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Monotonicity Analysis of Generalized Discrete Fractional Proportional h-Differences with Applications

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Abstract Monotonicity analysis is an important aspect of fractional mathematics. In this paper, we perform a monotonicity analysis for a generalized class of nabla discrete fractional proportional difference on the $h\mathbb{Z}$ scale of time. We first define the sums and differences of order $0 < \alpha \leq 1$ on the time scale for a general form of nabla fractional along with Riemann-Liouville h-fractional proportional sums and differences. We formulate the Caputo fractional proportional differences and present the relation between them and the fractional proportional differences. Afterward, we introduce and prove the monotonicity results for nabla and Caputo discrete h-fractional proportional differences. Finally, we provide two numerical examples to verify the theoretical results along with a proof for a new version of the fractional proportional difference of the mean value theorem on $h\mathbb{Z}$ as an application.

Keywords Monotonicity analysis, h-fractional proportional difference, Caputo fractional proportional difference, fractional proportional Mean Value Theorem(MVT)

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1. Introduction

Fractional calculus has become an important and active area of research. Many researchers use this topic to model and successfully solve diverse types of problems that appear in science and engineering [1–5]. Fractional calculus extends traditional calculus by allowing derivatives and integrals of non-integer orders, providing

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a powerful tool for modeling complex, real-world phenomena such as memory effects and anomalous diffusion. In differential equations, fractional derivatives offer greater flexibility and accuracy in describing processes with long-range dependencies and hereditary properties, making them essential in fields such as physics, biology, and engineering [6–9]. While continuous fractional calculus is well established, discrete fractional calculus still has high potential in modern applications. Lately, this specific field has been under the spotlight of the research community. Different properties of discrete fractional operators are studied to reveal the potential in various aspects of such a topic [10–13].

One important aspect of discrete fractional calculus is the study of fractional sums and differences with nabla operators. The theory and applications have been extensively considered with new developments over the last few decades [12,14–17]. For example in [18], the Laplace transform on the fractional proportional operators is studied and a generalization of fractional proportional sums and differences is given. Wei et al. [19] consider the series representation for nabla discrete fractional sums and differences. A new discrete fractional solution of the modified Bessel differential equation is introduced in [20]. The monotonicity properties of discrete delta and nabla fractional operators have been an active topic of research recently [21–24]. In [25], the monotonicity properties for nabla fractional sums and differences of order $0 < \alpha \leq 1$ on the time scale $h\mathbb{Z}$, where $0 < h \leq 1$, are studied. Monotonicity results for Riemann-Liouville and Caputo fractional proportional differences on the time scale $\mathbb{Z}$ are presented in [26]. In [27], a new method for negative nabla and delta fractional proportional differences is introduced. Authors of [28] present a comprehensive study on the monotonicity analysis for delta and nabla discrete fractional operators of the Liouville-Caputo family, which directly aligns with the exploration of fractional operators in this work. Additionally, [29] investigates unexpected properties of fractional difference operators, particularly finite and eventual monotonicity, offering insights that are relevant to the current study's focus on operator behavior.

Motivated by the aforementioned work, we present the monotonicity properties for a generalized class of discrete h-fractional proportional differences on the time scale $h\mathbb{Z}$. We start by defining the general fractional sums and differences along with the Riemann-Liouville form. We then move to find and prove the relation between the nabla fractional sums and differences and the Caputo proportional differences. Monotonicity analysis is then conducted for both nabla and Caputo proportional differences. Moreover, we present a new form of the fractional proportional difference of the mean value theorem on $h\mathbb{Z}$ time scale. The work presented in this paper is a direct generalization of results presented in [25] and [26]. In order to validate the theoretical findings, we introduce two numerical examples that are considered direct illustrations of the basic results presented in this work. Additionally, we provide a comprehensive proof of an updated version of the fractional proportional difference of the mean value theorem.

The rest of the paper is organized as follows. In Section 2, basic definitions and preliminary work are introduced. Section 3 includes the main monotonicity results for nabla h-fractional proportional differences. In Section 4, two numerical examples that verify and support the theoretical results are presented and a fractional proportional version of the Mean Value Theorem is developed. Finally, Section 5 concludes the paper.

2. Preliminary definitions

The definitions presented in this section rely mostly on the results presented in [18,25,26]. The interested readers could turn to those references for a more detailed explanations.

Definition 2.1. Let f be a function that is defined on space $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...\}$. We can introduce the backward h-discrete proportional difference with order α and $0 < \rho \leq 1$ as:

$$(\nabla_h^\rho f)(t) = (1-\rho)f(t) + \rho(\nabla_h f)(t), \tag{2.1}$$

while the forward h-discrete proportional difference of order α is given by:

$$(\Delta^\rho f)(t) = (1-\rho)f(t) - \rho\Delta_h f(t), \tag{2.2}$$

where $t \in \mathbb{N}_{a+h,h}$, $0 < h \leq 1$ and $a \geq 0$ is an integer.

Definition 2.2. Considering the time scale $h\mathbb{Z}$, we can define the backward difference operator as:

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}, \tag{2.3}$$

while the forward operator can be presented as:

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}. \tag{2.4}$$

Definition 2.3. On $h\mathbb{Z}$, the backward jump operator can be introduced as

$$\rho(t) = t - h, \tag{2.5}$$

while the forward jump operator is presented by

$$\sigma(t) = t + h. \tag{2.6}$$

Definition 2.4. Let $\alpha \in \mathbb{R}$ and $0 < h \leq 1$, we define the h-factorial of t as:

$$t_h^{\overline{\alpha}} = h^\alpha \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}, \tag{2.7}$$

where $t \in \mathbb{R} - \{..., -2h, -h, 0\}$ and $0_h^{\overline{\alpha}} = 0$.

Definition 2.5. For $t \in \mathbb{N}_{a,h}$, $0 < \rho \leq 1$ and $0 < h \leq 1$, the exponential function f is given by

$$f(t) = \widehat{e}_{\frac{\rho-1}{\rho},h}(t,a) = \left(\frac{\rho}{\rho - (\rho-1)h}\right)^{\frac{t-a}{h}}. \tag{2.8}$$

Definition 2.6. (Nabla h-fractional proportional sums)

Let f be a function such that $f : \mathbb{N}_{a,h} \to \mathbb{R}$, where $0 < \rho \leq 1$, $\alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. The left h-fractional proportional sum for f is defined as:

$$({}_a\nabla_h^{-\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t \widehat{e}_p(t - \tau + \alpha h, 0)(t - \rho_h(\tau))_h^{\overline{\alpha-1}} f(\tau) \nabla_h \tau$$

$$=\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t+(k-\alpha)h,0)(t-\rho_h(kh))_h^{\overline{\alpha-1}}f(kh)h, \quad (2.9)$$

where $t\in\mathbb{N}_{a,h}$.

Now let f be a function such that $f :_{b,h}\mathbb{N}\to\mathbb{R}$ where ${}_{b,h}\mathbb{N}=\{b,b-h,b-2h,...\}$. Then the right h-fractional proportional sum for f is defined as:

$$\begin{aligned}({}_h\nabla_b^{-\alpha,\rho}f)(t) =&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_t^b\widehat{e}_p(\tau-t+\alpha h,0)(\tau-\rho_h(t))_h^{\overline{\alpha-1}}f(\tau)\Delta_h\tau\\ =&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\sum_{k=t/h}^{b/h-1}\widehat{e}_p((k+\alpha)h-t,0)(kh-\rho_h(t))_h^{\overline{\alpha-1}}f(kh)h, \quad (2.10)\end{aligned}$$

where $t\in\ {}_{b,h}\mathbb{N}$.

p is given as $p=\frac{\rho-1}{\rho}$ throughout the definition.

We can generate the left and right Riemann-Liouville fractional sums for order α by simply setting $\rho=1$ in Definition 2.6 and making use of Definition 2.5.

Definition 2.7. (The Riemann-Liouville sums for the h-fractional proportionals)
Let f be a function such that $f:\mathbb{N}_{a,h}=\{a,a+h,a+2h,\ldots\}\to\mathbb{R}$, and $0<h\leq 1$. Then the left Riemann-Liouville h-fractional sum with order $\alpha>0$ can be given as:

$$\begin{aligned}({}_a^R\nabla_h^{-\alpha}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t(t-\rho_h(s))_h^{\overline{\alpha-1}}f(s)\nabla_h\\ &= \frac{1}{\Gamma(\alpha)}\sum_{k=a/h+1}^{t/h}(t-\rho_h(kh))_h^{\overline{\alpha-1}}f(kh)h, \quad t\in\mathbb{N}_{a+h,h}. \quad (2.11)\end{aligned}$$

Now for a function f such that $f:\ {}_{b,h}\mathbb{N}=\{b,b-h,b-2h,\ldots\}\to\mathbb{R}$ and ending at b, the right Riemann-Liouville h-fractional sum with order $\alpha>0$ can be defined as:

$$\begin{aligned}({}_h^R\nabla_b^{-\alpha}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b(s-\rho_h(t))_h^{\overline{\alpha-1}}f(s)\Delta_h s\\ &= \frac{1}{\Gamma(\alpha)}\sum_{k=t/h}^{b/h-1}(kh-\rho_h(t))_h^{\overline{\alpha-1}}f(kh)h. \quad (2.12)\end{aligned}$$

Definition 2.8. (Nabla h-fractional proportional differences)
For a function f, $0<\rho\leq 1$, $0<h\leq 1$, and $\alpha\in\mathbb{C}$, where $Re(\alpha)>0$, we can define the left h-fractional proportional difference as:

$$\begin{aligned}({}_a\nabla_h^{\alpha,\rho}f)(t) &= \nabla_h^{\rho}\ {}_a\nabla_h^{-(1-\alpha),\rho}f(t)\\ &= \frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t\widehat{e}_p(t-\tau+h(1-\alpha),0)(t-\rho_h(\tau))_h^{\overline{-\alpha}}f(\tau)\nabla_h\tau\\ &= \frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h.\end{aligned} \quad (2.13)$$

The right h-fractional proportional difference that ends at b can also be defined as:

$$\begin{aligned}({}_h\nabla_b^{\alpha,\rho} f)(t) &= -\Delta_h^\rho \, {}_h\nabla_b^{-(1-\alpha),\rho} f(t) \\ &= \frac{-\Delta_h^\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_t^b \widehat{e}_p(\tau - t + h(1-\alpha), 0)(\tau - \rho_h(t))_h^{\overline{-\alpha}} f(\tau)\Delta_h \tau \\ &= \frac{-\Delta_h^\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)} \sum_{k=t/h}^{b/h-1} \widehat{e}_p((k+1-\alpha)h - t, 0)(kh - \rho_h(t))_h^{\overline{-\alpha}} f(kh)h.\end{aligned} \tag{2.14}$$

Clearly, substituting $\rho = 1$ in Definition 2.8 yields the Riemann-Liouville h-fractional differences with order α.

Definition 2.9. (The Riemann-Liouville differnces for h-fractional proportionals)
The left Riemann-Liouville h-fractional difference with order $\alpha > 0$ (which starts at a) can be defined as:

$$\begin{aligned}({}_a^R\nabla_h^\alpha f)(t) &= (\nabla_h \, {}_a\nabla_h^{-(1-\alpha)} f)(t) \\ &= \frac{\nabla_h}{\Gamma(1-\alpha)} \sum_{k=a/h+1}^{t} (t - \rho_h(kh))_h^{\overline{-\alpha}} f(kh)h,\end{aligned} \tag{2.15}$$

where $t \in \mathbb{N}_{a+h,h}$. The right Riemann-Liouville h-fractional difference with order $\alpha > 0$ (which ends at b) can be presented as:

$$\begin{aligned}({}_h^R\nabla_b^\alpha f)(t) &= (-\Delta_h {}_h\nabla_b^{-(1-\alpha)} f)(t) \\ &= \frac{-\Delta_h}{\Gamma(1-\alpha)} \sum_{k=t}^{b/h-1} (kh - \rho_h(t))_h^{\overline{-\alpha}} f(kh)h,\end{aligned} \tag{2.16}$$

where $t \in {}_{b-h,h}\mathbb{N}$.

The Caputo fractional difference operator is favored in real-world applications because it allows initial conditions to be specified in the same manner as in classical differential equations, making it more intuitive for modeling physical processes such as biological growth or physical systems. Unlike the Riemann-Liouville operator, which requires fractional-order initial conditions that can be challenging to interpret in practical contexts, the Caputo operator is well suited to initial value problems where traditional initial conditions are defined at a specific point in time. This makes it particularly applicable in fields such as control theory, viscoelastic materials, and diffusion processes. While the Riemann-Liouville operator is more versatile in theoretical analysis, the Caputo operator is more practical for real-world systems, where clear and measurable initial conditions are crucial. For a more detailed comparison of these operators, the reader is referred to [32].

Definition 2.10. (The Caputo h-fractional proportional differences)
Let f be a function that is defined on $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \dots\}$ and also on ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \dots\}$. For $0 < \alpha \le 1$, $0 < h \le 1$, and $a < b \in \mathbb{R}$, we can define the left Caputo h-fractional proportional difference with order α (which starts at a) as:

$$({}_a^C\nabla_h^{\alpha,\rho} f)(t)$$

$$
\begin{aligned}
&={}_a\nabla_h^{-(1-\alpha),\rho}(\nabla_h^\rho f)(t)\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_{a_h(\alpha)}^{t}\widehat{e}_p(t-s+h(1-\alpha),0)(t-\rho_h(s))_h^{\overline{-\alpha}}(\nabla_h^\rho f)(s)\nabla_h s\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(\nabla_h^\rho f)(kh)h, \quad (2.17)
\end{aligned}
$$

where $t\in\mathbb{N}_{a+h,h}$. Also, the right Caputo h-fractional proportional difference with order α (which ends at b) can be presented as:

$$
\begin{aligned}
&({}_h^C\nabla_b^{\alpha,\rho}f)(t)\\
&={}_h\nabla_b^{-(1-\alpha),\rho}(\ -\Delta_h^\rho f)(t)\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_{t}^{b_h(\alpha)}\widehat{e}_p(s-t+h(1-\alpha),0)(s-\rho_h(t))_h^{\overline{-\alpha}}(\ -\Delta_h^\rho f)(s)\Delta_h s\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=t}^{b/h-1}\widehat{e}_p(h(k+1-\alpha)-t,0)(hk-\rho_h(t))_h^{\overline{-\alpha}}(\ -\Delta_h^\rho f)(hk)h,
\end{aligned}
\quad (2.18)
$$

where $t\in\ {}_{b-h,h}\mathbb{N}$.

Proposition 2.1. *(Nabla and Caputo h-fractional proportional differences relation)*
The relation between nabla and Caputo h-fractional proportional differences can be defined for $0<\rho\leq 1$, $0<h\leq 1$, $\alpha\in\mathbb{C}$, *and* $Re(\alpha)>0$ *as follows:*

$$(i)\quad ({}_a^C\nabla_h^{\alpha,\rho}f)(t) \;=\; ({}_a\nabla_h^{\alpha,\rho}f)(t)-\frac{\rho^\alpha}{\Gamma(1-\alpha)}\widehat{e}_p(t,a+h\alpha)(t-a)_h^{\overline{-\alpha}}f(a). \quad (2.19)$$

$$(ii)\quad ({}_h^C\nabla_b^{\alpha,\rho}f)(t) \;=\; ({}_h\nabla_b^{\alpha,\rho}f)(t)-\frac{\rho^\alpha}{\Gamma(1-\alpha)}\widehat{e}_p(b-h\alpha,t)(b-t)_h^{\overline{-\alpha}}f(b). \quad (2.20)$$

Proof.

$$
\begin{aligned}
(i)\quad &({}_a^C\nabla_h^{\alpha,\rho}f)(t)\\
&={}_a\nabla_h^{-(1-\alpha),\rho}(\nabla_h^\rho f)(t)\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(\nabla_h^\rho f)(kh)h\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}\\
&\quad\times[(1-\rho)f(kh)+\rho(\nabla_h f)(kh)[\,h\\
&=\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(1-\rho)f(kh)h\\
&\quad+\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}\rho(\nabla_h f)(kh)h
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(1-\rho)f(kh)h\\
&+\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}\\
&\times\rho(\frac{f(kh)-f(kh-h)}{h})h\\
=&\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(1-\rho)f(kh)h\\
&+\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}\rho\frac{f(kh)}{h}h\\
&-\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=(a-h)/h+1}^{(t-h)/h}\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}}\\
&\times\rho\frac{f(kh)}{h}h\\
=&\frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}(1-\rho)f(kh)h\\
&+\frac{1}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}\rho f(kh)h\\
&-\frac{1}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h-1}\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}}\\
&\times\rho f(kh)h\\
&-\frac{\rho}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\widehat{e}_p(t-h-h(a/h-1+\alpha),0)(t-h-(a/h)h+h)_h^{\overline{-\alpha}}\\
&\times f((a/h)h)h\\
=&\frac{1-\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\\
&+\frac{\rho}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\\
&-\frac{\rho}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h-1}\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}}\\
&\times f(kh)h\\
&-\frac{\rho}{h\rho^{1-\alpha}\Gamma(1-\alpha)}\widehat{e}_p(t-h-h(a/h-1+\alpha),0)(t-h-(a/h)h+h)_h^{\overline{-\alpha}}\\
&\times f((a/h)h)h
\end{aligned}
$$

$$=\frac{1-\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h$$
$$+\frac{\rho\nabla_h}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h$$
$$-\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\widehat{e}_p(t,a+h\alpha)(t-a)_h^{\overline{-\alpha}}f(a)$$
$$=\frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h$$
$$-\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\widehat{e}_p(t,a+h\alpha)(t-a)_h^{\overline{-\alpha}}f(a)$$
$$=({}_a\nabla_h^{\alpha,\rho}f)(t)-\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\widehat{e}_p(t,a+h\alpha)(t-a)_h^{\overline{-\alpha}}f(a).$$

(ii) The proof for this part can be obtained in the same way as for (i), therefore it is omitted. □

Remark 2.1. Setting ρ to be 1 and making use of Definition 2.5 will lead to defining the relation between Riemann-Liouville and Caputo h-fractional proportional differences and can be given as:

$$(i)\quad ({}_a^C\nabla_h^{\alpha}f)(t) = ({}_a^R\nabla_h^{\alpha}f)(t)-\frac{1}{\Gamma(1-\alpha)}(t-a)_h^{\overline{-\alpha}}f(a). \tag{2.21}$$

$$(ii)\quad ({}_h^C\nabla_b^{\alpha}f)(t) = ({}_h^R\nabla_b^{\alpha}f)(t)-\frac{1}{\Gamma(1-\alpha)}(b-t)_h^{\overline{-\alpha}}f(b). \tag{2.22}$$

3. The generalized monotonicity results

We start this section by introducing the $h\mathbb{Z}$ version for the monotonicity definitions that are presented in [25].

Definition 3.1. Let f be a function defined as $f:\ \mathbb{N}_{a,h}\to\mathbb{R}$ where $f(a)\geq 0$, $0<\alpha\leq 1$ and $0<h\leq 1$. If $f(t+h)\geq\ \alpha\ f(t)\ \ \forall t\in\mathbb{N}_{a,h}$, then the function $f(t)$ is $\alpha-$increasing on $\mathbb{N}_{a,h}$.

Definition 3.2. Let f be a function defined as $f:\ \mathbb{N}_{a,h}\to\mathbb{R}$ where $f(a)\leq 0$, $0<\alpha\leq 1$ and $0<h\leq 1$. If $f(t+h)\leq\ \alpha\ f(t)\ \ \forall t\in\mathbb{N}_{a,h}$, then the function $f(t)$ is $\alpha-$decreasing on $\mathbb{N}_{a,h}$.

Remark 3.1. For Definition 3.1, when $\alpha=1$, the concepts of increasing and $\alpha-$increasing coincide for function f. The same can be noticed for the decreasing concept in Definition 3.2.

Using Definitions 3.1 and 3.2, we have the following results:

Theorem 3.1. *Let f be a function defined as $f:\ \mathbb{N}_{a-h,h}\to\ \mathbb{R}$. Now assume that $(\ {}_{a-h}\nabla_h^{\alpha,\rho}\ f\ )\ (t)\ \geq 0$ where $0<\alpha\leq 1$, $0<h\leq 1$ and also $0<\rho\leq 1,\ t\in\mathbb{N}_{a-h,h}$. Then the function $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)-$ increasing.*

Proof. We start by using Definition 2.8, which gives us the expression for the operator ${}_{a-h}\nabla_h^{\alpha,\rho} f(t)$. This operator involves a weighted summation over discrete intervals, adjusted by parameters α, ρ, and h.

We recall that:

$$\begin{aligned}
&\left({}_{a-h}\nabla_h^{\alpha,\rho} f\right)(t)\\
=&\frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=(a-h)/h+1}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\\
=&\frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\sum_{k=a/h}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h.
\end{aligned}$$

This sum represents a discrete fractional difference operator acting on f. Let us define $S(t)$ for simplicity:

$$S(t)=\sum_{k=a/h}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h.$$

Now we know that $\nabla_h^{\rho}S(t)\geq 0$, which implies that:

$$\nabla_h^{\rho}S(t)=(1-\rho)S(t)+\rho\nabla_h S(t)\geq 0. \tag{3.1}$$

We now expand $\nabla_h S(t)$. Using the definition of the discrete backward difference operator ∇_h, we have:

$$\nabla_h S(t)=\frac{S(t)-S(t-h)}{h}.$$

By substituting $S(t)$ and using basic definitions presented in Section 2, we can do the following:

$$\begin{aligned}
\nabla_h S(t)=&\left[\sum_{k=a/h}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\right.\\
&\left.-\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\right]\frac{1}{h}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)\\
&+\left[\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\right.\\
&\left.-\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\right]\frac{1}{h}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)\\
&+\sum_{k=a/h}^{t/h-1}f(kh)h\left[\widehat{e}_p(t-h(k-1+\alpha),0)(t-kh+h)_h^{\overline{-\alpha}}\right.\\
&\left.-\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-kh)_h^{\overline{-\alpha}}\right]\frac{1}{h}
\end{aligned}$$

$$
\begin{aligned}
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)\\
&+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k+1-\alpha}\frac{\Gamma(\frac{t-kh+h}{h}-\alpha)}{\Gamma(\frac{t-kh+h}{h})}h^{-\alpha}\right.\\
&\left.-\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(\frac{t-kh}{h}-\alpha)}{\Gamma(\frac{t-kh}{h})}h^{-\alpha}\right]\frac{1}{h}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)\\
&+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\left(\frac{\rho}{\rho-(\rho-1)h}\right)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(t/h-k+1-\alpha)}{\Gamma(t/h-k+1)}h^{-\alpha}\right.\\
&\left.-\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(t/h-k-\alpha)}{\Gamma(t/h-k)}h^{-\alpha}\right]\frac{1}{h}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\left(\frac{\rho}{\rho-(\rho-1)h}\right)\left(\frac{t/h-k-\alpha}{t/h-k}\right)-1\right]\\
&\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\left(\frac{\Gamma(t/h-k-\alpha)}{\Gamma(t/h-k)}\right)\frac{h^{-\alpha}}{h}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\frac{\rho(t-hk-\alpha)-t+hk}{\rho-(\rho-1)h}\right]\\
&\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\left(\frac{\Gamma(t/h-k+1+(-\alpha-1))}{\Gamma(t/h-k+1)}\right)h^{-\alpha-1}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)+\sum_{k=a/h}^{t/h-1} f(kh)h\left[t-hk-\alpha-\frac{t-hk}{\rho}\right]\\
&\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}\\
=&\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)-\frac{1}{\rho}\sum_{k=a/h}^{t/h-1}[t-hk-\rho(t-hk-\alpha)]\\
&\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h. \qquad (3.2)
\end{aligned}
$$

At this point, the terms involving ρ, α, and h highlight the influence of the discrete step size h and the fractional order α on the growth of $S(t)$.

Next, let's examine the $(1-\rho)S(t)$ term from Equation 3.1:

$$
\begin{aligned}
(1-\rho)S(t)=&(1-\rho)\sum_{k=a/h}^{t/h}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h\\
=&(1-\rho)h\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t)+\sum_{k=a/h}^{t/h-1}[\rho(hk+h\alpha-t)+t-hk-h\alpha]\\
&\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h. \qquad (3.3)
\end{aligned}
$$

This term provides another measure of how the operator scales with respect to ρ. Combining this with $\nabla_h S(t)$, we substitute both Equations 3.2 and 3.3 back into

Equation 3.1 to get:

$$\begin{aligned}\nabla_h^\rho S(t) =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \Bigg[\sum_{k=a/h}^{t/h-1}[t-hk \\ & -\rho(t-hk-\alpha)]\,\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ & -\sum_{k=a/h}^{t/h-1}[\rho(hk+h\alpha-t)+t-hk-h\alpha]\,\widehat{e}_p(t-h(k-1+\alpha),0) \\ & (t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h\Bigg] \\ =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \sum_{k=a/h}^{t/h-1}\Bigg[[t-hk \\ & -\rho(t-hk-\alpha)] - [\rho(hk+h\alpha-t)+t-hk-h\alpha]\Bigg] \\ & \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \sum_{k=a/h}^{t/h-1}[\rho\alpha-\rho h\alpha \\ & +h\alpha]\,\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\,((1-\rho)h+\rho)\sum_{k=a/h}^{t/h-1} \\ & \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \geq 0. \end{aligned} \tag{3.4}$$

Finally, by substituting $t=a$, we find:

$$\nabla_h^\rho S(a) = ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(a) \geq 0. \tag{3.5}$$

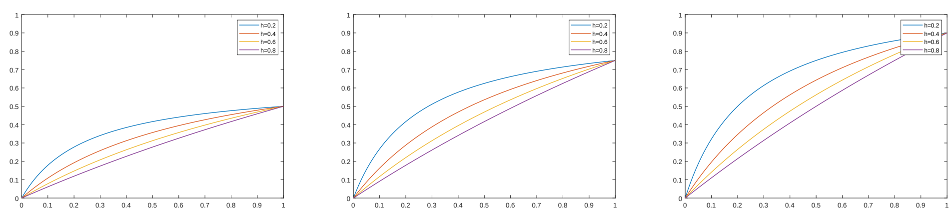
Thus, $f(a)\geq 0$. Similarly, by setting $t=a+h$, we establish that:

$$\begin{aligned} & \nabla_h^\rho S(a+h) \\ =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(a+h) - \alpha((1-\rho)h+\rho) \\ & \widehat{e}_p(2h-h\alpha,0)(a+h-\rho_h(a))_h^{\overline{-\alpha-1}}f(a)h \\ =& ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(a+h) - \alpha((1-\rho)h+\rho) \\ & \widehat{e}_p(2h-h\alpha,0)(2h)_h^{\overline{-\alpha-1}}f(a)h \\ =& ((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\frac{h-h\alpha}{h}}\left(\frac{\Gamma(h/h-\alpha)}{\Gamma(h/h)}\right)h^{-\alpha}f(a+h) \\ & -\alpha((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\frac{2h-h\alpha}{h}}\left(\frac{\Gamma(2h/h-\alpha-1)}{\Gamma(2h/h)}\right)h^{-\alpha-1}f(a)h \\ =& ((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha}\left(\frac{\Gamma(1-\alpha)}{\Gamma(1)}\right)h^{-\alpha}f(a+h) \end{aligned}$$

$$
\begin{aligned}
&-\alpha((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{2-\alpha}\left(\frac{\Gamma(1-\alpha)}{\Gamma(2)}\right)h^{-\alpha}f(a)\\
=&((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha}\Gamma(1-\alpha)h^{-\alpha}f(a+h)\\
&-\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha}\Gamma(1-\alpha)h^{-\alpha}f(a)\\
\geq&0.
\end{aligned}
$$

Hence, $f(a+h)\geq\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)f(a)$.

By continuing this process inductively, we conclude that $f(t+h)\geq\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)\cdot f(t)$ for any t, showing that $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$-increasing. □



(a) $\alpha=0.5$ (b) $\alpha=0.75$ (c) $\alpha=0.9$

Figure 1. Monotonicity factor for different values of α, h, and ρ.

The main monotonicity result is illustrated in Figure 1. The monotonicity factor is calculated for different values of α, h and ρ. Namely in Figure 1(a), we calculate the monotonicity factor in the case where $\alpha=0.5$, $h=0.2,0.4,0.6$ and 0.8 and $0\leq\rho\leq1$. In Figures 1(b) and 1(c) the same sittings are used with $\alpha=0.75,0.9$, respectively.

Remark 3.2. Theorem 3.1 is a generalization for the results presented in [25] and [26]. We note the following:

(i) When we set $\rho=1$ in theorem 3.1, we get that $f(t)$ is $\alpha-$increasing which is presented as the main result in [25].

(ii) When we set $h=1$ in theorem 3.1, we get that $f(t)$ is $\alpha\rho-$increasing which is presented as the main result in [26].

Using Theorem 3.1 and Proposition 2.1, we can obtain a straightforward result of monotonicity for the Caputo h-fractional proportional difference presented as follows:

Corollary 3.1. *For a function f defined as $f:\mathbb{N}_{a-h,h}\to\mathbb{R}$ where $0<h\leq1$, $0<\rho\leq1$ and $0<\alpha\leq1$, we suppose that the following holds for $t\in\mathbb{N}_{a-h,h}$:*

$$({}^{C}_{a-h}\nabla_h^{\alpha,\rho}f)(t)=-\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\widehat{e}_p(t,a-h(1-\alpha))(t-a+h)_h^{\overline{-\alpha}}f(a-h). \quad (3.6)$$

Then the function $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)-$ increasing.

Theorem 3.2. *Let f be a function defined as $f: \mathbb{N}_{a-h,h} \to \mathbb{R}$ and $f(a) \geq 0$. Assume that $f(t)$ is increasing on $\mathbb{N}_{a,h}$ where $0 < h \leq 1$, $0 < \rho \leq 1$ and $0 < \alpha \leq 1$. Then we can get that:*

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \ \geq 0, \quad \forall t \in \mathbb{N}_{a-h,h}. \tag{3.7}$$

Proof. We will proceed by showing that $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t)$ is non-negative for all $t \in \mathbb{N}_{a-h,h}$ by analyzing its components.

Recall from Theorem 3.1 that:

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \nabla_h^{\rho} S(t), \quad t \in \mathbb{N}_{a-h,h}.$$

Thus, in order to show that $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0$, we only need to prove that $S(t)$ is increasing on $\mathbb{N}_{a,h}$, which implies that $\nabla_h^{\rho} S(t) \geq 0$.

First, consider the case when $t = a$. We substitute $t = a$ into the equation for $\nabla_h^{\rho} S(t)$:

$$\nabla_h^{\rho} S(a) = ((1-\rho)h + \rho)\, \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(a).$$

By the assumptions $0 < h \leq 1$, $0 < \rho \leq 1$, $0 < \alpha \leq 1$ and $f(a) \geq 0$, we get $\nabla_h^{\rho} S(a) \geq 0$.

Now, assume that $\nabla_h^{\rho} S(i) \geq 0$ for all $i < t$. We need to show that $\nabla_h^{\rho} S(t) \geq 0$ for $t > a$.

Since $f(t)$ is assumed to be increasing, then we can get that $f(t) \geq f(t-h) \geq f(a) \geq 0$, $\forall t \in \mathbb{N}_{a,h}$.

Since $f(t)$ is increasing, we know:

$$f(t) \geq f(t-h) \geq f(a) \geq 0, \quad \forall t \in \mathbb{N}_{a,h}.$$

We now use the recursive formula for $\nabla_h^{\rho} S(t)$:

Recalling Equation 3.4,

$$\nabla_h^{\rho} S(t) = ((1-\rho)h + \rho)\, \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\, ((1-\rho)h + \rho) \sum_{k=a/h}^{t/h-1} \widehat{e}_p(t - h(k-1+\alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha-1}} f(kh)h \geq 0,$$

we have

$$\begin{aligned}
\nabla_h^{\rho} S(t) &= ((1-\rho)h + \rho)\, \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\, ((1-\rho)h + \rho) \Big[\widehat{e}_p(t - h(t/h - 1 \\
&\quad - 1 + \alpha), 0)(t - \rho_h(t-h))_h^{\overline{-\alpha-1}} f(t-h)h + \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t - h(k-1+\alpha), 0) \\
&\quad (t - \rho_h(kh))_h^{\overline{-\alpha-1}} f(kh)h\Big] \\
&= ((1-\rho)h + \rho)\, \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\, ((1-\rho)h + \rho) \Big[\widehat{e}_p(2h - h\alpha, 0) \\
&\quad (2h)_h^{\overline{-\alpha-1}} f(t-h)h + \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t - h(k-1+\alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha-1}} f(kh)h
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h \\
&+ \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h\Big] \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\,((1-\rho)h+\rho)\,\Big[\widehat{e}_p(2h-h\alpha,0) \\
&(2h)_h^{\overline{-\alpha-1}} f(t-h)h + \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} \\
&(f(t-h)-f(kh))h \\
&+ \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h\Big].
\end{aligned}
$$

Given that $f(t)$ is increasing, then $f(t-h)-f(kh) \geq 0$, $\forall k$, where $k = \frac{a}{h}, \frac{a}{h}+1, ..., \frac{t}{h}-2$. Hence, we can do the following:

$$
\begin{aligned}
\nabla_h^{\rho} S(t) &\geq ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\,((1-\rho)h+\rho)\,\Big[\widehat{e}_p(2h-h\alpha,0) \\
&\quad (2h)_h^{\overline{-\alpha-1}} f(t-h)h + \sum_{k=a/h}^{t/h-2} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h\Big] \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t) - \alpha\,((1-\rho)h+\rho) \sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h \\
&\quad (k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t) - ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} \\
&\quad f(t-h) + ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t-h) - \alpha\,((1-\rho)h+\rho) \sum_{k=a/h}^{t/h-1} \\
&\quad \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} f(t-h)h \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}(f(t)-f(t-h)) + ((1-\rho)h+\rho)\,\widehat{e}_p(h \\
&\quad -h\alpha,0)(h)_h^{\overline{-\alpha}} f(t-h) - \alpha\,((1-\rho)h+\rho)\, f(t-h)h \sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h(k-1+\alpha),0) \\
&\quad (t-\rho_h(kh))_h^{\overline{-\alpha-1}} \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}(f(t)-f(t-h)) + ((1-\rho)h+\rho)\, f(t-h) \\
&\quad \Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h \sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}\Big] \\
&\geq ((1-\rho)h+\rho)\, f(t-h)\Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h \sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h(k-1+\alpha),0) \\
&\quad (t-\rho_h(kh))_h^{\overline{-\alpha-1}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= ((1-\rho)h+\rho)\, f(t-h)\Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h\Big(\widehat{e}_p(t-a+h-h\alpha),0)\\
&\quad (t-\rho_h(a))_h^{\overline{-\alpha-1}} + \widehat{e}_p(t-a-h\alpha),0)(t-\rho_h(a+h))_h^{\overline{-\alpha-1}} + \cdots + \widehat{e}_p(2h-h\alpha),0)\\
&\quad (t-\rho_h(t-h))_h^{\overline{-\alpha-1}}\Big)\Big]\\
&= ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\Big[\widehat{e}_p(h-h\alpha,0)\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \widehat{e}_p(t-a+h-h\alpha,0)\\
&\quad \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)} - \widehat{e}_p(t-a-h\alpha,0)\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha-1)}{\Gamma(\frac{t-a}{h})} - \cdots - \widehat{e}_p(2h-h\alpha,0)\\
&\quad \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)}\Big)\Big]\\
&= ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\Big[\widehat{e}_p(h-h\alpha,0)\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \widehat{e}_p(2h-h\alpha,0)\\
&\quad \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)} - \widehat{e}_p(3h-h\alpha,0)\alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \cdots - \widehat{e}_p(t-a+h-h\alpha,0)\\
&\quad \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1))}\Big)\Big].
\end{aligned}
$$

By adding the parts and continuing in the same manner, we get

$$
\begin{aligned}
\nabla_h^{\rho} S(t) &\geq ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\Bigg(\widehat{e}_p(t-a-h\alpha,0)\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}))}\\
&\quad - \widehat{e}_p(t-a+h-h\alpha,0)\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1))}\Bigg)\\
&= ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\widehat{e}_p(t-a-h\alpha,0)\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}))}\Bigg(1-\\
&\quad - \left(\frac{\rho}{\rho-(\rho-1)h}\right)\alpha\frac{1}{\frac{t-a}{h}}\Bigg)\\
&= (\rho + \frac{h(\rho-1)(t-a)}{a+h\alpha-t})f(t-h)\widehat{e}_p(t-a-h\alpha,0)h^{-\alpha}\frac{\Gamma(\frac{t-a+h}{h}-\alpha)}{\Gamma(\frac{t-a+h}{h}))}\\
&= (\rho + \frac{h(\rho-1)(t-a)}{a+h\alpha-t})f(t-h)\widehat{e}_p(t-a-h\alpha,0)(t-a+h)^{\overline{-\alpha}} \geq 0.
\end{aligned}
$$

Since $\nabla_h^{\rho} S(t) \geq 0$ for all $t \in \mathbb{N}_{a-h,h}$, it follows that:

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0, \quad \forall t \in \mathbb{N}_{a-h,h}.$$

This completes the proof. □

Theorem 3.3. *Let f be a function defined as $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ and $f(a) > 0$. Assume that $f(t)$ is strictly increasing on $\mathbb{N}_{a,h}$ where $0 < h \leq 1$, $0 < \rho \leq 1$ and $0 < \alpha \leq 1$. Then we can get that:*

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \ > 0, \quad \forall t \in \mathbb{N}_{a-h,h}. \tag{3.8}$$

The proof of this result follows the same steps as the one for Theorem 3.2 and therefore we omit it.

Theorem 3.4. *Let f be a function defined as $f:\ \mathbb{N}_{a-h,h} \rightarrow\ \mathbb{R}$. Now assume that $\left(\ {}_{a-h}\nabla_h^{\alpha,\rho}\ f\ \right)(t)\ \leq 0$ where $0<\alpha\leq 1$, $0<h\leq 1$ and also $0<\rho\leq 1,\ t\in\mathbb{N}_{a-h,h}$. Then the function $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)-$ decreasing.*

Proof. To start this proof, we define a function k such that $k:\ \mathbb{N}_{a-h,h}\rightarrow\ \mathbb{R}$ and $k(t)=-f(t)$. Using this, we get the following:

$$(\ {}_{a-h}\nabla_h^{\alpha,\rho})k)(t)=(\ {}_{a-h}\nabla_h^{\alpha,\rho})-f)(t)=-(\ {}_{a-h}\nabla_h^{\alpha,\rho})f)(t)\geq 0.$$

Now by using Theorem 3.1 for the function $k(t)$, we directly have our wanted result. □

Theorem 3.5. *Let f be a function defined as $f:\ \mathbb{N}_{a-h,h}\rightarrow\mathbb{R}$ and $f(a)\leq 0$. Assume that $f(t)$ is decreasing on $\mathbb{N}_{a,h}$ where $0<h\leq 1$, $0<\rho\leq 1$ and $0<\alpha\leq 1$. Then we can get that:*

$$({}_{a-h}\nabla_h^{\alpha,\rho}f)(t)\ \leq 0,\quad \forall t\in\mathbb{N}_{a-h,h}. \tag{3.9}$$

Proof. By defining a function $g(t)=-f(t)$, we can apply Theorem 3.2 and the result follows. □

The following Theorem is a direct result for the work presented in [18].

Theorem 3.6. *Let us suppose that f is a function defined as $f:\ \mathbb{N}_{a+h,h}\rightarrow\mathbb{R}$ where $0<h\leq 1$, $0<\rho\leq 1$ and $0<\alpha\leq 1$. Given the previous assumption, the next result holds:*

$$\left(\ {}_{a}\nabla_h^{-\alpha,\rho}\ {}_{a-h}\nabla_h^{\alpha,\rho}f\right)(t)=f(t)-\frac{h^{1-\alpha}\widehat{e}_p(t,a)}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}f(a).$$

4. Applications

In this section, two applications are presented to verify and illustrate the theoretical results generated in this work.

4.1. Numerical examples

The nature of a function is a vital aspect when considering real-life problems especially those related to engineering. For instance, monotonic functions are indispensable when dealing with growth and decay behavior in material, biological and population engineering [30]. In particular, several biological models have been developed to simulate the growth and decay of microscopic organisms, diseases, infections spread and more. The flexibility and the wide range of possibilities that the fractional calculus provides through the newly adopted operators allow such models to be handled efficiently and enable the possibility to model more complex real life applications. In this subsection, we first verify our monotonicity results numerically using a logistic growth function as an example. In the second example, we verify our results by presenting an example of another type of growth function called Generalized Weibull function which is a special case of Koya-Goshu growth function used for modeling biological growths [30, 31].

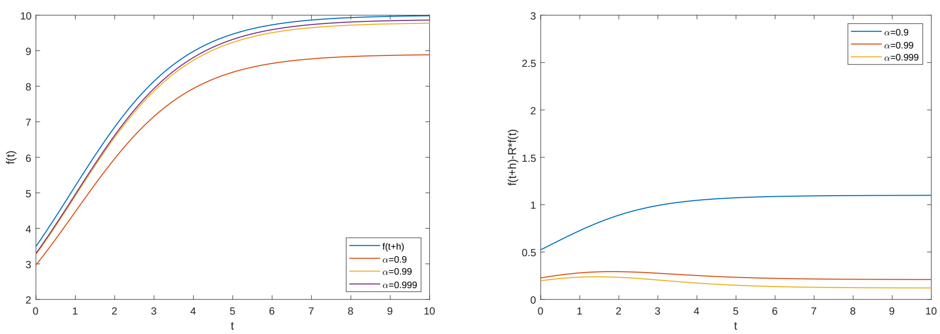
Example 4.1. Let us consider the classical Logistic growth function

$$f(t)=\frac{A}{1+Be^{-kt}}, \tag{4.1}$$

where $k > 0$, A, B and k are parameters. For the sake of our numerical example, let us choose $A = 10$, $B = 2$ and $k = 0.7$.

The function $f(t)$ has a positive derivative and for $0 < \alpha \leq 1$, $0 < h \leq 1$ and also $0 < \rho \leq 1$ we get $\left({}_{a-h}\nabla_h^{\alpha,\rho} f \right)(t) \geq 0$. Applying Theorem 3.1, $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$ – increasing. In order to verify this numerically and taking into account Definition 3.1, we just need to show that

$$f(t+h) \geq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(t). \tag{4.2}$$



(a) (b)

Figure 2. In Figure 2(a), the curves for function $f(t+h)$ compared to $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(t)$ for different values of α while $h = 0.1$ and $\rho = 0.9$. The differences between the values of the two functions are illustrated in Figure 2(b).

In Figure 2, the results for the numerical illustrations of Equation 4.2 are presented for $h = 0.1$, $\rho = 0.9$ and $\alpha = 0.9, 0.99, 0.999$, respectively. The numerical results support those presented theoretically in Theorem 3.1 and can easily apply for other theoretical results with minor modifications on the function and the parameters.

Example 4.2. In this example, we consider the Generalized Weibull function which is used to model growth in biological applications [30, 31]. The function can be derived by solving the ordinary differential equation:

$$\frac{df}{dt} = r_t f(t), \tag{4.3}$$

where r_t is the rate function given by

$$r_t = \left(\frac{k}{\delta}\right)\left(\frac{t-\mu}{\delta}\right)^{-1}\left(\frac{A}{f(t)} - 1\right), \tag{4.4}$$

where k, δ, μ, A are parameters and t represents time. Solving Equation 4.3 will give the general Weibull function:

$$f(t) = A(1 - Be^{(} - k(\frac{t-\mu}{\delta})^{v}). \tag{4.5}$$

For the sake of our example, we suppose that the value of the parameters $A = 10$, $B = 0.5$, $k = 0.01$, $\mu = 0.02$, $\delta = 3$ and $v = 2$.

The function $f(t)$ has a positive derivative and for $0 < \alpha \leq 1$, $0 < h \leq 1$ and also $0 < \rho \leq 1$ we get $\left({}_{a-h}\nabla_h^{\alpha,\rho} m \right)(t) \leq 0$. Applying Theorem 3.1, $m(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)-$ decreasing. In order to verify this numerically and taking into account Definition 3.1, we just need to show that

$$f(t+h) \geq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(t). \tag{4.6}$$

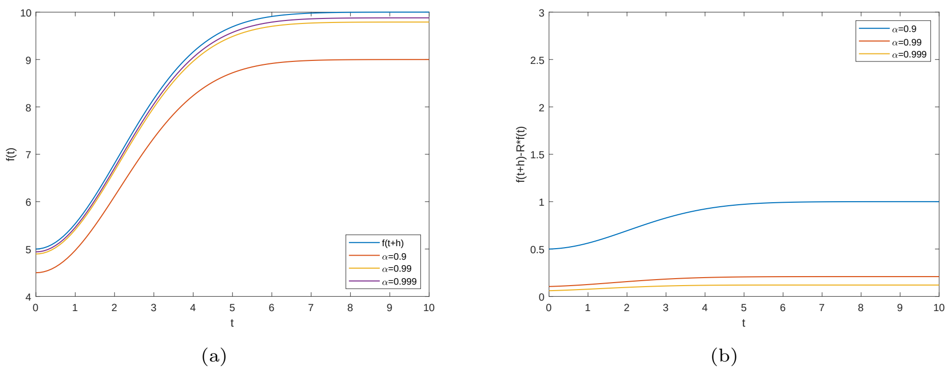


Figure 3. In Figure 3(a), the curves for function $f(t+h)$ compared to $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(t)$ for different values of α while $h = 0.1$ and $\rho = 0.9$. The differences between the values of the two functions are illustrated in Figure 3(b).

In Figure 3, the results for the numerical illustrations of Equation 4.6 are presented for $h = 0.1$, $\rho = 0.9$ and $\alpha = 0.9, 0.99, 0.999$, respectively. The numerical results are consistent with theoretical result presented in Theorem 3.4.

Both examples illustrate mathematical models commonly applied in real-world scenarios. The logistic growth function, used in biology, medicine, and economics, describes constrained population growth or the adoption of new technologies, where growth accelerates initially but slows as limits are reached. The generalized Weibull function, applied in biological growth models, reliability engineering, and environmental science, captures more complex, non-linear growth or decay patterns, such as tumor growth, system failure rates, or extreme weather events. These models, supported by numerical verification, are vital in predicting and understanding dynamic processes across various fields.

4.2. Generalized Mean Value Theorem

In this subsection, we present simple calculus application for the results generated in this work by introducing another version of the Mean Value Theorem (MVT). Before we start, let us first rewrite Equation 3.10 in Theorem 3.6 in a simplified form as follows

$$\left({}_a\nabla_h^{-\alpha,\rho} \; {}_{a-h}\nabla_h^{\alpha,\rho} f\right)(t) = f(t) - R_h^{\rho}(\alpha, t, a) f(a), \tag{4.7}$$

where $R_h^{\rho}(\alpha, t, a) = \dfrac{h^{1-\alpha}\widehat{e}_p(t,a)}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}$.

Theorem 4.1. *(The MVT for fractional h-proportional differences)*
Suppose that there exist two functions f and g that are defined on $\mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N} = \{a, a+h, a+2h, \cdots, b-2h, b-h, b\}$ such that $b = a+kh$ for any $k \in \mathbb{N}$. Furthermore, suppose that the function g is strictly increasing with $g(a) > 0$, $0 < h \leq 1$, $0 < \rho \leq 1$ and $0 < \alpha \leq 1$. Then $\exists\ s_1, s_2 \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}$ such that the following holds:

$$\frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(s_1)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(s_1)} \leq \frac{f(b) - R_h^{\rho}(\alpha,t,a)f(a)}{g(b) - R_h^{\rho}(\alpha,t,a)g(a)} \leq \frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(s_2)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(s_2)}. \tag{4.8}$$

Proof. We start by showing that $g(b) - R_h^{\rho}(\alpha,t,a)g(a) > 0$. Since the function g is assumed strictly increasing, then we can show that $g(b) - R_h^{\rho}(\alpha,t,a)g(a) > 0$ by using Theorem 3.3. We get

$$({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) > 0, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

Now, applying the operator of fractional sum for both sides we get:

$${}_{a}\nabla_h^{-\alpha,\rho}({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) > 0, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

Using Equation 4.7, we get

$${}_{a}\nabla_h^{-\alpha,\rho}({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) = g(t) - R_h^{\rho}(\alpha,t,a)g(a) > 0, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

When $t = b$, we get what we need as follows

$$g(b) - R_h^{\rho}(\alpha,b,a)g(a) > 0.$$

We now turn to prove this theorem using contradiction by assuming that Equation 4.8 doesn't hold. Then we have either

$$\frac{f(b) - R_h^{\rho}(\alpha,t,a)f(a)}{g(b) - R_h^{\rho}(\alpha,t,a)g(a)} < \frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(t)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(t)}, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}, \tag{4.9}$$

or we can get

$$\frac{f(b) - R_h^{\rho}(\alpha,t,a)f(a)}{g(b) - R_h^{\rho}(\alpha,t,a)g(a)} > \frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(t)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(t)}, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}. \tag{4.10}$$

But again since the function g is strictly increasing, then using Theorem 3.3 we have:

$$({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) > 0, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

Then, Equation 4.9 becomes

$$\frac{f(b) - R_h^{\rho}(\alpha,t,a)f(a)}{g(b) - R_h^{\rho}(\alpha,t,a)g(a)}({}_{a-h}\nabla_h^{-\alpha,\rho} g)(t) < ({}_{a-h}\nabla_h^{-\alpha,\rho} f)(t), \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}. \tag{4.11}$$

Applying the operator of fractional sum and making use of Equation 4.7 at $t = b$, we get

$$\frac{f(b) - R_h^{\rho}(\alpha,t,a)f(a)}{g(b) - R_h^{\rho}(\alpha,t,a)g(a)}(g(b) - R_h^{\rho}(\alpha,b,a)g(a)) < f(b) - R_h^{\rho}(\alpha,b,a)f(a). \tag{4.12}$$

This leads to $f(b) < f(b)$ and this is a contradiction. Following the same way using Equation 4.10 also leads to a contradiction. Hence, the proof is complete. $\square$

We illustrate the proposed Generalized MVT using the following numerical example:

Example 4.3. Let us consider the logistic growth function presented by Example 4.1:

$$f(t) = \frac{A}{1 + Be^{-kt}}, \tag{4.13}$$

where $k > 0$, A, B and k are parameters. For the sake of our numerical example, let us choose $A = 10$, $B = 2$ and $k = 0.7$. Let us use the generalized MVT on interval $t \in [0, 10]$ with $\alpha = 0.8$, $h = 0.1$, $\rho = 0.9$, and the function $g(t) = t$.

In order to check Equation 4.8, we first compute the fractional differences $({}_{a-h}\nabla_h^{-\alpha,\rho} f)(t)$ and $({}_{a-h}\nabla_h^{-\alpha,\rho} g)(t)$ at $s_1 = 3$ and $s_2 = 7$. We get the following:

For $s_1 = 3$,

$$\frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(s_1)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(s_1)} \approx 1.5,$$

and for $s_2 = 7$,

$$\frac{({}_{a-h}\nabla_h^{-\alpha,\rho} f)(s_1)}{({}_{a-h}\nabla_h^{-\alpha,\rho} g)(s_1)} \approx 0.7.$$

Now we compute the middle term of the Generalized MVT:

$$\frac{f(b) - R_h^\rho(\alpha, t, a) f(a)}{g(b) - R_h^\rho(\alpha, t, a) g(a)}.$$

Using the given parameters, the coefficient $R_h^\rho(\alpha, t, a) \approx 0.7$, So we get:

$$\frac{f(b) - R_h^\rho(\alpha, t, a) f(a)}{g(b) - R_h^\rho(\alpha, t, a) g(a)} = \frac{9.96 - (0.7 \cdot 3.33)}{10 - 0.7 \cdot 0} = 0.763.$$

Clearly, the result 0.763 lies between 1.5 and 0.7, satisfying the Generalized MVT. This confirms the validity of the Generalized MVT for fractional proportional differences in this case. The computed fractional differences at the points $s_1 = 3$ and $s_2 = 7$ accurately bound the middle term, demonstrating the application of the theorem to a logistic growth function.

5. Conclusions

In this work, we conducted a monotonicity analysis and presented the results for a generalized class of discrete fractional proportional h-differences. We first defined the sums and differences that are associated with $\left({}_a\nabla_h^{\alpha,\rho} f\right)(t)$ for $0 < \alpha \leq 1$, $0 < h \leq 1$ and $0 < \rho \leq 1$. We used the basic definitions in Section 2 to re-establish the relation between nabla and Caputo fractional proportional h-differences. Afterwards, we moved to present the main monotonicity results and we proved that if the proportional h-differences satisfies the general form $\left({}_{a-h}\nabla_h^{\alpha,\rho} f\right)(t) > 0$, then the function $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$ −increasing. Moreover, the monotonicity results for Caputo fractional proportional differences is introduced. At the end, we presented two applications to support the theoretical results. The first is a numerical illustration for the result presented in Theorem 3.1 and is presented as two numerical examples. The second is a direct application that follows the monotonicity results in calculus where a general version of the Mean Value Theorem on $\mathbb{Z}$ for fractional proportional h-differences is proposed. A good topic that follows is to investigate the monotonicity analysis for fractional proportional h- differences with memory.

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