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Some Hermite-Hadamard Type Inequalities with the Aid of Newly Defined Double Post-Quantum Integrals

Hasan Kara[1,†], Hüseyin Budak[2,3], Fatih Hezenci[1] and Muhammad Uzair Awan[4]

Abstract In the present paper, we first give four post-quantum integrals for functions of two variables, denoted by ${}_{ac}T_{p,q}$, ${}_{a}^{d}T_{p,q}$, ${}_{c}^{b}T_{p,q}$ and ${}^{bd}T_{p,q}$. Afterwards, each of these newly defined integrals is illustrated. Moreover, some new Hermite-Hadamard inequalities are established based on these definitions. We also show the correctness of these inequalities with the aid of some numerical examples.

Keywords Hermite-Hadamard inequality, q-integral, T_q-integral, quantum calculus, co-ordinated convexity

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1. Introduction

Quantum calculus has been the cornerstone of physics and mathematics. In particular, quantum integrals have solved many problems in the literature. After F. H. Jackson described the q-Jackson integral in [12], this topic has attracted the attention of many mathematicians. Quantum calculus and related properties are discussed in [14] by P. Cheung and V. Kac. Tariboon and Ntouyas obtained several q-analogues of classical mathematics topics in [20]. Agarwal described the q-fractional derivative in [1]. Noor et al. created new q-analogues of the inequalities using the q-differentiable convex function in [19]. Tariboon and Notuyas introduced q_a-definite integral in [20]. Alp et al. obtained q_a-Hermite-Hadamard inequalities with convex functions on quantum integral in [5]. Bermudo et al. introduced a new quantum integral concept called q^b-integral and presented the related Hermite-Hadamard type inequalities in [6]. In addition, the authors established new inequalities that include q_a and q^b integrals together. In the paper [18], Latif offered q_{ac}-integral and properties this of integral for two variables functions. The author also worked on the Hermite-Hadamard inequality based on this definition. However, Alp and

† The corresponding author.
Email address: hasan64kara@gmail.com (H. Kara), hsyn.budak@gmail.com (H. Budak), fatihezenci@gmail.com (F. Hezenci), awan.uzair@gmail.com (M. U. Awan)
1 Department of Mathematics, Faculty of Science and Arts, Duzce University, Türkiye
2 Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India
3 Department of Mathematics, Faculty of Sciences and Art, Kocaeli University, Kocaeli 41001, Türkiye
4 Department of Mathematics, Government College University, Faisalabad, Pakistan

Sarıkaya revised the minor errors in this inequality and established a new inequality in [3]. Budak et al. described new q^b-integrals for two variables in [9]. The authors also proved three different Hermite-Hadamard inequalities through the three new integrals, namely q_a^d, q_c^b and q^{bd}-integrals. Nowadays, there are many studies on integral inequalities and quantum integrals (see, [2], [8], [10], [11], [17], [22]).

On the other hand, using the facts of trpezoid areas in [4], Alp and Sarıkaya investigated the generalized quantum integral, expressed as the ${}_aT_q$-integral. In addition, the Hermite-Hadamard inequality in the case of this definition is also constructed by the authors. In the paper [16], Kara et al. presented the generalized quantum integral stated as the ${}^bT_q$-integral involving areas of the trapezoids. With the help of the given definition of this paper, the researchers obtained the new Hermite-Hadamard inequalities. Moreover, Kara and Budak established new T_q-integrals by two variables in [15]. In addition to these, they were also proved to correspond to four Hermite-Hadamard type inequalities on co-ordinates. Vivas-Cortez et al. considered to the generalized post-quantum integral, called the ${}_aT_{p,q}$-integral in [21]. More precisely, the authors also investigated new Hermite-Hadamard-type inequalities in the case of ${}_aT_{p,q}$-integral. Budak et al. demonstrated a new concept of post quantum integral, namely ${}^bT_{p,q}$-integral, in [7]. They also provided several Hermite-Hadamard inequalities to the case of ${}^bT_{p,q}$-integral by using convex functions. The definitions and theorems mentioned in this paragraph and that used in the article are detailed in the following section.

2. T_q-integrals and $T_{p,q}$-integrals

This section presents the desired definitions and related inequalities.

From the fact of the area of trapezoids, Alp and Sarikaya investigated the generalized quantum integral, which is called the ${}_aT_q$-integral as follows:

Definition 2.1 (see, [4])**.** Suppose that $\Psi : [a, b] \to \mathbb{R}$ is a continuous function. For $\varkappa \in [a, b]$, it follows that

$$\int_a^b \Psi(s)\ {}_a d_q^T s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \Psi(q^n b + (1-q^n)a) - \Psi(b)\right],$$

where $0 < q < 1$.

Because of the ${}_aT_q$-integrals, Hermite-Hadamard inequalities are also as follows:

Theorem 2.1 (see, [4])**.** *[${}_aT_q$-Hermite-Hadamard Inequalities] If $\Psi : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ and $0 < q < 1$, then we get the following double inequality*

$$\Psi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \Psi(\varkappa)\ {}_a d_q^T \varkappa \le \frac{\Psi(a)+\Psi(b)}{2}. \tag{2.1}$$

By using the area of trapezoids, Kara et al. [16] introduced the following generalized quantum integral which is called ${}^bT_q$-integral.

Definition 2.2 (see, [16]). Let us consider that $\Psi:[a,b]\to\mathbb{R}$ is a continuous function. For $\varkappa\in[a,b]$, ${}^{b}T_q$-integral is equal to

$$\int_a^b \Psi(s)\ {}^{b}d_q^T s=\frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\Psi(q^n a+(1-q^n)b)-\Psi(a)\right].$$

Here, $0<q<1$.

In addition, the Hermite-Hadamard inequalities based on ${}^{b}T_q$-integrals are as follows:

Theorem 2.2 (${}^{b}T_q$-Hermite-Hadamard Inequalities). *Let $\Psi:[a,b]\to\mathbb{R}$ denote a convex continuous function on $[a,b]$ and $0<q<1$. Then, the following double inequality holds:*

$$\Psi\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b\Psi(\varkappa)\ {}^{b}d_q^T\varkappa\le\frac{\Psi(a)+\Psi(b)}{2}. \tag{2.2}$$

Vivas-Cortez et al. presented the following definition, designated ${}_{a}T_{p,q}$, with the help of the trapezoid areas.

Definition 2.3 (see, [21]). Let $\Psi:[a,b]\to\mathbb{R}$ be continuous function. For $x\in[a,(1-p)a+pb]$, the following equality holds:

$$\begin{aligned}&\int_a^x\Psi(s)\ {}_{a}d_{p,q}^T s \qquad (2.3)\\&=\frac{(p-q)(x-a)}{2q}\left[(p+q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right.\\&\left.-\Psi\left(\frac{x+(p-1)a}{p}\right)\right],\end{aligned}$$

where $0<q<p\le1$.

Based on this definition, the following Hermite-Hadamard inequality is established.

Theorem 2.3 (${}_{a}T_{p,q}$-Hermite-Hadamard Inequalities). *[21] If $\Psi:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$ and $0<q<p\le1$, then the inequality is obtained as follows:*

$$\Psi\left(\frac{a+b}{2}\right)\le\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}\Psi(\varkappa)\ {}_{a}d_{p,q}^T\varkappa\le\frac{\Psi(a)+\Psi(b)}{2}. \tag{2.4}$$

On the other hand, Budak et al. presented the definition called ${}^{b}T_{p,q}$ below.

Definition 2.4. [7] Assume that $\Psi:[a,b]\to\mathbb{R}$ is continuous function. For $x\in[pa+(1-p)b,b]$, the following equality

$$\int_x^b\Psi(s)\ {}^{b}d_{p,q}^T s \tag{2.5}$$

$$=\frac{(p-q)(b-x)}{2q}\left[(p+q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Psi\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)b\right)\right.$$
$$\left.-\Psi\left(\frac{x+(p-1)}{p}\right)\right]$$

is valid. Here, $0<q<p\leq 1$.

Moreover, with the help of the definition of ${}^{b}T_{p,q}$, Budak proved the following Hermite-Hadamard inequalities.

Theorem 2.4 (${}^{b}T_{p,q}$-Hermite-Hadamard Inequalities)**.** *[7] Let $\Psi:[a,b]\to\mathbb{R}$ be differentiable convex function on $[a,b]$ and $0<q<p\leq 1$. Then, the following double inequality holds:*

$$\Psi\left(\frac{a+b}{2}\right)\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\Psi(\varkappa)\ {}^{b}d_{p,q}^{T}\varkappa\leq\frac{\Psi(a)+\Psi(b)}{2}. \tag{2.6}$$

To be precise, Kara and Budak defined four T_q-integrals for two variables in [15]. More precisely, the authors investigated the Hermite-Hadamard inequalities based on these definitions. In the third part, we introduce post-quantum integrals for two variables. We illustrate each of these definitions with examples. In the fourth section, using these definitions, we obtain four Hermite-Hadamard inequalities. We will also give examples of these inequalities and support them for a better understanding of the interested reader.

3. New post-quantum integrals based on the functions of two variables

In this section, we introduce new $T_{p,q}$-integrals to the case of two-variables functions.

Definition 3.1. Suppose that $\Psi:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is continuous function. Then the following ${}_{ac}T_{p,q}$ ${}_{a}^{d}T_{p,q}$, ${}_{c}^{b}T_{p,q}$ and ${}^{bd}T_{p,q}$-integrals on $[a,b]\times[c,d]$ equal to

$$\int_a^x\int_c^y\Psi(t,s)\ {}_{c}d_{p_2,q_2}^{T}s\ {}_{a}d_{p_1,q_1}^{T}t \tag{3.1}$$
$$=\frac{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}{4q_1q_2}$$
$$\times\left[(p_1+q_1)(p_2+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right.$$
$$\times\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$
$$-(p_2+q_2)\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\Psi\left(\frac{x+(p_1-1)a}{p_1},\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$
$$-(p_1+q_1)\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{y+(p_2-1)c}{p_2}\right)$$

$$+\Psi\left(\frac{x+(p_1-1)a}{p_1},\frac{y+(p_2-1)c}{p_2}\right)\Bigg],$$

for $(x,y)\in[a,(1-p_1)a+p_1b]\times[c,(1-p_2)c+p_2d]$;

$$\int_a^x\int_y^d\Psi(t,s)\ {}^{d}d_{p_2,q_2}^{T}s\ {}_{a}d_{p_1,q_1}^{T}t \tag{3.2}$$
$$=\frac{(p_1-q_1)(p_2-q_2)(x-a)(d-y)}{4q_1q_2}$$
$$\times\Bigg[(p_1+q_1)(p_2+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}$$
$$\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right)$$
$$-(p_2+q_2)\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\Psi\left(\frac{x+(p_1-1)a}{p_1},\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right)$$
$$-(p_1+q_1)\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{y+(p_2-1)d}{p_2}\right)$$
$$+\Psi\left(\frac{x+(p_1-1)a}{p_1},\frac{y+(p_2-1)d}{p_2}\right)\Bigg],$$

for $(x,y)\in[a,(1-p_1)a+p_1b]\times[p_2c+(1-p_2)d,d]$;

$$\int_x^b\int_c^y\Psi(t,s)\ {}_{c}d_{p_2,q_2}^{T}s\ {}^{b}d_{p_1,q_1}^{T}t \tag{3.3}$$
$$=\frac{(p_1-q_1)(p_2-q_2)(b-x)(y-c)}{4q_1q_2}$$
$$\times\Bigg[(p_1+q_1)(p_2+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}$$
$$\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$
$$-(p_2+q_2)\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\Psi\left(\frac{x+(p_1-1)b}{p_1},\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right)$$
$$-(p_1+q_1)\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{y+(p_2-1)c}{p_2}\right)$$
$$+\Psi\left(\frac{x+(p_1-1)b}{p_1},\frac{y+(p_2-1)c}{p_2}\right)\Bigg],$$

for $(x,y)\in[p_1a+(1-p_1)b,b]\times[c,(1-p_2)c+p_2d]$;

$$\int_x^b\int_y^d\Psi(t,s)\ {}^{d}d_{p_2,q_2}^{T}s\ {}^{b}d_{p_1,q_1}^{T}t \tag{3.4}$$

$$
\begin{aligned}
=&\frac{(p_1-q_1)(p_2-q_2)(b-x)(d-y)}{4q_1q_2}\\
&\times\left[(p_1+q_1)(p_2+q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\right.\\
&\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right)\\
&-(p_2+q_2)\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\Psi\left(\frac{x+(p_1-1)b}{p_1},\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right)\\
&-(p_1+q_1)\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\Psi\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{y+(p_2-1)d}{p_2}\right)\\
&\left.+\Psi\left(\frac{x+(p_1-1)b}{p_1},\frac{y+(p_2-1)d}{p_2}\right)\right],
\end{aligned}
$$

for $(x,y)\in[p_1a+(1-p_1)b,b]\times[p_2c+(1-p_2)d,d]$, respectively.

Example 3.1. Let us define a function $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=ts^2$. Then, by equality (3.1) for $q_1=q_2=1/3$, $p_1=p_2=2/3$, $x=(1-p_1)a+p_1b$ and $y=(1-p_2)c+p_2d$, we obtain

$$
\int_0^{\frac{2}{3}}\int_0^{\frac{2}{3}} ts^2\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}s\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t \tag{3.5}
$$

$$
\begin{aligned}
=&\frac{1}{9}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{9}{4}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^m\Psi\left(\left(\frac{1}{2}\right)^n,\left(\frac{1}{2}\right)^m\right)\right.\\
&\left.-\sum_{m=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^m\Psi\left(1,\left(\frac{1}{2}\right)^m\right)-\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\Psi\left(\left(\frac{1}{2}\right)^n,1\right)+\Psi(1,1)\right]\\
=&\frac{1}{9}\left[\frac{24}{7}-\frac{12}{7}-2+1\right]=\frac{5}{63}.
\end{aligned}
$$

Example 3.2. Assume $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=t^2s^2$. If we choose $q_1=q_2=1/3$ and $p_1=p_2=2/3$, $x=(1-p_1)a+p_1b$ and $y=p_2c+(1-p_2)d$ in equality (3.2), then we have

$$
\int_0^{\frac{2}{3}}\int_{\frac{1}{3}}^{1} t^2s^2\,{}^1d^T_{\frac{2}{3},\frac{1}{3}}s\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t \tag{3.6}
$$

$$
\begin{aligned}
=&\frac{1}{9}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{9}{4}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^m\left(\frac{1}{2}\right)^{2n}\left(1-2\left(\frac{1}{2}\right)^m+\left(\frac{1}{2}\right)^{2m}\right)\right.\\
&\left.-\sum_{m=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^m\left(1-2\left(\frac{1}{2}\right)^m+\left(\frac{1}{2}\right)^{2m}\right)-\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2n}0^2+0\right.\\
=&\frac{1}{9}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{9}{4}\left(\frac{1}{2}\right)^{3n}\left(\left(\frac{1}{2}\right)^m-2\left(\frac{1}{2}\right)^{2m}+\left(\frac{1}{2}\right)^{3m}\right)\right.
\end{aligned}
$$

$$-\sum_{m=0}^{\infty}\frac{3}{2}\left(\left(\frac{1}{2}\right)^{m}-2\left(\frac{1}{2}\right)^{2m}+\left(\frac{1}{2}\right)^{3m}\right)+0\Bigg]$$
$$=\frac{1}{9}\left[\frac{9}{4}\frac{8}{7}\left(2-\frac{8}{3}+\frac{8}{7}\right)-\frac{3}{2}\left(2-\frac{8}{3}+\frac{8}{7}\right)\right]=\frac{25}{441}.$$

Example 3.3. Let us note a function $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=ts^{3}$. If we take $q_1=q_2=1/3$ and $p_1=p_2=2/3$, $x=p_1a+(1-p_1)b$ and $y=(1-p_2)c+p_2d$ in equality (3.3), then we obtain

$$\int_{\frac{1}{3}}^{1}\int_{0}^{\frac{2}{3}} ts^{3}\ {}_0d^{T}_{\frac{2}{3},\frac{1}{3}}s\ {}^1d^{T}_{\frac{2}{3},\frac{1}{3}}t \tag{3.7}$$
$$=\frac{1}{9}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{9}{4}\left(\frac{1}{2}\right)^{n}\left(1-\left(\frac{1}{2}\right)^{n}\right)\left(\frac{1}{2}\right)^{4m}-\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^{n}\left(1-\left(\frac{1}{2}\right)^{n}\right)\right]$$
$$=\frac{1}{9}\left[\frac{8}{5}-1\right]=\frac{1}{15}.$$

Example 3.4. Consider $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=t^{2}s$. If we assign $q_1=q_2=1/3$ and $p_1=p_2=2/3$, $x=p_1a+(1-p_1)b$ and $y=p_2c+(1-p_2)d$ in equality (3.4), then we get the following equality

$$\int_{\frac{1}{3}}^{1}\int_{\frac{1}{3}}^{1} t^{2}s\ {}^1d^{T}_{\frac{2}{3},\frac{1}{3}}s\ {}^1d^{T}_{\frac{2}{3},\frac{1}{3}}t \tag{3.8}$$
$$=\frac{1}{9}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{9}{4}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{m}\Psi\left(1-\left(\frac{1}{2}\right)^{n},1-\left(\frac{1}{2}\right)^{m}\right)=\frac{5}{63}.$$

4. Some Hermite-Hadamard inequalities for new post quantum integrals with two variables

In this section, we will establish some new $T_{p,q}$-Hermite-Hadamard inequalities for co-ordinated convex functions.

Theorem 4.1. *Suppose that $\Psi:[a,b]\times[c,d]\subset\mathbb{R}^{2}\to\mathbb{R}$ is a co-ordinated convex function on $[a,b]\times[c,d]$. Then, we obtain the inequalities*

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{4.1}$$
$$\leq\frac{1}{2}\Bigg[\frac{1}{p_1(b-a)}\int_{a}^{(1-p_1)a+p_1b}\Psi\left(\varkappa,\frac{c+d}{2}\right)\ {}_ad^{T}_{p_1,q_1}\varkappa$$
$$+\frac{1}{p_2(d-c)}\int_{c}^{(1-p_2)c+p_2d}\Psi\left(\frac{a+b}{2},\gamma\right)\ {}_cd^{T}_{p_2,q_2}\gamma\Bigg]$$

$$\leq \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{(1-p_1)a+p_1 b} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma) \; {}_c d^T_{p_2,q_2}\gamma \; {}_a d^T_{p_1,q_1}\varkappa$$

$$\leq \frac{1}{4}\left[\frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,c)\, {}_a d^T_{p_1,q_1}\varkappa + \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,d)\, {}_a d^T_{p_1,q_1}\varkappa \right.$$

$$\left. + \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(a,\gamma)\, {}_c d^T_{p_2,q_2}\gamma + \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(b,\gamma)\, {}_c d^T_{p_2,q_2}\gamma \right]$$

$$\leq \frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}$$

for all $0 < q_1 < p_1 \leq 1$ *and* $0 < q_2 < p_2 \leq 1$.

Proof. Let $g_\varkappa : [c,d] \to \mathbb{R}$, $g_\varkappa(\gamma) = \Psi(\varkappa,\gamma)$ be a convex function on $[c,d]$. By using the inequality (2.4) for the interval $[c,(1-p_2)c+p_2 d]$ and $0 < q_2 < p_2 \leq 1$, we get

$$g_\varkappa\left(\frac{c+d}{2}\right) \leq \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} g_\varkappa(\gamma)\, {}_c d^T_{p_2,q_2}\gamma \leq \frac{g_\varkappa(c)+g_\varkappa(d)}{2}$$

i.e.

$$\Psi\left(\varkappa,\frac{c+d}{2}\right) \leq \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma)\, {}_c d^T_{p_2,q_2}\gamma \leq \frac{\Psi(\varkappa,c)+\Psi(\varkappa,d)}{2} \tag{4.2}$$

for all $\varkappa \in [a,b]$. From the facts of ${}_aT_{p_1,q_1}$-integrating the inequality (4.2) on $[a,(1-p_1)a+p_1 b]$ for $0 < q_1 < p_1 \leq 1$, we obtain

$$\begin{aligned} &\frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi\left(\varkappa,\frac{c+d}{2}\right) {}_a d^T_{q_1}\varkappa \qquad (4.3)\\ &\leq \frac{1}{p_1 p_2(b-a)(d-c)} \int_a^{(1-p_1)a+p_1 b} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma)\, {}_c d^T_{p_2,q_2}\gamma\, {}_a d^T_{p_1,q_1}\varkappa \\ &\leq \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \frac{\Psi(\varkappa,c)+\Psi(\varkappa,d)}{2}\, {}_a d^T_{p_1,q_1}\varkappa . \end{aligned}$$

Similarly, let us consider that $g_\gamma : [a,b] \to \mathbb{R}$, $g_\gamma(\varkappa) = \Psi(\varkappa,\gamma)$ is a convex function on $[a,b]$. With the help of the inequality (2.4) for $0 < q_1 < p_1 \leq 1$, we get

$$g_\gamma\left(\frac{a+b}{2}\right) \leq \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} g_\gamma(\varkappa)\, {}_a d^T_{p_1,q_1}\varkappa \leq \frac{g_\gamma(a)+g_\gamma(b)}{2},$$

which gives

$$\Psi\left(\frac{a+b}{2},\gamma\right) \leq \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,\gamma)\ {}_a d^T_{p_1,q_1}\varkappa \leq \frac{\Psi(a,\gamma)+\Psi(b,\gamma)}{2} \tag{4.4}$$

for all $\gamma \in [c,d]$. By using ${}_cT_{q_2}$-integrating the inequality (4.4) on $[c,d]$ for $0 < q_1 < p_1 \leq 1$, we have the following double inequality

$$\begin{aligned} &\frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi\left(\frac{a+b}{2},\gamma\right) {}_c d^T_{p_2,q_2}\gamma \\ \leq &\frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{(1-p_1)a+p_1 b} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma)\ {}_c d^T_{p_2,q_2}\gamma\ {}_a d^T_{p_1,q_1}\varkappa \\ \leq &\frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \frac{\Psi(a,\gamma)+\Psi(b,\gamma)}{2}\ {}_c d^T_{p_2,q_2}\gamma\,. \end{aligned} \tag{4.5}$$

If we collect the inequalities from (4.3) to (4.5), then we have the following inequalities

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi\left(\varkappa,\frac{c+d}{2}\right) {}_a d^T_{p_1,q_1}\varkappa \right. \\ &\left. +\frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi\left(\frac{a+b}{2},\gamma\right) {}_c d^T_{p_2,q_2}\gamma \right] \\ \leq &\frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{(1-p_1)a+p_1 b} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma)\ {}_c d^T_{p_2,q_2}\gamma\ {}_a d^T_{p_1,q_1}\varkappa \\ \leq &\frac{1}{4p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(a,\gamma)\ {}_c d^T_{p_2,q_2}\gamma + \frac{1}{4p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(b,\gamma)\ {}_c d^T_{p_2,q_2}\gamma \\ &+ \frac{1}{4p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,c)\ {}_a d^T_{p_1,q_1}\varkappa + \frac{1}{4p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,d)\ {}_a d^T_{p_1,q_1}\varkappa, \end{aligned} \tag{4.6}$$

which proves the second and third inequalities in (4.1).

From the fact of the first inequality in (2.4), we get

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi\left(\varkappa,\frac{c+d}{2}\right) {}_a d^T_{p_1,q_1}\varkappa \tag{4.7}$$

and

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi\left(\frac{a+b}{2},\gamma\right) {}_c d^T_{p_2,q_2}\gamma\,. \tag{4.8}$$

If we add the inequality (4.7) and (4.8), then we obtain the first inequality in (4.1).

Finally, by using the second inequality in (2.4), we have

$$\frac{1}{4p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(\varkappa,c)\,{}_ad_{p_1,q_1}^T\varkappa\leq\frac{1}{4}\frac{\Psi(a,c)+\Psi(b,c)}{2},\tag{4.9}$$

$$\frac{1}{4p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(\varkappa,d)\,{}_ad_{p_1,q_1}^T\varkappa\leq\frac{1}{4}\frac{\Psi(a,d)+\Psi(b,d)}{2},\tag{4.10}$$

$$\frac{1}{4p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(a,\gamma)\,{}_cd_{p_2,q_2}^T\gamma\leq\frac{1}{4}\frac{\Psi(a,c)+\Psi(a,d)}{2},\tag{4.11}$$

and

$$\frac{1}{4p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(b,\gamma)\,{}_cd_{p_2,q_2}^T\gamma\leq\frac{1}{4}\frac{\Psi(b,c)+\Psi(b,d)}{2}.\tag{4.12}$$

By summing the inequalities from (4.9) to (4.12), we obtain

$$\begin{aligned}&\frac{1}{4p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(\varkappa,c)\,{}_ad_{p_1,q_1}^T\varkappa+\frac{1}{4p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(\varkappa,d)\,{}_ad_{p_1,q_1}^T\varkappa\\&+\frac{1}{4p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(a,\gamma)\,{}_cd_{p_2,q_2}^T\gamma+\frac{1}{4p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(b,\gamma)\,{}_cd_{p_2,q_2}^T\gamma\\\leq&\frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}.\end{aligned}$$

This finishes the proof of Theorem 4.1. □

From the facts of Theorem 2.3 and Theorem 2.4, we can obtain the following Theorem.

Theorem 4.2. *If $\Psi:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a coordinated convex function on $[a,b]\times[c,d]$, then we obtain the following inequalities*

$$\begin{aligned}&\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\\leq&\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi\left(\varkappa,\frac{c+d}{2}\right)\,{}_ad_{p_1,q_1}^T\varkappa\right.\\&\left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},\gamma\right)\,{}^dd_{p_2,q_2}^T\gamma\right]\end{aligned}\tag{4.13}$$

$$\leq \frac{1}{p_1 p_2 (b-a)(d-c)} \int_a^{(1-p_1)a+p_1 b} \int_{p_2 c+(1-p_2)d}^{d} \Psi(\varkappa,\gamma)\ {}^{d}d^T_{p_2,q_2}\gamma\ {}_{a}d^T_{p_1,q_1}\varkappa$$

$$\leq \frac{1}{4}\left[\frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,c)\ {}_{a}d^T_{p_1,q_1}\varkappa + \frac{1}{p_1(b-a)} \int_a^{(1-p_1)a+p_1 b} \Psi(\varkappa,d)\ {}_{a}d^T_{p_1,q_1}\varkappa \right.$$

$$\left. + \frac{1}{p_2(d-c)} \int_{p_2 c+(1-p_2)d}^{d} \Psi(a,\gamma)\ {}^{d}d^T_{p_2,q_2}\gamma + \frac{1}{p_2(d-c)} \int_{p_2 c+(1-p_2)d}^{d} \Psi(b,\gamma)\ {}^{d}d^T_{p_2,q_2}\gamma \right]$$

$$\leq \frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}$$

for all $0 < q_1 < p_1 \leq 1$ *and* $0 < q_2 < p_2 \leq 1$.

Proof. The proof is similar to that of Theorem 4.1 by using Theorem 2.3 and Theorem 2.4. □

Theorem 4.3. *Suppose that* $\Psi : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is a coordinated convex function on* $[a,b] \times [c,d]$. *Then, the following inequalities hold:*

$$\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{4.14}$$

$$\leq \frac{1}{2}\left[\frac{1}{p_1(b-a)} \int_{p_1 a+(1-p_1)b}^{b} \Psi\left(\varkappa, \frac{c+d}{2}\right) {}^{b}d^T_{p_1,q_1}\varkappa \right.$$

$$\left. + \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi\left(\frac{a+b}{2}, \gamma\right) {}_{c}d^T_{p_2,q_2}\gamma \right]$$

$$\leq \frac{1}{p_1 p_2 (b-a)(d-c)} \int_{p_1 a+(1-p_1)b}^{b} \int_c^{(1-p_2)c+p_2 d} \Psi(\varkappa,\gamma)\ {}_{c}d^T_{p_2,q_2}\gamma\ {}^{b}d^T_{p_1,q_1}\varkappa$$

$$\leq \frac{1}{4}\left[\frac{1}{p_1(b-a)} \int_{p_1 a+(1-p_1)b}^{b} \Psi(\varkappa,c)\ {}^{b}d^T_{p_1,q_1}\varkappa + \frac{1}{p_1(b-a)} \int_{p_1 a+(1-p_1)b}^{b} \Psi(\varkappa,d)\ {}^{b}d^T_{p_1,q_1}\varkappa \right.$$

$$\left. + \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(a,\gamma)\ {}_{c}d^T_{p_2,q_2}\gamma + \frac{1}{p_2(d-c)} \int_c^{(1-p_2)c+p_2 d} \Psi(b,\gamma)\ {}_{c}d^T_{p_2,q_2}\gamma \right]$$

$$\leq \frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}$$

for all $0 < q_1 < p_1 \leq 1$ *and* $0 < q_2 < p_2 \leq 1$.

Proof. The proof is similar to that of Theorem 4.1 by using Theorem 2.3 and Theorem 2.4. □

By the Theorem 2.4, we can also obtain the following Theorem.

Theorem 4.4. *Assume that* $\Psi : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is coordinated convex function on* $[a,b] \times [c,d]$. *Then, it follows*

$$\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{4.15}$$

$$\leq \frac{1}{2}\left[\frac{1}{p_1(b-a)}\int\limits_{p_1a+(1-p_1)b}^{b}\Psi\left(\varkappa,\frac{c+d}{2}\right)\,{}^{b}d^{T}_{p_1,q_1}\varkappa\right.$$
$$\left.+\frac{1}{p_2(d-c)}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},\gamma\right)\,{}^{d}d^{T}_{p_2,q_2}\gamma\right]$$
$$\leq \frac{1}{p_1p_2(b-a)(d-c)}\int\limits_{p_1a+(1-p_1)b}^{b}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi(\varkappa,\gamma)\ {}^{d}d^{T}_{p_2,q_2}\gamma\ {}^{b}d^{T}_{p_1,q_1}\varkappa$$
$$\leq \frac{1}{4}\left[\frac{1}{p_1(b-a)}\int\limits_{p_1a+(1-p_1)b}^{b}\Psi(\varkappa,c)\,{}^{b}d^{T}_{p_1,q_1}\varkappa+\frac{1}{p_1(b-a)}\int\limits_{p_1a+(1-p_1)b}^{b}\Psi(\varkappa,d)\,{}^{b}d^{T}_{p_1,q_1}\varkappa\right.$$
$$\left.+\frac{1}{p_2(d-c)}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi(a,\gamma)\,{}^{d}d^{T}_{p_2,q_2}\gamma+\frac{1}{p_2(d-c)}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi(b,\gamma)\,{}^{d}d^{T}_{p_2,q_2}\gamma\right]$$
$$\leq \frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}$$

for all $0<q_1<p_1\leq 1$ *and* $0<q_2<p_2\leq 1$.

Proof. The proof is similar to the proof of Theorem 4.1 by using Theorem 2.4. □

Remark 4.1. In Theorems 4.1-4.4,

1. if we chose $p=1$, then Theorems 4.1-4.4 reduce to 7-10 proved by Kara and Budak in [15];
2. if we consider $p=1$ and by taking the limit $q\to 1^-$, Theorems 4.1-4.4 reduce to Theorem 1 proved by Dragomir in [13].

Example 4.1. Let us consider $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=ts^2$. Then, $\Psi(t,s)$ is a co-ordinated convex function of two variables on $[0,1]\times[0,1]$. By using Theorem 4.1 with $q_1=q_2=1/3$ and $p_1=p_2=2/3$, the first expression of (4.1) is

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\Psi\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{8}.$$

By utilizing equality (2.3) in the second expression of (4.1), it follows

$$\frac{1}{p_1(b-a)}\int\limits_{a}^{(1-p_1)a+p_1b}\Psi\left(t,\frac{c+d}{2}\right)\,{}_{a}d^{T}_{p_1,q_1}t \tag{4.16}$$
$$=\frac{3}{2}\int\limits_{0}^{\frac{2}{3}}\Psi\left(t,\frac{1}{2}\right)\,{}_{0}d^{T}_{\frac{2}{3},\frac{1}{3}}t=\frac{3}{8}\left[\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n}-1\right)\right]=\frac{1}{8},$$

$$\frac{1}{p_2(d-c)}\int\limits_{c}^{(1-p_2)c+p_2d}\Psi\left(\frac{a+b}{2},s\right)\,{}_{c}d^{T}_{p_2,q_2}s \tag{4.17}$$

$$= \frac{3}{4}\int_0^{\frac{2}{3}} s^2 \, {}_0d^T_{\frac{2}{3},\frac{1}{3}} s = \frac{3}{4}\left[\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{2n} - 1^2\right)\right] = \frac{5}{28}.$$

From the facts of the equalities (4.16) and (4.17), we get

$$\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi\left(t,\frac{c+d}{2}\right)\,{}_ad^T_{p_1,q_1}t + \frac{1}{p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi\left(\frac{a+b}{2},s\right)\,{}_cd^T_{p_2,q_2}s\right] = \frac{1}{2}\left(\frac{1}{8}+\frac{5}{28}\right) = \frac{17}{112}.$$

From equality (3.5), the third expression of the inequalities (4.1) derives the following equality

$$\frac{1}{p_1p_2(b-a)(d-c)}\int_a^{(1-p_1)a+p_1b}\int_c^{(1-p_2)c+p_2d}\Psi(t,s)\,\,{}_cd^T_{p_2,q_2}s\,\,{}_ad^T_{p_1,q_1}t$$

$$= \frac{9}{4}\int_0^{\frac{2}{3}}\int_0^{\frac{2}{3}} ts^2 \,{}_0d^T_{\frac{2}{3},\frac{1}{3}}s\,\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t = \frac{9}{4}\left(\frac{5}{63}\right) = \frac{45}{252}.$$

Let us consider the following equalities,

$$\frac{1}{p_1\,(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(t,c)\,{}_ad^T_{p_1,q_1}t = \frac{3}{2}\int_0^{\frac{2}{3}}\Psi(t,0)\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t = 0, \tag{4.18}$$

$$\frac{1}{p_1\,(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(t,d)\,{}_ad^T_{p_1,q_1}t \tag{4.19}$$

$$=\frac{3}{2}\int_0^{\frac{2}{3}}\Psi(t,1)\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t = \frac{3}{2}\int_0^{\frac{2}{3}} t\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2},$$

$$\frac{1}{p_2\,(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(a,s)\,{}_cd^T_{p_2,q_2}s = \frac{3}{2}\int_0^{\frac{2}{3}}\Psi(0,s)\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}s = 0, \tag{4.20}$$

and

$$\frac{1}{p_2\,(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(b,s)\,{}_cd^T_{p_2,q_2}s \tag{4.21}$$

$$=\frac{3}{2}\int_0^{\frac{2}{3}}\Psi(1,s)\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}s = \frac{3}{2}\int_0^{\frac{2}{3}} s^2\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}s = \frac{3}{2}\cdot\frac{5}{21} = \frac{5}{14}.$$

If we combine all equalities from (4.18) to (4.21), then we obtain the following equality

$$\frac{1}{4}\left[\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(t,c)\,{}_ad^T_{p_1,q_1}t+\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi(t,d)\,{}_ad^T_{p_1,q_1}t\right.$$
$$\left.+\frac{1}{p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(a,s)\,{}_cd^T_{p_2,q_2}s+\frac{1}{p_2(d-c)}\int_c^{(1-p_2)c+p_2d}\Psi(b,s)\,{}_cd^T_{p_2,q_2}s\right]$$
$$=\frac{1}{4}\left(0+\frac{1}{2}+0+\frac{5}{14}\right)=\frac{1}{4}\cdot\frac{12}{14}=\frac{3}{14}.$$

Finally,

$$\frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}=\frac{\Psi(0,0)+\Psi(0,1)+\Psi(1,0)+\Psi(1,1)}{4}=\frac{1}{4}.$$

Consequently, the statements of the Theorem 4.1 are provided as follows

$$\frac{1}{8}<\frac{17}{112}<\frac{45}{252}<\frac{3}{14}<\frac{1}{4}.$$

Example 4.2. Suppose that $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ described by the function $\Psi(t,s)=t^2s^2$. Suppose also that $\Psi(t,s)$ is a co-ordinated convex function on $[0,1]\times[0,1]$. By applying Theorem 4.2 with $q_1=q_2=1/3$ and $p_1=p_2=2/3$, the first statement of (4.13) becomes

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\Psi\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{16}.$$

The following equalities

$$\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi\left(t,\frac{c+d}{2}\right)\,{}_ad^T_{p_1,q_1}t=\frac{3}{8}\int_0^{\frac{2}{3}}t^2\,{}_0d^T_{\frac{2}{3},\frac{1}{3}}t=\frac{3}{8}\cdot\frac{5}{21}=\frac{5}{56}\quad(4.22)$$

and

$$\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},s\right)\,{}^dd^T_{p_2,q_2}s\quad(4.23)$$
$$=\frac{3}{8}\int_{\frac{1}{3}}^{1}s^2\,{}^1d^T_{\frac{2}{3},\frac{1}{3}}s=\frac{3}{8}\left[\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(1-\left(\frac{1}{2}\right)^n\right)^2\right)\right]=\frac{3}{8}\cdot\frac{5}{21}=\frac{5}{56}$$

are valid by using the post-quantum integrals (2.3) and (2.5), respectively. From the equalities (4.22) and (4.23), we obtain

$$\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_a^{(1-p_1)a+p_1b}\Psi\left(t,\frac{c+d}{2}\right)\,{}_ad^T_{p_1,q_1}t\right.$$

$$+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},s\right)\ {}^{d}d^T_{p_2,q_2}s\Bigg]$$
$$=\frac{1}{2}\left(\frac{5}{56}+\frac{5}{56}\right)=\frac{5}{56}.$$

By the equality (3.6), we have

$$\frac{1}{p_1p_2(b-a)(d-c)}\int_{a}^{(1-p_1)a+p_1b}\int_{p_2c+(1-p_2)d}^{d}\Psi(t,s)\ {}^{d}d^T_{p_2,q_2}s\ {}_{a}d^T_{p_1,q_1}t$$
$$=\frac{9}{4}\int_{0}^{\frac{2}{3}}\int_{\frac{1}{3}}^{1}t^2s^2\ {}^{1}d^T_{\frac{2}{3},\frac{1}{3}}s\ {}_{0}d^T_{\frac{2}{3},\frac{1}{3}}t=\frac{9}{4}\cdot\frac{25}{441}=\frac{25}{196}.$$

Use the fact that

$$\frac{1}{p_1\,(b-a)}\int_{a}^{(1-p_1)a+p_1b}\Psi(t,c)\,{}_{a}d^T_{p_1,q_1}t=\frac{3}{2}\int_{0}^{\frac{2}{3}}\Psi(t,0)\,{}_{0}d^T_{\frac{2}{3},\frac{1}{3}}t=0, \tag{4.24}$$

$$\frac{1}{p_1\,(b-a)}\int_{a}^{(1-p_1)a+p_1b}\Psi(t,d)\,{}_{a}d^T_{p_1,q_1}t \tag{4.25}$$
$$=\frac{3}{2}\int_{0}^{\frac{2}{3}}t^2\,{}_{0}d^T_{\frac{2}{3},\frac{1}{3}}t=\frac{3}{2}\left[\frac{1}{3}\left(\sum\frac{3}{2}\left(\frac{1}{2}\right)^{3n}-1\right)\right]=\frac{3}{2}\cdot\frac{5}{21}=\frac{5}{14},$$

$$\frac{1}{p_2\,(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi(a,s)\,{}^{d}d^T_{p_2,q_2}s=\frac{3}{2}\int_{\frac{1}{3}}^{1}\Psi(0,s)\,{}^{1}d^T_{\frac{2}{3},\frac{1}{3}}s=0, \tag{4.26}$$

$$\frac{1}{p_2\,(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi(b,s)\,{}^{d}d^T_{p_2,q_2}s=\frac{3}{2}\int_{\frac{1}{3}}^{1}s^2\,{}^{1}d^T_{\frac{2}{3},\frac{1}{3}}s=\frac{3}{2}\cdot\frac{5}{21}=\frac{5}{14}. \tag{4.27}$$

By using the equalities from (4.24) to (4.27), we derive

$$\frac{1}{4}\Bigg[\frac{1}{p_1\,(b-a)}\int_{a}^{(1-p_1)a+p_1b}\Psi(t,c)\,{}_{a}d^T_{p_1,q_1}t+\frac{1}{p_1\,(b-a)}\int_{a}^{(1-p_1)a+p_1b}\Psi(t,d)\,{}_{a}d^T_{p_1,q_1}t$$
$$+\frac{1}{p_2\,(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi(a,s)\,{}^{d}d^T_{p_2,q_2}s+\frac{1}{p_2\,(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi(b,s)\,{}^{d}d^T_{p_2,q_2}s\Bigg]$$

$$=\frac{1}{4}\left[0+\frac{5}{14}+0+\frac{5}{14}\right]=\frac{5}{28}.$$

The last one is as follows

$$\frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}=\frac{\Psi(0,0)+\Psi(0,1)+\Psi(1,0)+\Psi(1,1)}{4}=\frac{1}{4}.$$

As you seen from the above,

$$\frac{1}{16}<\frac{5}{56}<\frac{25}{196}<\frac{5}{28}<\frac{1}{4},$$

which shows the result defined in Theorem 4.2.

Example 4.3. Let us note that a function $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ is defined as $\Psi(t,s)=ts^3$ and $\Psi(t,s)$ is a co-ordinated convex function on $[0,1]\times[0,1]$. By using Theorem 4.3 with $q_1=q_2=1/3$ and $p_1=p_2=2/3$, we get

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\Psi\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{16}.$$

With the help of the equalities (2.5), we have

$$\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}\Psi\left(t,\frac{c+d}{2}\right)\,{}^{b}d^{T}_{p_1,q_1}t \tag{4.28}$$

$$=\frac{3}{16}\int_{\frac{1}{3}}^{1}t\,{}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}t=\frac{3}{16}\left[\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^{n}\left(1-\left(\frac{1}{2}\right)^{n}\right)\right]=\frac{3}{16}\cdot\frac{1}{3}=\frac{1}{16}$$

and by using equality (2.3), it follows

$$\frac{1}{p_2(d-c)}\int_{c}^{(1-p_2)c+p_2d}\Psi\left(\frac{a+b}{2},s\right)\,{}_{c}d^{T}_{p_2,q_2}s \tag{4.29}$$

$$=\frac{3}{4}\int_{0}^{\frac{2}{3}}s^3\,{}_{0}d^{T}_{\frac{2}{3},\frac{1}{3}}s=\frac{3}{4}\left[\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{3n}-1\right]=\frac{3}{4}\cdot\frac{3}{15}=\frac{3}{20}.$$

If we add the equalities (4.28)-(4.29) and multiply the result by $\frac{1}{2}$, then we obtain

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}\Psi\left(t,\frac{c+d}{2}\right)\,{}^{b}d^{T}_{p_1,q_1}t\right.\\&\left.+\frac{1}{p_2(d-c)}\int_{c}^{(1-p_2)c+p_2d}\Psi\left(\frac{a+b}{2},s\right)\,{}_{c}d^{T}_{p_2,q_2}s\right]\\=&\frac{1}{2}\left(\frac{1}{16}+\frac{3}{20}\right)=\frac{1}{2}\cdot\frac{17}{80}=\frac{17}{160}.\end{aligned}$$

From the fact of Example (3.3), the following equalities hold:

$$\frac{1}{p_1p_2(b-a)(d-c)} \int\limits_{p_1a+(1-p_1)b}^{b} \int\limits_{c}^{(1-p_2)c+p_2d} \Psi(t,s)\ {}_c d_{p_2,q_2}^T s\ {}^b d_{p_1,q_1}^T t$$
$$= \frac{9}{4} \int\limits_{\frac{1}{3}}^{1} \int\limits_{0}^{\frac{2}{3}} ts^3\ {}_0 d_{\frac{2}{3},\frac{1}{3}}^T s\ {}^1 d_{\frac{2}{3},\frac{1}{3}}^T t = \frac{9}{4}\cdot\frac{1}{15} = \frac{3}{20}.$$

Let us calculate the following equalities

$$\frac{1}{p_1(b-a)} \int\limits_{p_1a+(1-p_1)b}^{b} \Psi(t,c)\, {}^b d_{p_1,q_1}^T t = \frac{3}{2}\int\limits_{\frac{1}{3}}^{1} \Psi(t,0)\, {}^1 d_{\frac{2}{3},\frac{1}{3}}^T t = 0, \tag{4.30}$$

$$\frac{1}{p_1(b-a)} \int\limits_{p_1a+(1-p_1)b}^{b} \Psi(t,d)\, {}^b d_{p_1,q_1}^T t \tag{4.31}$$
$$= \frac{3}{2}\int\limits_{\frac{1}{3}}^{1} t\, {}^1 d_{\frac{2}{3},\frac{1}{3}}^T t = \frac{3}{2}\left[\frac{1}{3}\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(1-\left(\frac{1}{2}\right)^n\right)\right] = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2},$$

$$\frac{1}{p_2(d-c)} \int\limits_{c}^{(1-p_2)c+p_2d} \Psi(a,s)\, {}_c d_{p_2,q_2}^T s = \frac{3}{2}\int\limits_{0}^{\frac{2}{3}} \Psi(0,s)\, {}_0 d_{\frac{2}{3},\frac{1}{3}}^T s = 0, \tag{4.32}$$

and

$$\frac{1}{p_2(d-c)} \int\limits_{c}^{(1-p_2)c+p_2d} \Psi(b,s)\, {}_c d_{p_2,q_2}^T s \tag{4.33}$$
$$= \frac{3}{2}\int\limits_{0}^{\frac{2}{3}} s^3\, {}_0 d_{\frac{2}{3},\frac{1}{3}}^T s = \frac{3}{2}\left[\frac{1}{3}\left(\sum_{n=0}^{\infty}\frac{3}{2}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{3n}-1\right)\right] = \frac{3}{2}\cdot\frac{13}{96} = \frac{13}{64}.$$

If we consider the expressions from (4.30) to (4.33), then we can find the following equality

$$\frac{1}{4}\left[\frac{1}{p_1(b-a)} \int\limits_{p_1a+(1-p_1)b}^{b} \Psi(t,c)\, {}^b d_{p_1,q_1}^T t + \frac{1}{p_1(b-a)} \int\limits_{p_1a+(1-p_1)b}^{b} \Psi(t,d)\, {}^b d_{p_1,q_1}^T t\right.$$
$$\left. + \frac{1}{p_2(d-c)} \int\limits_{c}^{(1-p_2)c+p_2d} \Psi(a,s)\, {}_c d_{p_2,q_2}^T s + \frac{1}{p_2(d-c)} \int\limits_{c}^{(1-p_2)c+p_2d} \Psi(b,s)\, {}_c d_{p_2,q_2}^T s\right]$$

$$=\frac{1}{4}\left(0+\frac{1}{2}+0+\frac{13}{64}\right)=\frac{1}{4}\cdot\frac{45}{64}=\frac{45}{256}.$$

The last statements become

$$\frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4}=\frac{\Psi(0,0)+\Psi(0,1)+\Psi(1,0)+\Psi(1,1)}{4}=\frac{1}{4}.$$

It is clear that

$$\frac{1}{16}<\frac{17}{160}<\frac{3}{20}<\frac{45}{256}<\frac{1}{4},$$

which shows the result of Theorem 4.3.

Example 4.4. Consider a function $\Psi:[0,1]\times[0,1]\to\mathbb{R}$ by $\Psi(t,s)=t^2 s$. Then, $\Psi(t,s)$ is a co-ordinated convex function of two variables on $[0,1]\times[0,1]$. By applying Theorem 4.4 with $q_1=q_2=1/3$ and $p_1=p_2=2/3$, we have

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\Psi\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{8}.$$

Through the equality (2.5), the following expressions are obtained

$$\frac{1}{p_1(b-a)}\int\limits_{p_1a+(1-p_1)b}^{b}\Psi\left(t,\frac{c+d}{2}\right)\,{}^{b}d^{T}_{p_1,q_1}t=\frac{3}{4}\int\limits_{\frac{1}{3}}^{1}t^2\,{}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}t=\frac{3}{4}\cdot\frac{5}{21}=\frac{5}{28}$$

and

$$\frac{1}{p_2(d-c)}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},s\right)\,{}^{d}d^{T}_{p_2,q_2}s=\frac{3}{8}\int\limits_{\frac{1}{3}}^{1}s\,{}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}s=\frac{3}{8}\cdot\frac{1}{3}=\frac{1}{8}.$$

From the obtained results, the statements can be rewritten as follows

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{p_1(b-a)}\int_{p_1a+(1-p_1)b}^{b}\Psi\left(t,\frac{c+d}{2}\right)\,{}^{b}d^{T}_{p_1,q_1}t\right.\\&\left.+\frac{1}{p_2(d-c)}\int_{p_2c+(1-p_2)d}^{d}\Psi\left(\frac{a+b}{2},s\right)\,{}^{d}d^{T}_{p_2,q_2}s\right]\\=&\frac{1}{2}\left(\frac{5}{28}+\frac{1}{8}\right)=\frac{1}{2}\cdot\frac{17}{56}=\frac{17}{112}.\end{aligned}$$

By utilizing Example (3.8), we get

$$\begin{aligned}&\frac{1}{p_1p_2(b-a)(d-c)}\int\limits_{p_1a+(1-p_1)b}^{b}\int\limits_{p_2c+(1-p_2)d}^{d}\Psi(t,s)\;{}^{d}d^{T}_{p_2,q_2}s\,{}^{b}d^{T}_{p_1,q_1}t\\=&\frac{9}{4}\int\limits_{\frac{1}{3}}^{1}\int\limits_{\frac{1}{3}}^{1}t^2s\;{}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}s\;{}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}t=\frac{9}{4}\cdot\frac{5}{63}=\frac{5}{28}.\end{aligned}$$

The result of the four integrals in the expression of inequality (4.15) is as follows

$$\frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} \Psi(t,c)\, {}^{b}d^{T}_{p_1,q_1}t = \frac{3}{2}\int_{\frac{1}{3}}^{1} \Psi(t,0)\, {}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}t = 0, \tag{4.34}$$

$$\frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} \Psi(t,d)\, {}^{b}d^{T}_{p_1,q_1}t = \frac{3}{2}\int_{\frac{1}{3}}^{1} t^2\, {}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}t = \frac{3}{2}\cdot\frac{5}{21} = \frac{5}{14}, \tag{4.35}$$

$$\frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \Psi(a,s)\, {}^{d}d^{T}_{p_2,q_2}s = \frac{3}{2}\int_{\frac{1}{3}}^{1} \Psi(0,s)\, {}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}s = 0, \tag{4.36}$$

and

$$\frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \Psi(b,s)\, {}^{d}d^{T}_{p_2,q_2}s = \frac{3}{2}\int_{\frac{1}{3}}^{1} s\, {}^{1}d^{T}_{\frac{2}{3},\frac{1}{3}}s = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}. \tag{4.37}$$

If we add the equality from (4.34) to (4.37) and multiply this sum by $\frac{1}{4}$, we get

$$\begin{aligned}
&\frac{1}{4}\Bigg[\frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} \Psi(t,c)\, {}^{b}d^{T}_{p_1,q_1}t + \frac{1}{p_1(b-a)} \int_{p_1a+(1-p_1)b}^{b} \Psi(t,d)\, {}^{b}d^{T}_{p_1,q_1}t \\
&+ \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \Psi(a,\gamma)\, {}^{d}d^{T}_{p_2,q_2}s + \frac{1}{p_2(d-c)} \int_{p_2c+(1-p_2)d}^{d} \Psi(b,\gamma)\, {}^{d}d^{T}_{p_2,q_2}s\Bigg] \\
=&\frac{1}{4}\left(0+\frac{5}{14}+0+\frac{1}{2}\right) = \frac{1}{4}\cdot\frac{6}{7} = \frac{3}{14}.
\end{aligned}$$

If the last expression of the inequality (4.15) is calculated, it will be as follows

$$\frac{\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)}{4} = \frac{\Psi(0,0)+\Psi(0,1)+\Psi(1,0)+\Psi(1,1)}{4} = \frac{1}{4}.$$

As a result, we have

$$\frac{1}{8} < \frac{17}{112} < \frac{5}{28} < \frac{3}{14} < \frac{1}{4}. \tag{4.38}$$

The statement (4.38) proves the correctness of Theorem 4.4.

5. Conclusion

In this article, we first defined $T_{p,q}$-integrals for functions of two variables. Also, Hermite-Hadamard inequalities related to definitions were obtained. In future studies, researchers can derive new inequalities by using generalized convexities such as co-ordinated preinvex, co-ordinated s-convex, and co-ordinated h-convex functions on coordinates. This article will also motivate further research on quantum integrals and post-quantum integrals.

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