\mu_i\frac{C}{h(t)^{\alpha-1}}\int_0^\infty yJ_i(y)\mathrm{d}y \leq \sum_{i=1}^{m_0}\mu_i\frac{C^2}{(c_0/2)^{\alpha-1}(t+\theta)^{\alpha-1}} \leq \frac{K_2-c_0}{(t+\theta)^{\alpha-1}}
\end{aligned}$$

if

$$K_2 \geq c_0 + \frac{C^2}{(c_0/2)^{\alpha-1}}\sum_{i=1}^{m}\mu_i. \tag{3.8}$$

Hence, when θ, K_1 and K_2 are chosen such that (3.5) and (3.8) hold, then

$$\begin{aligned}
&\sum_{i=1}^{m}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^\infty J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
&\geq (1-\epsilon)c_0 - 2\sum_{i=1}^{m}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x - \frac{K_2-c_0}{(t+\theta)^{\alpha-1}}\\
&= c_0 - K_2\epsilon(t) - 2\sum_{i=1}^{m}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
&= h'(t) \quad \text{for all } t>0,
\end{aligned}$$

which finishes the proof of (3.7).

Claim 2: With θ, K_1, K_2 chosen such that (3.5) and (3.8) hold, and K_2 suitably further enlarged (see (3.9) below), $\theta_0 \gg 1$ and $0<\epsilon_0\ll 1$, we have, for all $t>0$ and $x\in(-\underline{h}(t),\underline{h}(t))$,

$$\underline{U}_t(t,x) \preceq D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y - D\circ\underline{U}(t,x)+F(\underline{U}(t,x)).$$

A simple calculation gives

$$\begin{aligned}
\underline{U}_t &= -\epsilon'(t)[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*]\\
&\quad -(1-\epsilon)h'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
&= (\alpha-1)(t+\theta)^{-\alpha}[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*]
\end{aligned}$$

$$-(1-\epsilon)[c_0+\delta'(t)][\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))],$$

and using the equation satisfied by Φ we deduce

$$\begin{aligned}
&-(1-\epsilon)c_0[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
=&(1-\epsilon)\Big[D\circ\int_{-\infty}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\Phi(y-\underline{h}(t))\mathrm{d}y-D\circ\Phi(x-\underline{h}(t))\\
&\qquad+D\circ\int_{-\underline{h}(t)}^{\infty}\mathbf{J}(-x-y)\circ\Phi(-y-\underline{h}(t))\mathrm{d}y-D\circ\Phi(-x-\underline{h}(t))\Big]\\
&+(1-\epsilon)\big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\big]\\
=&D\circ\Big[\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)\Big]\\
&+(1-\epsilon)\Big[D\circ\int_{-\infty}^{-\underline{h}(t)}\mathbf{J}(x-y)\circ[\Phi(y-\underline{h}(t))-\mathbf{u}^*]\mathrm{d}y\\
&\qquad+D\circ\int_{\underline{h}(t)}^{\infty}\mathbf{J}(-x-y)\circ[\Phi(-y-\underline{h}(t))\mathrm{d}y-\mathbf{u}^*]\mathrm{d}y\Big]\\
&+(1-\epsilon)\big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\big]\\
\preceq&D\circ\Big[\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)\Big]\\
&+(1-\epsilon)\big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\big].
\end{aligned}$$

Hence

$$\underline{U}_t\preceq D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)+F(\underline{U}(t,x))+A_1(t,x)+A_2(t,x),$$

where

$$\begin{aligned}
A_1(t,x):=&(\alpha-1)(t+\theta)^{-\alpha}[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*],\\
A_2(t,x):=&-(1-\epsilon)\delta'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
&+(1-\epsilon)[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))]-F(\underline{U}(t,x)).
\end{aligned}$$

To finish the proof of Claim 2, it remains to check that

$$A_1(t,x)+A_2(t,x)\preceq\mathbf{0}\quad\text{for}\quad t>0,\ x\in(-\underline{h}(t),\underline{h}(t)).$$

We next prove this inequality for x in the following three intervals, separately:

$$I_1(t):=[\underline{h}(t)-K_0,\underline{h}(t)],\ I_2(t):=[-\underline{h}(t),-\underline{h}(t)+K_0],\ I_3(t):=[-\underline{h}(t)+K_0,\underline{h}(t)-K_0].$$

For $x\in I_1(t)$, by (3.2),

$$\mathbf{0}\succ\Phi(-x-\underline{h}(t))-\mathbf{u}^*\succeq\Phi(K_0-2\underline{h}(t))-\mathbf{u}^*\succeq\Phi(-\underline{h}(t))-\mathbf{u}^*\succeq\frac{-C}{h(t)^{\alpha-1}}\mathbf{1}$$

Then by $(\mathbf{f}_2)$, there exists $L>0$ such that

$$F(\Phi(-x-\underline{h}(t)))=F(\Phi(-x-\underline{h}(t)))-F(\mathbf{u}^*)\preceq L\frac{C}{h(t)^{\alpha-1}}\mathbf{1}$$

and

$$F(\underline{U}(t,x)) \succeq (1-\epsilon)F\Big[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*\Big]$$
$$\succeq (1-\epsilon)\big[F(\Phi(x-\underline{h}(t)))-L\frac{C}{h(t)^{\alpha-1}}\mathbf{1}\big].$$

Thus from the definition of $\delta(t)$, (3.4) and (3.6), we deduce

$$\begin{aligned} A_2(t,x) \preceq &(1-\epsilon)\big[\delta'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]+F(\Phi(x-\underline{h}(t))) \\ &+F(\Phi(-x-\underline{h}(t)))-F\big(\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*\big)\big] \\ \preceq &(1-\epsilon)\left[-\delta'(t)\epsilon_1+2L\frac{C}{h(t)^{\alpha-1}}\right]\mathbf{1} \\ \preceq &(1-\epsilon)\left[-K_2(t+\theta)^{1-\alpha}\epsilon_1+\frac{2LC}{h(t)^{\alpha-1}}\right]\mathbf{1} \\ \preceq &(1-\epsilon)(t+\theta)^{1-\alpha}\big[-K_2\epsilon_1+2LC(2/c_0)^{\alpha-1}\big]\mathbf{1}. \end{aligned}$$

Moreover,

$$A_1(t,x) \preceq (\alpha-1)(t+\theta)^{-\alpha}\mathbf{u}^* \leq 2|\mathbf{u}^*|(1-\epsilon)(\alpha-1)(t+\theta)^{-\alpha}\mathbf{1},$$

where $|\mathbf{u}^*| := \max_{1\leq i\leq m} u_i^*$ and by enlarging θ_0 we have assumed that $\epsilon(t) \leq \theta_0^{-\alpha} < 1/2$. Hence

$$A_1(t,x)+A_2(t,x) \preceq (1-\epsilon)(t+\theta)^{1-\alpha}\Big[-K_2\epsilon_1+2LC(2/c_0)^{\alpha-1}+2|\mathbf{u}^*|\alpha\theta_0^{-1}\Big]\mathbf{1} \preceq \mathbf{0}$$

if additionally

$$K_2 \geq \epsilon_1^{-1}\Big[2LC(2/c_0)^{\alpha-1}+2|\mathbf{u}^*|\alpha\theta_0^{-1}\Big]. \tag{3.9}$$

This proves the desired inequality for $x \in I_1(t)$.

Since $A_1(t,x)+A_2(t,x)$ is even in x, the desired inequality is also valid for $x \in I_2(t) = -I_1(t)$. It remains to prove the desired inequality for $x \in I_3(t)$.

We apply Lemma 3.2 with $u = \Phi(x-\underline{h}(t))$ and $v = \Phi(-x-\underline{h}(t))$. Let

$$P(t,x) = (p_i(t,x)) := \mathbf{u}^*-\Phi(x-\underline{h}(t)), \quad Q(t,x) = (q_i(t,x)) := \mathbf{u}^*-\Phi(-x-\underline{h}(t)).$$

Then by (3.3) we have

$$P(t,x),\ Q(t,x) \in [\mathbf{0},\epsilon_0\mathbf{u}^*] \text{ for } x \in I_3(t), t>0. \tag{3.10}$$

Moreover, since $\min\{x-\underline{h}(t), -x-\underline{h}(t)\} \leq -\underline{h}(t)$ always holds, by (3.2) and (3.4), if we denote $C_3 := C(c_0/2)^{1-\alpha}$, then for $x \in I_3(t),\ t>0,\ \ j,k \in \{1,...,m\}$,

$$p_j(t,x)q_k(t,x) \leq \frac{C\epsilon_0}{\underline{h}(t)^{\alpha-1}} \leq C_3\epsilon_0\epsilon(t). \tag{3.11}$$

Let A_2^i denote the i-th component of A_2. Now due to $\delta'(t)<0$ and $\Phi' \prec\!\!\prec \mathbf{0}$, we have, by (3.10), (3.11) and Lemma 3.2, assuming $\epsilon_0>0$ is sufficiently small,

$$A_2^i(t,x) \leq g_i(\mathbf{u}^*-P,\mathbf{u}^*-Q) \leq \frac{\epsilon}{2}\mathbf{u}^*\cdot\nabla f_i(\mathbf{u}^*)$$

for $x \in I_3(t),\ t>0,\ i\in\{1,...,m\}$ and all $\theta_0 \gg 1$. Since

$$A_1^i(t,x) \le (\alpha-1)(t+\theta)^{-\alpha}u_i^* \le \alpha|u_i^*|\theta_0^{-1}\epsilon(t),$$

we thus obtain

$$A_1^i+A_2^i \le \epsilon\Big(\mathbf{u}^*\cdot\nabla f_i(\mathbf{u}^*)/2+\alpha u_i^*\theta_0^{-1}\Big)<0 \ \text{ for } x\in I_3(t),\ t>0,\ i\in\{1,...,m\},\ \theta_0\gg 1,$$

provided that ϵ_0 is sufficiently small. The proof of Claim 2 is now complete.

Claim 3: There exists $t_0>0$ such that

$$\begin{cases} g(t+t_0)\le -\underline{h}(t),\ h(t+t_0)\ge \underline{h}(t) \text{ for } t\ge 0,\\ U(t+t_0,x)\succeq \underline{U}(t,x) \text{ for } t\ge 0,\ x\in[-\underline{h}(t),\underline{h}(t)]. \end{cases} \tag{3.12}$$

It is clear that

$$\underline{U}(t,\pm\underline{h}(t)) = (1-\epsilon(t))[\Phi(-2\underline{h}(t))-\mathbf{u}^*] \prec \mathbf{0} \text{ for } \ t\ge 0.$$

Since spreading happens for (U,g,h), there exists a large constant $t_0>0$ such that

$$\begin{aligned} &g(t_0) < -K_1 = -\underline{h}(0) \text{ and } \underline{h}(0) = K_1 < h(t_0),\\ &U(t_0,x)\succeq (1-\theta^{1-\alpha})\mathbf{u}^* \succeq \underline{U}(0,x) \ \text{ for } \ x\in[-\underline{h}(0),\underline{h}(0)]. \end{aligned}$$

which together with the inequalities proved in Claims 1 and 2 allows us to apply the comparison principle (Lemma 2.3 and Remark 2.4 in [11]) to conclude that (3.12) is valid.

Claim 4: There exists $C>0$ such that

$$\delta(t) \ge -C\Big[1+\int_0^t (1+x)^{1-\alpha}dx + \int_0^{\frac{c_0}{2}t} x^2\hat{J}(x)dx + t\int_{\frac{c_0}{2}t}^{\infty} x\hat{J}(x)dx\Big].$$

Clearly, for large θ,

$$\int_0^t \epsilon(\tau)\mathrm{d}\tau = \int_0^t (x+\theta)^{1-\alpha}\mathrm{d}x < \int_0^t (x+1)^{1-\alpha}\mathrm{d}x.$$

By changing order of integrations we have

$$\begin{aligned} &\int_0^t\int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau \le \int_0^t\int_{-\infty}^{-\frac{c_0}{2}\tau}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\\ &=\int_0^t\int_{\frac{c_0}{2}\tau}^\infty\Big[y-\frac{c_0}{2}\tau\Big]J_i(y)\mathrm{d}y\mathrm{d}\tau \le \int_0^t\int_{\frac{c_0}{2}\tau}^\infty yJ_i(y)\mathrm{d}y\mathrm{d}\tau\\ &=\frac{c_0}{2}\int_0^{\frac{c_0}{2}t} y^2J_i(y)dy + t\int_{\frac{c_0}{2}t}^\infty yJ_i(y)dy. \end{aligned}$$

The desired inequality now follows directly from the definition of $\delta(t)$. $\square$

3.2. Bound from above

Next we prove an upper bound for $h(t)-c_0t$. Let us note that we do not need the condition $(\mathbf{J}^\alpha)$ (for J_i with $i\in A_0$) in the following result.

Lemma 3.3. *Under the assumptions of Theorem B (i), if* $(\mathbf{J_1})$ *holds, and additionally* F *is* C^2 *and* $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \lll \mathbf{0}$, *then there exists* $C>0$ *such that*

$$h(t)-c_0t\le C \quad \textit{for all} \;\; t>0. \tag{3.13}$$

Proof. As in the proof of Lemma 3.1, (c_0,Φ^{c_0}) denotes the unique solution pair of (1.4)-(1.5) in Theorem A, and to simplify notations we write $\Phi^{c_0}(x)=\Phi(x)=(\phi_i(x))$.

For fixed $\beta>1$, and some large constants $\theta>0$ and $K_1>0$ to be determined, define

$$\begin{cases} \bar h(t):=c_0t+\delta(t), \quad t\ge 0,\\ \overline{U}(t,x):=(1+\epsilon(t))\Phi(x-\bar h(t)), \quad t\ge 0,\; x\le \bar h(t), \end{cases}$$

where $\epsilon(t):=(t+\theta)^{-\beta}$ and

$$\delta(t):=K_1+\frac{c_0}{1-\beta}[(t+\theta)^{1-\beta}-\theta^{1-\beta}].$$

Clearly, there is a large constant $t_0>0$ such that

$$U(t+t_0,x)\preceq (1+\frac{1}{2}\epsilon(0))\mathbf{u}^* \;\text{ for }\;\; t\ge 0,\; x\in[g(t),h(t)].$$

Due to $\Phi(-\infty)=\mathbf{u}^*$, we may choose sufficient large $K_1>0$ such that $\underline{h}(0)=K_1>2h(t_0)$, $-\underline{h}(0)=-K_1<2g(t_0)$, and for $x\in[g(t_0),h(t_0)]$,

$$\overline{U}(0,x)=(1+\epsilon(0))\Phi(-K_1/2)\succ(1+\frac{1}{2}\epsilon(0))\mathbf{u}^*\succeq U(t_0,x). \tag{3.14}$$

Claim 1: We have, with $\overline{U}=(\bar u_i)$,

$$\bar h'(t)\ge\sum_{i=1}^m\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y \;\text{ for }\;\; t>0.$$

A direct calculation shows

$$\sum_{i=1}^m\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\le\sum_{i=1}^m\mu_i\int_{-\infty}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y$$
$$=(1+\epsilon)\sum_{i=1}^m\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y=(1+\epsilon)c_0=\bar h'(t),$$

as desired.

Claim 2: If $\theta>0$ is sufficiently large, then for $t>0$ and $x\in(g(t+t_0),\underline{h}(t))$, we have

$$\overline{U}_t(t,x)\succeq D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x)). \tag{3.15}$$

By (1.4), we have

$$\begin{aligned}\overline{U}_t(t,x) =& -(1+\epsilon)[c_0+\delta'(t)]\Phi'(x-\bar{h}(t))+\epsilon'(t)\Phi(x-\underline{h}(t))\\ =& -(1+\epsilon)c_0\Phi'(x-\bar{h}(t))-(1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t))\\ & -\beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t))\\ \succeq& D\circ\int_{g(t_0+t)}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))+A(t,x)\end{aligned}$$

with

$$\begin{aligned}A(t,x):=&(1+\epsilon)F(\Phi(x-\bar{h}(t)))-F((1+\epsilon)\Phi(x-\bar{h}(t)))\\ &-(1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t))-\beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t)).\end{aligned}$$

To prove the claim, we need to show

$$A(t,x)\succeq\mathbf{0}\quad\text{for } x\in[g(t_0+t),\bar{h}(t)] \text{ and } t>0.$$

Let ϵ_0, ϵ_1 and K_0 be given as in the proof of Lemma 3.1. For $x\in[\bar{h}(t)-K_0,\bar{h}(t)]$ and $t>0$, by (3.6), we have

$$\begin{aligned}A(t,x)\succeq& -(1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t))-\beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t))\\ =& -(1+\epsilon)c_0(t+\theta)^{-\beta}\Phi'(x-\bar{h}(t))-\beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t))\\ \succeq& c_0(t+\theta)^{-\beta}\epsilon_1\mathbf{1}-\beta(t+\theta)^{-\beta-1}\mathbf{u}^*\succeq(t+\theta)^{-\beta-1}\big[c_0\theta\epsilon_1\mathbf{1}-\beta\mathbf{u}^*\big]\succeq\mathbf{0},\end{aligned}$$

provided θ is large enough.

We next estimate $A(t,x)$ for $x\in[g(t+t_0),\underline{h}(t)-K_0]$. Define

$$G(u)=(g_i(u)):=(1+\epsilon)F(u)-F((1+\epsilon)u),\quad u,v\in\mathbb{R}^m.$$

Then for $u,v\in[\mathbf{0},\mathbf{u}^*]$ and $i\in\{1,...,m\}$,

$$\begin{aligned}g_i(u)=&g_i(\mathbf{u}^*)+\nabla g_i(\tilde{u})\cdot(u-\mathbf{u}^*)\\ =&-f_i((1+\epsilon)\mathbf{u}^*)+(1+\epsilon)\nabla f_i(\tilde{u})\cdot(u-\mathbf{u}^*)-(1+\epsilon)\nabla f_i((1+\epsilon)\tilde{u})\cdot(u-\mathbf{u}^*)\\ =&-f_i((1+\epsilon)\mathbf{u}^*)+(1+\epsilon)\bigg[\nabla f_i(\tilde{u})-\nabla f_i((1+\epsilon)\tilde{u})\bigg]\cdot(u-\mathbf{u}^*)\end{aligned}$$

for some $\tilde{u}=\tilde{u}^i\in[u,\mathbf{u}^*]$. Since $F\in C^2$, there exists $C_1>0$ such that

$$|\partial_{jk}f_i(u)|\le C_1\quad\text{for}\quad u\in[0,\hat{\mathbf{u}}],\ i,j,k\in\{1,...,m\}.$$

Therefore

$$g_i(u)\ge-f_i((1+\epsilon)\mathbf{u}^*)-(1+\epsilon)b_1\sum_{j=1}^m(u_j^*-u_j)$$

with

$$b_1:=C_1|\epsilon\tilde{u}|\le C_1\epsilon|\mathbf{u}^*|:=C_2\epsilon.$$

Thus

$$g_i(u) \geq -\epsilon \nabla f_i(\mathbf{u}^*) \cdot \mathbf{u}^* + o(\epsilon) - 2C_2\epsilon \sum_{j=1}^{m} (u_j^* - u_j).$$

By (3.3) we have

$$-\epsilon_0 \mathbf{u}^* \preceq \Phi(x - \bar{h}(t)) - \mathbf{u}^* \lll \mathbf{0} \quad \text{for} \quad x \in [g(t_0 + t), \underline{h}(t) - K_0],\ t > 0. \tag{3.16}$$

Using (3.3), $\delta' > 0$, $\Phi' \preceq \mathbf{0}$ and $\epsilon = (t+\theta)^{-\beta} \leq \theta^{-\beta}$, we obtain

$$\begin{aligned} A^i(t,x) \geq& (1+\epsilon) f_i(\Phi(x - \bar{h}(t))) - f_i((1+\epsilon)\Phi(x - \bar{h}(t))) - \beta(t+\theta)^{-\beta-1}\phi_i(x - \underline{h}(t)) \\ &= g_i(\Phi(x - \bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi_i(x - \underline{h}(t)) \\ &\geq \epsilon\Big[-\mathbf{u}^* \cdot \nabla f_i(\mathbf{u}^*) + o(1) - 2\epsilon_0 C_2 \sum_{j=1}^{m} u_j^* - \beta\theta^{-\beta-1}u_i^* \Big] \\ &> 0 \quad \text{for} \quad x \in [g(t_0+t), \underline{h}(t) - K_0],\ t > 0,\ i \in \{1, ..., m\}, \end{aligned}$$

provided θ is large enough and $\epsilon_0 > 0$ is small enough, since $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \lll \mathbf{0}$. We have now proved (3.15).

Due to the inequalities proved in Claims 1 and 2, (3.14) and

$$\overline{U}(t, g(t+t_0)) > 0, \quad \overline{U}(t, \bar{h}(t)) = (1+\epsilon)\Phi(\bar{h}(t) - \bar{h}(t)) = 0 \quad \text{for} \quad t \geq 0,$$

we are now able to apply the comparison principle (see Lemma 2.3 and Remark 2.4 in [11]) to conclude that

$$\begin{aligned} &h(t+t_0) \leq \bar{h}(t), && t \geq 0, \\ &U(t+t_0, x) \preceq \overline{U}(t,x), && t \geq 0,\ x \in [g(t+t_0), \underline{h}(t)]. \end{aligned}$$

The desired inequality (3.13) follows directly from $\delta(t) \leq K_1 + \frac{c_0}{\beta-1}\theta^{1-\beta}$ and $h(t+t_0) \leq \bar{h}(t)$. The proof is complete. □

3.3. Completion of the proof of Theorem 1.1

Proof of Theorem 1.1. Since every J_i with $i \in A_+$ satisfies $(\mathbf{J^3})$ and every J_i with $i \in A_0$ satisfies $(\mathbf{J}^\alpha)$ with $\alpha > 2$, from the proof of Lemmas 3.1 and 3.3, it is easily seen that

$$C_0 := \sup_{t>0} \big[|\bar{h}(t) - c_0 t| + |\underline{h}(t) - c_0 t| \big] < \infty. \tag{3.17}$$

Hence for large fixed $\theta > 0$ and all large t, say $t \geq t_0$,

$$[g(t), h(t)] \supset [-\underline{h}(t-t_0), \underline{h}(t-t_0)] \supset [-c_0 t + C, c_0 t - C] \text{ with } C := C_0 + c_0 t_0,$$

and

$$U(t,x) \succeq \underline{U}(t,x) \succeq (1-\epsilon(t))\big[\Phi^{c_0}(x - c_0 t + C) + \Phi^{c_0}(-x - c_0 t + C) - \mathbf{u}^*\big]$$

for $x \in [-c_0 t + C, c_0 t - C]$, where $\epsilon(t) = (t+\theta)^{1-\alpha}$. This inequality for $U(t,x)$ also holds for $x \in [g(t), h(t)]$ if we assume that $\Phi^{c_0}(x) = 0$ for $x > 0$, since when x lies outside of $[-c_0 t + C, c_0 t - C]$ the right side is $\prec \mathbf{0}$.

By considering (1.1) with initial function $u_0(-x)$, from the proof of Lemma 3.3 we see that the following analogous inequalities hold:

$$g(t) \geq -\bar{h}(t-t_0),\ U(t,x) \preceq (1+\epsilon(t))\Phi^{c_0}(-x-\bar{h}(t-t_0))$$

for $t > t_0$ and $x \in [g(t), h(t)]$. We thus have

$$[g(t), h(t)] \subset [-\bar{h}(t-t_0), \bar{h}(t-t_0)] \subset [-c_0t - C, c_0t + C],$$

and

$$U(t,x) \preceq \overline{U}(t,x) \preceq (1-\epsilon(t)) \min\Big\{\Phi^{c_0}(x - c_0t - C), \Phi^{c_0}(-x - c_0t - C)\Big\}$$

for $t > t_0$ and $x \in [g(t), h(t)]$. The proof is complete. □

4. Growth rate of $c_0t - h(t)$ and $c_0t + g(t)$ for kernels of type $(\mathbf{J}^\gamma_\infty)$

Recall that $(U(t,x), g(t), h(t))$ is the unique positive solution of (1.1), and we assume that spreading happens. Under the assumptions of Theorem B (i), we have

$$-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c_0 > 0.$$

In this section we determine the order of growth for $c_0t - h(t)$ and $c_0t + g(t)$ when the kernel functions $\{J_i : i \in A_+\}$ have a dominating one J_{i^*}, and there are $\gamma \in (2,3]$ and $\omega \in [\gamma - 1, \infty)$ such that

$$\begin{cases} J_{i^*}(x) \approx |x|^{-\gamma} \text{ for } |x| \gg 1, \\ J_i \text{ satisfies } (\mathbf{J}^\omega) \text{ for all } i \in A_0, \end{cases} \tag{4.1}$$

Clearly, (4.1) implies that every J_i $(i = 1, ..., m_0)$ satisfies $(\mathbf{J}^\omega)$. In particular, $(\mathbf{J_1})$ holds.

The main result of this section is the following theorem.

Theorem 4.1. *In Theorem B, if additionally $\{J_i : i \in A_+\}$ have a dominating one J_{i^*} and (4.1), (1.6) hold, then for $t \gg 1$,*

$$c_0t + g(t),\ c_0t - h(t) \approx \begin{cases} t^{3-\gamma} & \text{if } \gamma \in (2,3], \\ \ln t & \text{if } \gamma = 3. \end{cases}$$

It is clear that the conclusion of Theorem 1.2 follows directly from Theorem 4.1.

By $(\mathbf{f_1})$ and the Perron-Frobenius theorem, we know that the matrix $\nabla F(0) - \widetilde{D}$ with $\widetilde{D} = \text{diag}(d_1, ..., d_m)$ has a principal eigenvalue $\tilde{\lambda}_1$ with a corresponding eigenvector $V^* = (v_1^*, \cdots, v_m^*) \succ\succ \mathbf{0}$, namely

$$V^*\Big([\nabla F(0)]^T - \widetilde{D}\Big) = \tilde{\lambda}_1 V^*. \tag{4.2}$$

To prove Theorem 4.1, the difficult part is to find the lower bound for $c_0t - h(t)$, which will be established according to the following two cases: (i) $\tilde{\lambda}_1 < 0$, (ii) $\tilde{\lambda}_1 \geq 0$.

As before, we will only estimate $c_0t - h(t)$, since the estimate for $c_0t + g(t)$ follows by making the variable change $x \to -x$ in the initial functions.

4.1. The case $\tilde{\lambda}_1 < 0$

Lemma 4.1. *Suppose that the conditions in Theorem 4.1 are satisfied. If $\tilde{\lambda}_1 < 0$, then there exists $\sigma = \sigma(\gamma) > 0$ such that for all large $t > 0$,*

$$c_0 t - h(t) \geq \begin{cases} \sigma\, t^{3-\gamma} & \text{if } \gamma \in (2,3), \\ \sigma \ln t & \text{if } \gamma = 3. \end{cases} \tag{4.3}$$

Proof. Let $\beta := \gamma - 2 \in (0,1]$, and (c_0, Φ) be the solution of (1.4)-(1.5). Define

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \overline{U}(t,x) := (1+\epsilon(t))\Phi(x - \bar{h}(t)) + \rho(t,x), & t \geq 0,\ x \leq \bar{h}(t), \end{cases}$$

where

$$\epsilon(t) := K_1(t+\theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \epsilon(\tau)\mathrm{d}\tau, \ \ \rho(t,x) := K_4 \xi(x - \bar{h}(t))\epsilon(t)V^*,$$

with $\xi \in C^2(\mathbb{R})$ satisfying

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon},\ \ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}, \tag{4.4}$$

and the positive constants $\theta,\ K_1, K_2, K_3, K_4,\ \tilde{\epsilon}$ are to be determined.

We are going to show that, it is possible to choose these constants and some $t_0 > 0$ such that

$$\overline{U}_t(t,x) \succeq D \circ \int_{g(t+t_0)}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \overline{U}(t,y)\mathrm{d}y - \overline{U}(t,x) + F(\overline{U}(t,x)) \tag{4.5}$$

$$\text{for } t > 0,\ x \in (g(t+t_0), \bar{h}(t)),$$

$$\bar{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y \qquad \text{for } t > 0, \tag{4.6}$$

$$\overline{U}(t, g(t+t_0)) \succeq \mathbf{0}, \quad \overline{U}(t, \bar{h}(t)) \succeq \mathbf{0} \qquad \text{for } t \geq 0, \tag{4.7}$$

$$\overline{U}(0,x) \succeq U(t_0,x),\ \bar{h}(0) \geq h(t_0) \qquad \text{for } x \in [g(t_0), h(t_0)]. \tag{4.8}$$

If these inequalities are proved, then by the comparison principle (Lemma 2.3 in [11]), we obtain

$$\bar{h}(t) \geq h(t+t_0),\ \overline{U}(t,x) \succeq U(t+t_0,x) \text{ for } t > 0,\ x \in [g(t+t_0), h(t+t_0)],$$

and the desired inequality for $c_0 t - h(t)$ follows easily from the definition of $\bar{h}(t)$.

Therefore, to complete the proof, it suffices to prove the above inequalities. We divide the arguments below into several steps.

Firstly, by Theorem B, there is $C_1 > 1$ such that

$$-g(t), h(t) \leq (c_0+1)t + C_1 \quad \text{for} \quad t \geq 0. \tag{4.9}$$

Let us also note that (4.7) holds trivially.

Step 1. Choose $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.8) holds.

For later analysis, we need to find $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.8) holds and at the same time they have less than linear growth in θ.

Let $W^* \succ\succ \mathbf{0}$ be an eigenvector corresponding to the maximal eigenvalue $\tilde{\lambda}$ of $\nabla F(\mathbf{u}^*)$. By our assumptions on F, we have $\tilde{\lambda} < 0$. Hence there exists small $\epsilon_* > 0$ such that for any $k \in (0, \epsilon_*]$,

$$F(\mathbf{u}^* + kW^*) = kW^*\Big([\nabla F(\mathbf{u}^*)]^T + o(1)\mathbf{I}_m\Big) \preceq \frac{k}{2}\tilde{\lambda}W^* \prec\prec \mathbf{0},$$
$$F(\mathbf{u}^* - kW^*) = -kW^*\Big([\nabla F(\mathbf{u}^*)]^T + o(1)\mathbf{I}_m\Big) \succeq -\frac{k}{2}\tilde{\lambda}W^* \succ\succ \mathbf{0}.$$

It follows that, for $\tilde{\sigma} = \tilde{\lambda}/2$,

$$\overline{W}(t) = \mathbf{u}^* + \epsilon_* e^{\tilde{\sigma}t}W^*, \quad \underline{W}(t) = \mathbf{u}^* - \epsilon_* e^{\tilde{\sigma}t}W^*$$

are a pair of upper and lower solutions of the ODE system $W' = F(W)$ with initial data $W(0) \in [\mathbf{u}^* - \epsilon_* W^*, \mathbf{u}^* + \epsilon_* W^*]$.

By $(\mathbf{f_4})$, the unique solution of the ODE system

$$W' = F(W), \ W(0) = (\|u_{10}\|_\infty, \cdots, \|u_{m0}\|_\infty)$$

satisfies $\lim_{t\to\infty} W(t) = \mathbf{u}^*$. Hence there exists $t_* > 0$ such that

$$W(t_*) \in [\mathbf{u}^* - \epsilon_* W^*, \mathbf{u}^* + \epsilon_* W^*].$$

Using the above defined upper solution $\overline{W}(t)$ we obtain

$$W(t + t_*) \preceq \mathbf{u}^* + \epsilon_* e^{\tilde{\sigma}t}W^* \preceq (1 + \tilde{\epsilon}_* e^{\tilde{\sigma}t})\mathbf{u}^* \text{ for } t \geq 0,$$

where $\tilde{\epsilon}_* > 0$ is chosen such that $\epsilon_* W^* \leq \tilde{\epsilon}_* \mathbf{u}^*$. By the comparison principle we deduce

$$U(t + t_*, x) \preceq W(t + t_*) \preceq (1 + \tilde{\epsilon}_* e^{\tilde{\sigma}t})\mathbf{u}^* \text{ for } t \geq 0, \ x \in [g(t + t_*), h(t + t_*)].$$

Hence

$$U(t_0, x) \preceq (1 + \frac{\epsilon(0)}{2})\mathbf{u}^* \text{ for } x \in [g(t_0), h(t_0)]$$

provided that

$$t_0 = t_0(\theta) := \frac{\beta}{|\tilde{\sigma}|}\ln\theta + \frac{\ln(2\tilde{\epsilon}_*/K_1)}{|\tilde{\sigma}|} + t_*.$$

By (4.1), we have

$$\int_{\mathbb{R}} J(x)|x|^{\omega-1}\mathrm{d}x < \infty.$$

Then by Theorem 1.3, there is C_2 such that

$$\mathbf{u}^* - \Phi(x) \leq \frac{C_2}{|x|^{\omega-1}}\mathbf{u}^* \text{ for } \ x \leq -1.$$

Hence, for $K > 1$ we have

$$(1 + \epsilon(0))\Phi(-K) - (1 + \epsilon(0)/2)\mathbf{u}^* \succeq (1 + \epsilon(0))\big[1 - C_2 K^{1-\omega}\big]\mathbf{u}^* - (1 + \epsilon(0)/2)\mathbf{u}^*$$

$$= \left[K_1\theta^{-\beta}/2 - C_2K^{1-\omega}(1+K_1\theta^{-\beta})\right]\mathbf{u}^* \succeq \mathbf{0}$$

provided that

$$K^{\omega-1} \geq 2C_2 + \frac{2C_2}{K_1}\theta^{\beta}.$$

Therefore, for all $K_1 \in (0,1]$, $\theta \geq 1$ and $K \geq (4C_2/K_1)^{1/(\omega-1)}\theta^{\beta/(\omega-1)}$, we have

$$(1+\epsilon(0))\Phi(-K) - (1+\epsilon(0)/2)\mathbf{u}^* \succeq \mathbf{0}.$$

Now define

$$K_2(\theta) := 2\max\left\{(4C_2/K_1)^{1/(\omega-1)}\theta^{\beta/(\omega-1)}, (c_0+1)t_0(\theta) + C_1\right\}. \tag{4.10}$$

Then for $K_2 = K_2(\theta)$ we have

$$\overline{h}(0) = K_2 > K_2/2 \geq (c_0+1)t_0 + C_1 \geq h(t_0),$$

and for $x \in [g(t_0), h(t_0)]$,

$$\overline{U}(0,x) = (1+\epsilon(0))\Phi(x-K_2) \succeq (1+\epsilon(0))\Phi(-K_2/2) \succeq (1+\epsilon(0)/2)\mathbf{u}^*.$$

Thus (4.8) holds if t_0 and K_2 are chosen as above, for any $\theta \geq 1$, $K_1 \in (0,1]$.

Step 2. We verify that (4.6) holds if θ, K_1, K_3 and K_4 are chosen suitably.

Denote

$$C_3 := \sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)\mathrm{d}y\mathrm{d}x = \sum_{i=1}^{m_0}\mu_i\int_0^{+\infty} J_i(y)y\mathrm{d}y. \tag{4.11}$$

With $\rho = (\rho_i)$, a direct calculation shows

$$\begin{aligned}
&\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
=&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
&-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
=&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)[(1+\epsilon)\phi_i(x)+\rho_i(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\
&-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar{h}(t)}\int_{0}^{+\infty} J_i(x-y)[(1+\epsilon)\phi_i(x)+\rho_i(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\
\leq&(1+\epsilon)c_0 + C_3K_4\epsilon|V^*| - \sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar{h}(t)}\int_{0}^{+\infty} J_i(x-y)(1+\epsilon)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
\leq&(1+\epsilon)c_0 + C_3K_4\epsilon|V^*| - \sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar{h}(t)}\int_{0}^{+\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x,
\end{aligned}$$

where

$$|V^*| := \max_{1\leq i\leq m} v_i^*.$$

By elementary calculus, for any $k > 1$,

$$\begin{aligned}&\int_{-\infty}^{-k}\int_0^{\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{-k}\int_{-x}^{\infty}\frac{1}{y^{2+\beta}}\mathrm{d}y\mathrm{d}x=\int_k^{\infty}\int_x^{\infty}\frac{1}{y^{2+\beta}}\mathrm{d}y\mathrm{d}x\\&=\int_k^{\infty}\int_k^{y}\frac{1}{y^{2+\beta}}\mathrm{d}x\mathrm{d}y=\int_k^{\infty}\frac{y-k}{y^{2+\beta}}\mathrm{d}y=\beta^{-1}(1+\beta)^{-1}k^{-\beta}.\end{aligned}\tag{4.12}$$

From (4.1) and (4.9), there exists $C_4 > 0$ such that

$$\begin{aligned}&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\&\ge C_4\left[\min_{1\le i\le m}\phi_i(g(t+t_0)-\bar h(t))\right]\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x\\&\ge\phi_*C_4\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\frac{\phi_*C_4}{\beta(1+\beta)}(|g(t+t_0)|+\bar h(t))^{-\beta}\\&\ge\frac{\phi_*C_4}{\beta(1+\beta)}[(c_0+1)(t+t_0)+C_1+c_0t+K_2]^{-\beta}\\&=\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}\left[t+\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)}\right]^{-\beta},\end{aligned}\tag{4.13}$$

where $\phi_*=\min\limits_{1\le i\le m}\phi_i(-1)\le\min\limits_{1\le i\le m}\phi_i(-K_2)\le\min\limits_{1\le i\le m}\phi_i(g(t+t_0)-\bar h(t))$. Therefore, for all large $\theta>0$ so that

$$\theta>\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)},\tag{4.14}$$

which is possible since $t_0(\theta)$ and $K_2(\theta)$ grow slower than linearly in θ, we have

$$\begin{aligned}&\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\&\le(1+\epsilon(t))c_0+C_4K_4\epsilon(t)|V^*|-\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}\\&=c_0+\epsilon(t)\left[c_0+C_4K_4|V^*|-\frac{\phi_*C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right]\le c_0-K_3\epsilon(t)=h'(t)\end{aligned}$$

provided that K_1, K_3 and K_4 are small enough so that

$$K_1(c_0+C_4K_4|V^*|+K_3)\le\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}.\tag{4.15}$$

Therefore (4.6) holds if we first fix K_1, K_3, K_4 small so that (4.15) holds, and then choose θ large such that (4.14) is satisfied.

Step 3. We show that (4.5) holds when K_3 and K_4 are chosen suitably small and θ is large.

From (1.4), we deduce

$$\overline{U}_t(t,x)=-(1+\epsilon)[c_0+\delta'(t)]\Phi'(x-\bar h(t))+\epsilon'(t)\Phi(x-\underline{h}(t))+\rho_t(t,x),$$

and

$$-(1+\epsilon)c_0\Phi'(x-\bar h(t))$$

$$
\begin{aligned}
=&(1+\epsilon)\Big[D\circ\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\Phi(y-\bar h(t))\mathrm{d}y-D\circ\Phi(x-\bar h(t))+F(\Phi(x-\bar h(t)))\Big]\\
=&D\circ\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ[\overline{U}(t,y)-\rho(t,y)]\mathrm{d}y-D\circ[\overline{U}(t,x)-\rho(t,x)]+(1+\epsilon)F(\Phi(x-\bar h(t)))\\
=&D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&+D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar h(t)))-F(\overline{U}(t,x)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{U}_t(t,x)=&D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&+A(t,x)
\end{aligned}
$$

with

$$
\begin{aligned}
A(t,x):=&D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar h(t)))-F(\overline{U}(t,x))\\
&-(1+\epsilon)\delta'(t)\Phi'(x-\bar h(t))+\epsilon'(t)\Phi(x-\underline{h}(t))+\rho_t(t,x).
\end{aligned}
$$

Therefore to complete this step, it suffices to show that we can choose K_3, K_4 and θ such that $A(t,x)\succeq\mathbf{0}$. We will do that for $x\in[\bar h(t)-\tilde\epsilon,\bar h(t)]$ and for $x\in[g(t_0+t),\bar h(t)-\tilde\epsilon]$ separately.

Claim 1. If $\tilde\epsilon>0$ in (4.4) is sufficiently small and θ is sufficiently large, then

$$
\begin{aligned}
&D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar h(t)))-F(\overline{U}(t,x))\\
&\succeq\frac{|\tilde\lambda_1|}{4}\rho(t,x)\succ\mathbf{0}\quad\text{ for }\quad x\in[\bar h(t)-\tilde\epsilon,\bar h(t)].
\end{aligned}\tag{4.16}
$$

Since $\tilde\lambda_1<0$ and $D\circ V^*=V^*\widetilde{D}$, using (4.2) we deduce, for $x\in[\bar h(t)-\tilde\epsilon,\bar h(t)]$,

$$
\begin{aligned}
&D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]\\
=&K_4\epsilon(t)\Big[D\circ V^*-D\circ\int_{-\infty}^{0}\mathbf{J}(x-\bar h(t)-y)\circ\xi(y)V^*\mathrm{d}y\Big]\\
\succeq&K_4\epsilon(t)\Big[D\circ V^*-D\circ\int_{-2\tilde\epsilon}^{0}\mathbf{J}(x-\bar h(t)-y)\circ V^*\mathrm{d}y\Big]\\
=&K_4\epsilon(t)\Big[V^*\nabla F(0)-\tilde\lambda_1V^*-D\circ\int_{\bar h(t)-x-2\tilde\epsilon}^{\bar h(t)-x}\mathbf{J}(y)\circ V^*\mathrm{d}y\Big]\\
\succeq&K_4\epsilon(t)\Big[V^*\nabla F(0)-\tilde\lambda_1V^*-D\circ\int_{-2\tilde\epsilon}^{\tilde\epsilon}\mathbf{J}(y)\circ V^*\mathrm{d}y\Big]\\
\succeq&K_4\epsilon(t)\Big[V^*\nabla F(0)-\frac{\tilde\lambda_1}{2}V^*\Big]=\rho(t,x)\nabla F(0)-\frac{\tilde\lambda_1}{2}\rho(t,x),
\end{aligned}
$$

provided $\tilde\epsilon\in(0,\epsilon_1]$ for some small $\epsilon_1>0$.

On the other hand, for $x \in [\bar h(t) - \tilde\epsilon, \bar h(t)]$, by $(\mathbf{f_2})$ we obtain

$$
\begin{aligned}
&(1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))\\
&\succeq F((1+\epsilon)\Phi(x-\bar h(t))) - F(\overline{U}(t,x)) = F(\overline{U}(t,x)-\rho(t,x)) - F(\overline{U}(t,x)),
\end{aligned}
$$

and

$$
\mathbf{0} \preceq \overline{U}(t,x) \preceq (1+\epsilon)\Phi(\tilde\epsilon) + K_4\epsilon V^* \preceq 2\Phi(\tilde\epsilon) + \theta^{-\beta}V^*,
$$

So the components of $\overline{U}(t,x)$ and $\rho(t,x)$ are small for small $\tilde\epsilon$ and large θ. It follows that

$$
\begin{aligned}
&F(\overline{U}(t,x)-\rho(t,x)) - F(\overline{U}(t,x)) = -\rho(t,x)[\nabla F(\overline{U}(t,x)) + o(1)\mathbf{I}_m]\\
&= -\rho(t,x)[\nabla F(0) + o(1)\mathbf{I}_m] \succeq -\rho(t,x)\nabla F(0) + \frac{\tilde\lambda_1}{4}\rho(t,x)
\end{aligned}
$$

for $x \in [\bar h(t) - \tilde\epsilon, \bar h(t)]$, provided that $\tilde\epsilon$ is small and θ is large. Hence, (4.16) holds.

Denote

$$
M := \max_{1\le i\le m}\sup_{x\le 0}|\phi_i'(x)|.
$$

For $x \in [\bar h - \tilde\epsilon, \bar h]$, by (4.16) we have

$$
\begin{aligned}
A(t,x) \succeq& \frac{|\tilde\lambda_1|}{4}\rho(t,x) - (1+\epsilon)\delta'(t)\Phi'(x-\bar h(t)) + \epsilon'(t)\Phi(x-\underline{h}(t)) + \rho_t(t,x)\\
\succeq& \epsilon(t)\left[\frac{|\tilde\lambda_1|}{4}K_4V^* - 2K_3M\mathbf{1} - \beta(t+\theta)^{-1}\mathbf{u}^* - K_4\beta(t+\theta)^{-1}V^*\right]\\
\succeq& \epsilon(t)\left[\frac{|\tilde\lambda_1|}{4}K_4V^* - 2K_3M\mathbf{1} - \theta^{-1}\beta\Big(\mathbf{u}^* + K_4V^*\Big)\right] \succeq \mathbf{0}
\end{aligned}
$$

provided that we first fix K_3 and K_4 so that (4.15) holds and at the same time

$$
\frac{|\tilde\lambda_1|}{4}K_4V^* - 2K_3M\mathbf{1} \succ\!\succ \mathbf{0}, \tag{4.17}
$$

and then choose θ sufficiently large.

Next, for fixed small $\tilde\epsilon > 0$, we estimate $A(t,x)$ for $x \in [g(t+t_0), \bar h(t) - \tilde\epsilon]$.

Claim 2. For any given $1 \gg \eta > 0$, there is $c_1 = c_1(\eta)$ such that

$$
(1+\epsilon)F(v) - F((1+\epsilon)v) \succeq c_1\epsilon\mathbf{1} \quad\text{for}\quad v \in [\eta\mathbf{1}, \mathbf{u}^*] \text{ and } 0 < \epsilon \ll 1. \tag{4.18}
$$

Indeed, by (1.6) there exists $c_1 > 0$ depending on η such that

$$
F(v) - v[\nabla F(v)]^T \succeq 2c_1\mathbf{1} \quad\text{for}\quad v \in [\eta\mathbf{1}, \mathbf{u}^*].
$$

Since

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{(1+\epsilon)F(v) - F((1+\epsilon)v)}{\epsilon} =& \lim_{\epsilon\to 0}\frac{\epsilon F(v) - [F(v+\epsilon v) - F(v)]}{\epsilon}\\
=& F(v) - v[\nabla F(v)]^T \succeq 2c_1\mathbf{1}
\end{aligned}
$$

uniformly for $v \in [\eta \mathbf{1}, \mathbf{u}^*]$, there exists $\epsilon_0 > 0$ small so that

$$\frac{(1+\epsilon)F(v) - F((1+\epsilon)v)}{\epsilon} \succeq c_1 \mathbf{1}$$

for $v \in [\eta \mathbf{1}, \mathbf{u}^*]$ and $\epsilon \in (0, \epsilon_0]$. This proves Claim 2.

By Claim 2 and the Lipschitz continuity of F, there exist positive constants C_l and C_f such that, for $v = \Phi(x - \bar{h}(t)) \in [\Phi(-\tilde{\epsilon}), \mathbf{u}^*]$,

$$\begin{aligned} &(1+\epsilon)F(v) - F((1+\epsilon)v + \rho) \\ &= (1+\epsilon)F(v) - F((1+\epsilon)v) + F((1+\epsilon)v) - F((1+\epsilon)v + \rho) \succeq C_l \epsilon \mathbf{1} - C_f K_4 \epsilon \mathbf{1} \end{aligned}$$

when $\epsilon = \epsilon(t)$ is small.

We also have

$$\begin{aligned} &D \circ \left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,x) \mathrm{d}y\right] \succeq -D \circ \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,x) \mathrm{d}y \\ &\succeq -K_4 \epsilon(t) D \circ V^* \succeq -C_d K_4 \epsilon(t) \mathbf{1} \text{ for some } C_d > 0, \end{aligned}$$

and

$$\begin{aligned} \rho_t(t,x) &= -\xi' \bar{h}' K_4 \epsilon(t) V^* + \xi K_4 \epsilon'(t) V^* \succeq -\xi_* K_4 \epsilon(t) V^* - K_4 \beta (t+\theta)^{-1} \epsilon(t) V^* \\ &\succeq -(\xi_* + \beta \theta^{-1}) K_4 \epsilon(t) V^* \text{ with } \xi_* := c_0 \max_{x \in \mathbb{R}} |\xi'(x)|. \end{aligned}$$

Using these we obtain, for $x \in [g(t_0 + t), \bar{h}(t) - \tilde{\epsilon}]$,

$$\begin{aligned} A(t,x) &\succeq -C_d K_4 \epsilon(t) \mathbf{1} + (1+\epsilon) F(\Phi(x - \bar{h}(t))) - F(\bar{U}(t,x)) + 2M\delta'(t)\mathbf{1} + \epsilon'(t)\mathbf{u}^* + \rho_t(t,x) \\ &\succeq C_l \epsilon(t) \mathbf{1} - (C_f + C_d) K_4 \epsilon(t) \mathbf{1} - 2MK_3 \epsilon(t) \mathbf{1} \\ &\quad - \beta(t+\theta)^{-1} \epsilon(t) \mathbf{u}^* - (\xi_* + \beta\theta^{-1}) K_4 \epsilon(t) V^* \\ &= \epsilon(t) \Big[C_l \mathbf{1} - K_4 (C_f + C_d) \mathbf{1} - 2MK_3 \mathbf{1} - \beta(t+\theta)^{-1} \mathbf{u}^* - (\xi_* + \beta\theta^{-1}) K_4 V^*\Big] \\ &\succeq \epsilon(t) \Big[C_l \mathbf{1} - K_4 (C_f + C_d) \mathbf{1} - 2MK_3 \mathbf{1} - \xi_* K_4 V^* - \beta\theta^{-1}\big(\mathbf{u}^* + K_4 V^*\big)\Big] \succeq \mathbf{0} \end{aligned}$$

provided that we first choose K_3 and K_4 small such that

$$C_l \mathbf{1} - K_4 (C_f + C_d) \mathbf{1} - 2MK_3 \mathbf{1} - \xi_* K_4 V^* \gg \mathbf{0}$$

while keeping both (4.15) and (4.17) hold, and then choose $\theta > 0$ sufficiently large.

Therefore, (4.5) holds when K_3, K_4 and θ are chosen as above. The proof of the lemma is now complete. □

4.2. The case $\tilde{\lambda}_1 \geq 0$

Lemma 4.2. *Suppose that the conditions in Theorem 4.1 are satisfied. If $\tilde{\lambda}_1 \geq 0$, then* (4.3) *still holds.*

Proof. This is a modification of the proof of Lemma 4.1. We will use similar notations. Let $\beta = \gamma - 2 \in (0, 1]$, and (c_0, Φ) be the solution of (1.4)-(1.5). For fixed $\tilde{\epsilon} > 0$, let $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon}, \ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}.$$

Define

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), & t \ge 0, \\ \overline{U}(t,x) := (1+\epsilon(t))\Phi\big(x - \bar{h}(t) - \lambda(t)\big) - \rho(t,x), & t \ge 0,\ x \le \bar{h}(t), \end{cases}$$

where

$$\epsilon(t) := K_1(t+\theta)^{-\beta},\ \delta(t) := K_2 - K_3\int_0^t \epsilon(\tau)\mathrm{d}\tau,$$
$$\rho(t,x) := K_4\xi(x-\bar{h}(t))\epsilon(t)V^*,\ \lambda(t) := K_5\epsilon(t),$$

and the positive constants θ and K_1, K_2, K_3, K_4, K_5 are to be determined.

Let

$$C_{\tilde{\epsilon}} := \min_{1\le i\le m}\min_{x\in[-2\tilde{\epsilon},0]}|\phi_i'(x)|.$$

Then for $x \in [\bar{h}(t) - 2\tilde{\epsilon}, \bar{h}(t)]$ and $i \in \{1, ..., m\}$, with $\rho(t,x) = (\rho_i(t,x))$,

$$\bar{u}_i(t,x) \ge \phi_i\big(-\lambda(t)\big) - \rho_i(t,x) \ge C_{\tilde{\epsilon}}\lambda(t) - K_4\epsilon(t)v_i^* \ge \epsilon(t)(C_{\tilde{\epsilon}}K_5 - K_4 v_i^*) > 0$$

if

$$K_4 = C_{\tilde{\epsilon}}K_5/(2\max_{1\le i\le m} v_i^*), \tag{4.19}$$

which combined with $\xi(x) = 0$ for $|x| \ge 2\tilde{\epsilon}$ implies

$$\overline{U}(t,x) \succeq \mathbf{0} \text{ for } \ t \ge 0,\ x \le \bar{h}(t). \tag{4.20}$$

Let $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ be given by Step 1 in the proof of Lemma 4.1. Then $[g(t_0), h(t_0)] \subset (-\infty, K_2/2)$, and due to $\rho(0,x) = 0$ for $x \le h(t_0) < K_2/2 < K_2 = \bar{h}(0)$, we have

$$\begin{aligned} \overline{U}(0,x) =& (1+\epsilon(0))\Phi(x - K_2 - \lambda) \succeq (1+\epsilon(0))\Phi(-K_2/2) \\ \succeq& (1+\epsilon(0)/2)\mathbf{u}^* \succeq U(t_0,x) \ \text{ for } \ x \in [g(t_0), h(t_0)]. \end{aligned} \tag{4.21}$$

Step 1. We verify that by choosing K_1, K_3 and K_5 suitably small,

$$\bar{h}'(t) \ge \sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x \ \text{ for all } \ t > 0. \tag{4.22}$$

By direct calculations we have

$$\begin{aligned} &\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x \\ \le& \sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J_i(x-y)(1+\epsilon)\phi_i(x-\bar{h}(t)-\lambda(t))\mathrm{d}y\mathrm{d}x \\ =& (1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)\phi_i(x-\lambda(t))\mathrm{d}y\mathrm{d}x \\ &-(1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar{h}(t)}\int_{0}^{+\infty} J_i(x-y)\phi_i(x-\lambda(t))\mathrm{d}y\mathrm{d}x \end{aligned}$$

$$\le (1+\epsilon)c_0 + (1+\epsilon)\sum_{i=1}^{m_0}\mu_i \int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)[\phi_i(x-\lambda)-\phi_i(x)]\mathrm{d}y\mathrm{d}x$$
$$-(1+\epsilon)\sum_{i=1}^{m_0}\mu_i \int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x$$

Let $M_1 := \max_{1\le i\le m}\sup_{x\le 0}|\phi_i'(x)|$ and C_3 be given by (4.11). Then

$$(1+\epsilon)\sum_{i=1}^{m_0}\mu_i \int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)[\phi_i(x-\lambda(t))-\phi_i(x)]\mathrm{d}y\mathrm{d}x \le 2C_3M_1\lambda(t).$$

By (4.13),

$$\sum_{i=1}^{m_0}\mu_i \int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x$$
$$\ge \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}\left[t+\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)}\right]^{-\beta}.$$

Therefore, as in the proof of Lemma 4.1, for sufficiently large θ so that

$$\theta > \frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)} \tag{4.23}$$

holds, we have

$$\sum_{i=1}^{m_0}\mu_i \int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty} J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x$$
$$\le (1+\epsilon)c_0 + 2C_3M_1\lambda(t) - \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}$$
$$= c_0 + \epsilon(t)\left[c_0 + 2C_3M_1K_5 - \frac{\phi_* C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right] \le c_0 - K_3\epsilon(t) = \bar h'(t)$$

provided that K_1, K_3 and K_5 are suitably small so that

$$K_1(c_0+2C_3M_1K_5+K_3) \le \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}. \tag{4.24}$$

Step 2. We show that by choosing K_3, K_5 suitably small and θ sufficiently large, for $t>0$, $x\in[g(t+t_0),\bar h(t)]$,

$$\overline{U}_t(t,x) \succeq D\circ\int_{g(t+t_0)}^{\bar h(t)} \mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y - \overline{U}(t,x) + F(\overline{U}(t,x)). \tag{4.25}$$

Using the definition of $\overline{U}$, we have

$$\overline{U}_t(t,x) = -(1+\epsilon)(\bar h' + \lambda')\Phi'(x-\bar h-\lambda) + \epsilon'\Phi(x-\bar h-\lambda) - \rho_t$$
$$= -(1+\epsilon)[c_0+\delta'+\lambda']\Phi'(x-\bar h-\lambda) + \epsilon'\Phi(x-\bar h-\lambda) - \rho_t$$

and from (1.4), we obtain

$$
\begin{aligned}
&-(1+\epsilon)c_0\Phi'(x-\bar{h}-\lambda)\\
=&(1+\epsilon)\Big[D\circ\int_{-\infty}^{\bar{h}+\lambda}\mathbf{J}(x-y)\circ\Phi(y-\bar{h}-\lambda)\mathrm{d}y-D\circ\Phi(x-\bar{h}-\lambda)+F(\Phi(x-\bar{h}-\lambda))\Big]\\
\succeq&(1+\epsilon)\Big[D\circ\int_{-\infty}^{\bar{h}}\mathbf{J}(x-y)\circ\Phi(y-\bar{h}-\lambda)\mathrm{d}y-D\circ\Phi(x-\bar{h}-\lambda)+F(\Phi(x-\bar{h}-\lambda))\Big]\\
=&D\circ\int_{-\infty}^{\bar{h}}\mathbf{J}(x-y)\circ[\overline{U}(t,y)+\rho]\mathrm{d}y-D\circ[\overline{U}(t,x)+\rho]+(1+\epsilon)F(\Phi(x-\bar{h}-\lambda))\\
=&D\circ\int_{-\infty}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)\\
&-D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar{h}-\lambda))\\
\succeq&D\circ\int_{g(t+t_0)}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&-D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar{h}-\lambda))-F(\overline{U}(t,x)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{U}_t(t,x)\succeq&D\circ\int_{g(t+t_0)}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&+B(t,x)
\end{aligned}
$$

with

$$
\begin{aligned}
B(t,x):=&-D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar{h}}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]\\
&+(1+\epsilon)F(\Phi(x-\bar{h}-\lambda))-F(\overline{U})\\
&-(1+\epsilon)(\delta'+\lambda')\Phi'(x-\bar{h}-\lambda)+\epsilon'\Phi(x-\underline{h}-\lambda)-\rho_t.
\end{aligned}
$$

To show (4.25), it remains to choose suitable K_3, K_5 and θ such that $B(t,x)\succeq\mathbf{0}$ for $t>0$ and $x\in[g(t+t_0),\bar{h}(t)]$.

Claim: There exist small $\tilde{\epsilon}_0\in(0,\tilde{\epsilon}/2)$ and some $\tilde{J}_0>0$ depending on $\tilde{\epsilon}$ but independent of $\tilde{\epsilon}_0$, such that

$$
\begin{aligned}
&-D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar{h}}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar{h}-\lambda))-F(\overline{U}(t,x))\\
&\succeq\ \tilde{J}_0\,\rho(t,x)\ \ \text{for}\ \ x\in[\bar{h}(t)-\tilde{\epsilon}_0,\bar{h}(t)].
\end{aligned}
\tag{4.26}
$$

Indeed, for $x\in[\bar{h}(t)-\tilde{\epsilon}_0,\bar{h}(t)]$,

$$
\begin{aligned}
&D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\Big]\\
=&\ K_4\epsilon(t)\Big[D\circ V^*-D\circ\int_{-\infty}^{\bar{h}(t)}\mathbf{J}(x-y)\circ\xi(y-\bar{h}(t))V^*\mathrm{d}y\Big]\\
\preceq&\ K_4\epsilon(t)\Big[D\circ V^*-D\circ\int_{\bar{h}(t)-\tilde{\epsilon}}^{\bar{h}(t)}\mathbf{J}(x-y)\circ V^*\mathrm{d}y\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= K_4\epsilon(t)\Big[D\circ V^* - D\circ\int_{\bar h(t)-\tilde\epsilon-x}^{\bar h(t)-x}\mathbf{J}(y)\circ V^*\mathrm{d}y\Big]\\
&\preceq D\circ\rho\Big[1-\int_{-\tilde\epsilon+\tilde\epsilon_0}^{0}\mathbf{J}(y)\mathrm{d}y\Big]\preceq D\circ\rho\Big[1-\int_{-\tilde\epsilon/2}^{0}\mathbf{J}(y)\mathrm{d}y\Big].
\end{aligned}
$$

On the other hand, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$, we have

$$
\begin{aligned}
&(1+\epsilon)F(\Phi(x-\bar h-\lambda)-F(\overline{U})\succeq F((1+\epsilon)\Phi(x-\bar h-\lambda))-F(\overline{U})\\
&=F(\overline{U}+\rho)-F(\overline{U})=\rho\Big([\nabla F(\overline{U})]^T+o(1)\mathbf{I}_m\Big)=K_4\epsilon(t)V^*\Big([\nabla F(\mathbf{0})]^T+o(1)\mathbf{I}_m\Big)\\
&=K_4\epsilon(t)[V^*\tilde D+\tilde\lambda_1V^*+o(1)V^*]=K_4\epsilon(t)[D\circ V^*+\tilde\lambda_1V^*+o(1)V^*]\\
&=D\circ\rho+\tilde\lambda_1\rho+o(1)\rho.
\end{aligned}
$$

since both $\overline{U}(t,x)$ and $\rho(t,x)$ are close to $\mathbf{0}$ for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$ with $\tilde\epsilon_0$ small.

Hence, for such x and $\tilde\epsilon_0$, since $\tilde\lambda_1\geq 0$,

$$
\begin{aligned}
&-D\circ\Big[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\Big]+(1+\epsilon)F(\Phi(x-\bar h(t)))-F(\overline{U}(t,x))\\
&\succeq D\circ\rho\Big[-1+\int_{-\tilde\epsilon/2}^{0}\mathbf{J}(y)\mathrm{d}y\Big]+D\circ\rho+\tilde\lambda_1\rho+o(1)\rho\\
&\succeq \tilde J_0\,\rho(t,x),\qquad \text{with}\quad \tilde J_0:=\frac12\min_{1\leq i\leq m}d_i\int_{-\tilde\epsilon/2}^{0}J_i(y)\mathrm{d}y \text{ if } m_0=m.
\end{aligned}
$$

This proves (4.26) when $m_0=m$.

If $m_0<m$, we need to modify V^* in the definition of ρ slightly. In this case, for $\tilde\delta>0$ small we define

$$\tilde V^*:=V^*+\tilde\delta D=(v_i^*+\tilde\delta d_i).$$

Since $d_i=0$ for $i=m_0+1,...,m$ and $d_i>0$ for $i=1,...,m_0$, by $(\mathbf{f_1})$ (iv) we see that

$$W=(w_i):=D[\nabla F(\mathbf{0})]^T$$

satisfies $w_i>0$ for $i=m_0+1,...,m$. Let us write

$$W=W^1+W^2=(w_i^1)+(w_i^2)\text{ with }\begin{cases}w_i^1=0\text{ for }i=m_0+1,...,m,\\ w_i^2=0\text{ for }i=1,...,m_0.\end{cases}$$

Then

$$\tilde V^*\Big([\nabla F(\mathbf{0})]^T-\tilde D\Big)=\tilde\lambda_1V^*+\tilde\delta\widetilde{W}^1+\tilde\delta W^2\quad\text{with }\widetilde{W}^1:=W^1-D\tilde D.$$

It is important to observe that the vector $\widetilde{W}^1=(\tilde w_i^1)$ has its last $m-m_0$ components 0, namely $\tilde w_i^1=0$ for $i=m_0+1,...,m$.

Replacing V^* by $\tilde V^*$ in the definition of ρ, we see that the analysis above is not affected, except that, for $\tilde\epsilon_0>0$ small and $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$,

$$
\begin{aligned}
&(1+\epsilon)F(\Phi(x-\bar h-\lambda)-F(\overline{U})\succeq K_4\epsilon(t)\tilde V^*\Big([\nabla F(\mathbf{0})]^T+o(1)\mathbf{I}_m\Big)\\
&=K_4\epsilon(t)\Big([\tilde V^*\tilde D+\tilde\lambda_1V^*+o(1)V^*]+\tilde\delta\widetilde{W}^1+\tilde\delta W^2\Big)
\end{aligned}
$$

$$= K_4\epsilon(t)\Big(D\circ\tilde V^* + \tilde\lambda_1 V^* + o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big)$$
$$\succeq D\circ\rho + K_4\epsilon(t)\Big(o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big).$$

Hence, for such x and $\tilde\epsilon_0$, we now have

$$-D\circ\Big[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\Big] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))$$
$$\succeq D\circ\rho\Big[-1 + \int_{-\tilde\epsilon/2}^{0} \mathbf{J}(y)\mathrm{d}y\Big] + D\circ\rho + K_4\epsilon(t)\Big[o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big]$$
$$\succeq K_4\epsilon(t)\Big[\min_{1\le i\le m_0} v_i^* \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}yD + o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big].$$

We now fix $\tilde\delta > 0$ small enough such that

$$-\tilde\delta\widetilde{W}^1 \preceq \frac{1}{2}\min_{1\le i\le m_0} v_i^* d_i \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y,$$

and notice that

$$\widehat{W} := \frac{1}{2}\min_{1\le i\le m_0} v_i^* d_i \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y + \tilde\delta W^2 \gg \mathbf{0}.$$

Therefore there exists $\tilde J_0 > 0$ such that

$$\frac{1}{2}\widehat{W} \succeq \tilde J_0 \tilde V^*.$$

Then

$$K_4\epsilon(t)\Big[\min_{1\le i\le m_0} v_i^* d_i \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y + o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big]$$
$$\succeq K_4\epsilon(t)\Big[\widehat{W} + o(1)V^*\Big] \succeq K_4\epsilon(t)\frac{1}{2}\widehat{W} \succeq K_4\epsilon(t)\tilde J_0\tilde V^* = \tilde J_0\rho,$$

provided that $\tilde\epsilon_0 > 0$ is chosen sufficiently small.

Therefore for $\tilde\epsilon_0 > 0$ small and $x \in [\bar h(t) - \tilde\epsilon_0, \bar h(t)]$, we finally have

$$-D\circ\Big[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\Big] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))$$
$$\succeq \tilde J_0\,\rho(t,x), \quad \text{as desired.}$$

With $\tilde\delta > 0$ chosen as above, we will from now on denote

$$\hat V^* := \begin{cases} V^* & \text{if } m_0 = m, \\ \tilde V^* & \text{if } m_0 < m, \end{cases}$$

but keep the notation for ρ unchanged.

Clearly

$$-\rho_t(t,x) = \beta K_4 K_1 (t+\theta)^{-\beta-1}\hat V^* \succeq \mathbf{0}.$$

Recalling $M_1 := \max_{1\le i\le m}\sup_{x\le 0}|\phi_i'(x)|$, we obtain, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$ and small $\tilde\epsilon_0$,

$$\begin{aligned} B(t,x) &\succeq \tilde J_0 K_4\epsilon(t)\hat V^* + 2(\delta'(t)+\lambda'(t))M_1\mathbf{1} + \epsilon'(t)\mathbf{u}^* \\ &= \tilde J_0 K_4\epsilon(t)\hat V^* + 2\epsilon(t)(-K_3 - K_5\beta(t+\theta)^{-1})M_1\mathbf{1} - \beta(t+\theta)^{-1}\epsilon(t)\mathbf{u}^* \\ &\succeq \epsilon(t)\Big[\tilde J_0 K_4\hat V^* - 2(K_3+K_5\beta\theta^{-1})M_1\mathbf{1} - \beta\theta^{-1}\mathbf{u}^*\Big] \\ &= \epsilon(t)\Big[\tilde J_0 K_4\hat V^* - 2K_3M_1\mathbf{1} - \theta^{-1}\Big(K_5\beta M_1\mathbf{1} + \beta\mathbf{u}^*\Big)\Big] \succeq \mathbf{0} \end{aligned}$$

provided that K_3 is chosen small so that (4.24) holds,

$$\tilde J_0 K_4\hat V^* - 2K_3M_1\mathbf{1} \succ\!\succ \mathbf{0}, \tag{4.27}$$

and θ is chosen sufficiently large.

We next estimate $B(t,x)$ for $x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0]$. From Claim 2 in the proof of Lemma 4.1, and the Lipschitz continuity of F, there exist positive constants $C_l = C_l(\tilde\epsilon_0)$ and C_f such that, for $v=\Phi(x-\bar h(t-\lambda(t)))\in[\Phi(-\tilde\epsilon_0),\mathbf{u}^*]$,

$$\begin{aligned} &(1+\epsilon)F(v) - F((1+\epsilon)v-\rho) \\ &= (1+\epsilon)F(v) - F((1+\epsilon)v) + F((1+\epsilon)v) - F((1+\epsilon)v-\rho) \\ &\succeq C_l\epsilon\mathbf{1} - C_f\rho \succeq C_l\epsilon\mathbf{1} - C_fK_4\epsilon\hat V^* \end{aligned}$$

when $\epsilon=\epsilon(t)$ is small. Hence

$$\begin{aligned} &(1+\epsilon)F(\Phi(x-\bar h-\lambda)) - F(\bar U) \\ &\succeq C_l\epsilon\mathbf{1} - C_fK_4\epsilon\hat V^* \quad \text{for} \quad x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0],\ 0<\tilde\epsilon_0\ll 1. \end{aligned}$$

Clearly,

$$-D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ\rho(t,x)\mathrm{d}y\right] \succeq -K_4\epsilon(t)D\circ\hat V^*,$$

and

$$\rho_t(t,x) = -K_4\xi'\bar h'(t)\epsilon(t)\hat V^* + K_4\xi\epsilon'(t)\hat V^* \preceq \xi_*K_4\epsilon(t)\hat V^*$$

with $\xi_* := c_0\max_{x\in\mathbb{R}}|\xi'(x)|$.

We thus obtain, for $x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0]$ and $0<\tilde\epsilon_0\ll 1$,

$$\begin{aligned} B(t,x) &\succeq -K_4\epsilon(t)D\circ\hat V^* + (1+\epsilon)F(\phi(x-\bar h)) - F(\overline{U}) + 2M_1(\delta'+\lambda')\mathbf{1} + \epsilon'\mathbf{u}^* - \rho_t \\ &\succeq C_l\epsilon(t)\mathbf{1} - K_4\epsilon(t)(D\circ\hat V^* + C_f\hat V^* + \xi_*\hat V^*) + 2M_1(-K_3\epsilon(t) + K_5\epsilon'(t))\mathbf{1} + \epsilon'(t)\mathbf{u}^* \\ &\succeq \epsilon(t)\Big[C_l\mathbf{1} - K_4(D\circ\hat V^* + C_f\hat V^* + \xi_*\hat V^*) \\ &\quad - 2M_1(K_3+K_5\beta(t+\theta)^{-1})\mathbf{1} - \beta(t+\theta)^{-1}\mathbf{u}^*\Big] \\ &\succeq \epsilon(t)\Big[C_l\mathbf{1} - K_4\Big(D\circ\hat V^* + C_f\hat V^* + \xi_*\hat V^*\Big) - 2M_1K_3\mathbf{1} - \theta^{-1}\beta\Big(2M_1K_5\mathbf{1} + \mathbf{u}^*\Big)\Big] \\ &\succeq \mathbf{0} \end{aligned}$$

if we choose K_3 and K_5 small so that (4.24) and (4.27) hold and at the same time, due to (4.19)

$$C_l\mathbf{1} - K_4\Big(D\circ\hat V^* + C_f\hat V^* + \xi_*\hat V^*\Big) - 2M_1K_3\mathbf{1} \succ\!\succ \mathbf{0},$$

and then choose θ sufficiently large. Hence, (4.25) is satisfied if K_3 and K_5 are chosen small as above, and θ is sufficiently large.

From (4.20), we have

$$\overline{U}(t, g(t+t_0)) \succeq \mathbf{0}, \quad \overline{U}(t, \bar{h}(t)) \succeq \mathbf{0} \quad \text{for} \quad t \geq 0.$$

Together with (4.21), (4.22) and (4.25), this enables us to use the comparison principle (Lemma 2.3 of [11]) to conclude that

$$h(t+t_0) \leq \bar{h}(t), \; U(t+t_0, x) \preceq \overline{U}(t, x) \quad \text{for} \quad t \geq 0, \; x \in [g(t+t_0), \underline{h}(t)],$$

which implies (4.3). The proof of the lemma is now complete. □

4.3. Completion of the proof of Theorem 1.2

As already mentioned, Theorem 1.2 follows directly from Theorem 4.1. We now prove the latter.

By (4.1) and Lemma 3.1, there exists $C_0 > 0$ such that

$$\begin{aligned} h(t) - c_0 t \geq & - C\Big[1 + \int_0^t (1+x)^{2-\gamma} dx + \int_0^{\frac{c_0}{2}t} x^2 \hat{J}(x) dx + t\int_{\frac{c_0}{2}t}^{\infty} x\hat{J}(x)dx\Big] \\ \geq & - C\Big[1 + \int_1^{2t} x^{2-\gamma} dx + \int_0^1 \hat{J}(x)dx + C_0 \int_1^{\frac{c_0}{2}t} x^{2-\gamma} dx + C_0 t \int_{\frac{c_0}{2}t}^{\infty} x^{1-\gamma} dx\Big]. \end{aligned}$$

Therefore when $\gamma \in (2, 3)$ we have, for $t \geq 1$,

$$h(t) - c_0 t \geq - C\left[\tilde{C} + \ln(t+1) + \tilde{C}_1 t^{3-\gamma}\right] \geq -\hat{C}_1 t^{3-\gamma}$$

for some $\hat{C}_1, \tilde{C}, \tilde{C}_1 > 0$, and when $\gamma = 3$, for $t \geq 1$,

$$h(t) - c_0 t \geq - \hat{C}_2 \ln t$$

for some $\hat{C}_2 > 0$. This combined with Lemmas 4.1 and 4.2 gives the desired conclusion of Theorem 4.1. The proof is completed. □

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