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A Novel Variant of Milne's Rule Inequalities on Quantum Calculus for Convex Functions with Their Computational Analysis

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Abstract In this investigation, we introduce a novel approach for establishing Milne's type inequalities in the context of quantum calculus for differentiable convex functions. First, we prove a quantum integral identity. We derive numerous new Milne's rule inequalities for quantum differentiable convex functions. These inequalities are relevant in open Newton-Cotes formulas, as they facilitate the determination of bounds for Milne's rule applicable to differentiable convex functions in both classical and q-calculus. In addition, we conduct a computational analysis of these inequalities for convex functions and provide mathematical examples to demonstrate the validity of the newly established results within the framework of q-calculus.

Keywords Milne's inequality, q-calculus, convex functions

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1. Introduction

Convexity, a fundamental mathematical notion derived from ancient Greek philosophy, acquired substantial traction in the late nineteenth century, mainly due to the pioneering work of German mathematician Karl Hermann Amandus Schwarz, who introduced convex functions [13]. Convexity has numerous modern applications in economics, engineering, computer science, and mathematics, particularly in optimization problems and inequalities [19, 29]. Considerable study has demonstrated the strong relationship between convexity theory and integral inequalities, emphasizing their critical roles in differential equations and applied mathematics. This relationship is critical due to the broad range of applications and the significant

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impact of integral inequalities. Exploring numerous inequalities such as Gronwall, Simpson's type, Chebyshev, Jensen, Hölder, Milne, and Hermite-Hadamard (H-H) inequalities enriches the general comprehension of mathematical concepts. For those interested in delving deeper into these inequalities and their practical applications, references [1, 2, 24, 29, 36] provide valuable insights.

The H-H inequality for convex functions is one of several inequalities that can be deduced directly from the applications of convex functions. The H-H inequality, originated by C. Hermite and J. Hadamard, is a cornerstone in the field of convex functions, known for its geometric interpretation and numerous applications [21, 22].

$$\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)d\zeta \leq \frac{\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{2}. \tag{1.1}$$

This inequality has numerous advantages, particularly its widespread application in approximation theory. Its vast applications prompted mathematicians to begin developing it, which resulted in the publication of multiple new results. The trapezoidal and midpoint-type inequalities are reported in [14, 25] by employing the principles of differentiable convexity. Numerous studies have been accomplished over the past twenty years to find new bounds for the inequality on the left and right sides of (1.1). For more information, see [17, 30].

The q-H-H type inequality expands the traditional H-H inequality in mathematical analysis. It offers constraints on convex functions by considering their values at the interval's endpoints. Alp et al. [3] applied quantum calculus techniques to establish a novel variant of the H-H inequality (1.1), as follows:

$$\mathsf{F}\left(\frac{q\sigma+\rho}{1+q}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)_{\sigma}d_q\zeta \leq \frac{q\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{1+q}.$$

Burmudo et al. [6] proposed a novel formulation specifically designed for q values occurring within the interval $(0,1)$, presented as follows:

$$\mathsf{F}\left(\frac{\sigma+q\rho}{1+q}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)^{\rho}d_q\zeta \leq \frac{\mathsf{F}(\sigma)+q\mathsf{F}(\rho)}{1+q},$$

and

$$\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\zeta)_{\sigma}d_q\zeta + \int_{\sigma}^{\rho}\mathsf{F}(\zeta)^{\rho}d_q\zeta\right] \leq \frac{\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{2}.$$

In recent years, numerous researchers have focused on Milne's type inequality across various categories of mappings. Recognizing the versatility and effectiveness of convexity theory, they utilize it to tackle problems spanning multiple fields of pure and applied mathematics.

Budak et al. [10] revealed fractional versions of Milne-type inequalities for differentiable convex functions. Celik et al. [12] generalized Milne-type inequality for conformable fractional integrals. Also, they discussed different function classes. In [23], Hezenci et al. proposed a tempered fractional version of Milne-type inequality. Demir demonstrated multiple integral inequalities connected to Milne-type integral inequalities concerning the proportional Caputo-hybrid operator [15]. Ali et al. [5] established the error bounds for Milne's formula, a variant of the open Newton-Cotes formulas designed for differentiable convex functions within fractional and classical calculus frameworks.

On the contrary, many investigations in q-analysis trace back to Euler's pioneering work. Quantum calculus has numerous applications in number theory, combinatorics, orthogonal polynomials, fundamental hypergeometric functions, quantum theory, and the theory of relativity, among other branches of mathematics and physics. Due to the tremendous consideration, this topic has garnered significant attention from researchers and is regarded as an interdisciplinary field bridging mathematics and physics. For further insights into recent advancements in the theory of quantum calculus and the theory of inequalities within this field, interested readers are encouraged to explore [18, 20, 33].

Over the recent span, Tariboon and Ntouyas [34] revealed a comprehensive examination of q-derivatives and q-integrals within the domain $[\sigma, \rho] \subset \mathcal{R}$. Their research has yielded significant advancements by establishing quantum analogues of renowned mathematical results, such as H-H inequality, Hölder inequality, Ostrowski inequality, and other integral inequalities using classical convexity. Khan et al. [26] formulated quantum H-H inequality by employing the Green function. Noor et al. [28] examined the generalized form of q-integral inequalities. Kalsoom et al. [27] extended the formulation of the quantum Montgomery identity utilizing the quantum integral. Leveraging this, they introduced novel Ostrowski-type inequalities derived from the newly established identity. Advancing Simpson-Newton-type inequalities by applying Mercer's convexity in quantum calculus, Butt et al. [7] unveiled new quantum bounds through Hölder's inequality and the power mean inequality. Their investigation yields insight and expands upon earlier conclusions. For (α, m) convex function, Sial et al. [31] proposed a modified form of Simpson's and Newton's type inequalities. Numerous scholarly articles have been devoted to expanding and broadening the scope of quantum calculus. Here, we list some of them for interested readers [11, 32, 35].

Motivated by ongoing investigations, we develop Milne-type inequalities by leveraging the function's convexity property within the framework of q-calculus. We provide numerical examples to validate the effectiveness of these newly derived inequalities.

2. Preliminaries

In this section, we revisit the definitions and properties of q-derivatives and q-integrals. In this paper, q is considered a constant with $0 < q < 1$, and $[\sigma, \rho] \subseteq \mathcal{R}$ represents an interval with $\sigma < \rho$. The q-number is defined as follows:

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathcal{N}.$$

In 2013, Tariboon and Ntouyas [34] initiated the q_σ-derivative and integral. They also provided their properties. Here, we recollect the subsequent definitions from their work:

Definition 2.1 (See [34]). Let $\mathsf{F} : [\sigma, \rho] \to \mathcal{R}$ be a continuous function. Then the q_σ-derivative of F at $\varkappa \in (\sigma, \rho]$ is outlined as

$${}_\sigma D_q \mathsf{F}(\varkappa) = \frac{\mathsf{F}(\varkappa) - \mathsf{F}(\sigma + q(\varkappa - \sigma))}{(1 - q)(\varkappa - \sigma)}.$$

The q_σ-integral is described as

$$\int_\sigma^\varkappa \mathsf{F}(\zeta)_\sigma d_q\zeta = (1-q)(\varkappa-\sigma)\sum_{n=0}^{\infty} q^n \mathsf{F}\left(\sigma + q^n(\varkappa-\sigma)\right).$$

Bermudo et al. [6] unveiled a novel approach involving the q^ρ derivative and integral. They also examined several fundamental properties of these operators. Here, we recapitulate the definitions provided in their study:

Definition 2.2 (See [6]). Let $\mathsf{F} : [\sigma,\rho] \to \mathcal{R}$ be a continuous function. Then the q^ρ-derivative of F at $\varkappa \in [\sigma,\rho)$ is outlined as

$$^\rho D_q \mathsf{F}(\varkappa) = \frac{\mathsf{F}(\rho + q(\varkappa-\rho)) - \mathsf{F}(\varkappa)}{(1-q)(\rho-\varkappa)}.$$

The q^ρ-integral is described as

$$\int_x^\rho \mathsf{F}(\zeta)^\rho d_q\zeta = (1-q)(\rho-\varkappa)\sum_{n=0}^{\infty} q^n \mathsf{F}\left(\rho + q^n(\varkappa-\rho)\right).$$

In [4] and [31], the authors provide the following formulas of q-integration by parts:

Lemma 2.1. *For continuous functions $h, \mathsf{F} : [\sigma,\rho] \to \mathcal{R}$, the subsequent equality is valid:*

$$\begin{aligned}&\int_0^c h(\zeta)_\sigma D_q \mathsf{F}(\zeta\rho + (1-\zeta)\sigma) d_q\zeta \\ =& \left.\frac{h(\zeta)\mathsf{F}(\zeta\rho+(1-\zeta)\sigma)}{\rho-\sigma}\right|_0^c - \frac{1}{\rho-\sigma}\int_0^c D_q h(\zeta)\mathsf{F}(q\zeta\rho + (1-q\zeta)\sigma) d_q\zeta.\end{aligned}$$

Lemma 2.2. *For continuous functions $h, \mathsf{F} : [\sigma,\rho] \to \mathcal{R}$, the subsequent equality is valid:*

$$\begin{aligned}&\int_0^c h(\zeta)^\rho D_q \mathsf{F}(\zeta\sigma + (1-\zeta)\rho) d_q\zeta \\ =& \frac{1}{\rho-\sigma}\int_0^c D_q h(\zeta)\mathsf{F}(q\zeta\sigma + (1-q\zeta)\rho) d_q\zeta - \left.\frac{h(\zeta)\mathsf{F}(\zeta\sigma+(1-\zeta)\rho)}{\rho-\sigma}\right|_0^c.\end{aligned}$$

3. Main results

Initially, we establish the essential identity that will enable us to achieve the intended outcomes by utilising quantum differentiable functions.

Lemma 3.1. *Assume $\mathsf{F} : [\sigma,\rho] \to \mathcal{R}$ is a q-differentiable function. If $_\sigma D_q \mathsf{F}(\zeta)$ and $^\rho D_q \mathsf{F}(\zeta)$ are q-integrable on $[\sigma,\rho]$, then the following equality is valid:*

$$\begin{aligned}&\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa + \int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa\right] - \frac{1}{3}\left[2\mathsf{F}(\sigma) - \mathsf{F}\left(\frac{\sigma+\rho}{2}\right) + 2\mathsf{F}(\rho)\right] \\ =& \frac{\rho-\sigma}{2}\left[I_1 + I_2 - I_3 - I_4\right], \qquad (3.1)\end{aligned}$$

where

$$I_1 = \int_0^{\frac{1}{2}} \left(q\zeta - \frac{2}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta,$$
$$I_2 = \int_{\frac{1}{2}}^1 \left(q\zeta - \frac{1}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta,$$
$$I_3 = \int_0^{\frac{1}{2}} \left(q\zeta - \frac{2}{3}\right) {}_{\sigma}D_q \mathsf{F}(\sigma + \zeta(\rho - \sigma)) d_q\zeta,$$

and

$$I_4 = \int_{\frac{1}{2}}^1 \left(q\zeta - \frac{1}{3}\right) {}_{\sigma}D_q \mathsf{F}(\sigma + \zeta(\rho - \sigma)) d_q\zeta.$$

Proof. By utilizing the definition of q-integral, we acquire

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left(q\zeta - \frac{2}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &= \int_0^{\frac{1}{2}} q\zeta {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta - \frac{2}{3} \int_0^{\frac{1}{2}} {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta. \end{aligned}$$

Likewise,

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 \left(q\zeta - \frac{1}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &= \int_0^1 \left(q\zeta - \frac{1}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &\quad - \int_0^{\frac{1}{2}} \left(q\zeta - \frac{1}{3}\right) {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &= \int_0^1 q\zeta {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta - \frac{1}{3} \int_0^1 {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &\quad - \int_0^{\frac{1}{2}} q\zeta {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta + \frac{1}{3} \int_0^{\frac{1}{2}} {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta. \end{aligned}$$

Therefore,

$$\begin{aligned} I_1 + I_2 &= \int_0^1 q\zeta {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta - \frac{1}{3} \int_0^1 {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \\ &\quad - \frac{1}{3} \int_0^{\frac{1}{2}} {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta. \end{aligned} \tag{3.2}$$

By Lemma 2.2, we attain

$$\int_0^1 q\zeta {}^{\rho}D_q \mathsf{F}(\rho + \zeta(\sigma - \rho)) d_q\zeta \tag{3.3}$$

$$
\begin{aligned}
=&\frac{1}{\rho-\sigma}\int_0^1 q\mathsf{F}(q\zeta\sigma+(1-q\zeta)\rho)d_q\zeta-\left.\frac{q\zeta\mathsf{F}(\rho+\zeta(\sigma-\rho))}{\rho-\sigma}\right|_0^1\\
=&\frac{1-q}{\rho-\sigma}\sum_{n=0}^{\infty}q^{n+1}\mathsf{F}\left(q^{n+1}\sigma+\left(1-q^{n+1}\right)\rho\right)-\frac{q}{\rho-\sigma}\mathsf{F}(\sigma)\\
=&\frac{1-q}{\rho-\sigma}\sum_{n=0}^{\infty}q^{n}\mathsf{F}\left(q^{n}\sigma+\left(1-q^{n}\right)\rho\right)-\frac{1}{\rho-\sigma}\mathsf{F}(\sigma)\\
=&\frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa-\frac{1}{\rho-\sigma}\mathsf{F}(\sigma).
\end{aligned}
$$

Similarly,

$$
\int_0^1 {}^\rho D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))d_q\zeta=-\frac{1}{\rho-\sigma}\left[\mathsf{F}(\sigma)-\mathsf{F}(\rho)\right], \tag{3.4}
$$

$$
\int_0^{\frac{1}{2}} {}^\rho D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))d_q\zeta=-\frac{1}{\rho-\sigma}\left[\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)-\mathsf{F}(\rho)\right]. \tag{3.5}
$$

By substituting the equalities (3.3)-(3.5) into equation (3.2), we arrive at

$$
I_1+I_2=\frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa-\frac{1}{3(\rho-\sigma)}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]. \tag{3.6}
$$

Similarly, we have

$$
I_3+I_4=\frac{1}{3(\rho-\sigma)}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]-\frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa. \tag{3.7}
$$

The equalities (3.6) and (3.7) yield the following equality:

$$
\begin{aligned}
&\frac{\rho-\sigma}{2}\left[I_1+I_2-I_3-I_4\right]\\
=&\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa+\int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right].
\end{aligned}
$$

The proof of Lemma 3.1 is concluded. □

Theorem 3.1. *Consider the conditions outlined in Lemma 3.1 hold. If* $|{}^\rho D_q\mathsf{F}(\zeta)|$ *and* $|{}_\sigma D_q\mathsf{F}(\zeta)|$ *are convex on* $[\sigma,\rho]$*, then the subsequent inequality is valid:*

$$
\begin{aligned}
&\left|\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa+\int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\
\leq&\frac{\rho-\sigma}{2}\Big[\left(\mathcal{A}_1(q)+\mathcal{A}_3(q)\right)\left[|{}^\rho D_q\mathsf{F}(\sigma)|+|{}_\sigma D_q\mathsf{F}(\rho)|\right]\\
&+\left(\mathcal{A}_2(q)+\mathcal{A}_4(q)\right)\left[|{}^\rho D_q\mathsf{F}(\rho)|+|{}_\sigma D_q\mathsf{F}(\sigma)|\right]\Big],
\end{aligned}
\tag{3.8}
$$

where

$$
\mathcal{A}_1(q)=\int_0^{\frac{1}{2}}\zeta\left|q\zeta-\frac{2}{3}\right|d_q\zeta=\frac{4+q+q^2}{24[2]_q[3]_q},
$$

$$\mathcal{A}_2(q)=\int_0^{\frac{1}{2}}(1-\zeta)\left|q\zeta-\frac{2}{3}\right|d_q\zeta=\frac{4+9q+9q^2+2q^3}{24[2]_q[3]_q},$$

$$\mathcal{A}_3(q)=\int_{\frac{1}{2}}^1\zeta\left|q\zeta-\frac{1}{3}\right|d_q\zeta=\begin{cases}\frac{2-5q-5q^2}{8[2]_q[3]_q}, & 0<q\leq\frac{1}{3},\\ \frac{-74+153q+153q^2}{216[2]_q[3]_q}, & \frac{1}{3}<q\leq\frac{2}{3},\\ \frac{-2+5q+5q^2}{8[2]_q[3]_q}, & \frac{2}{3}<q<1,\end{cases}$$

and

$$\mathcal{A}_4(q)=\int_{\frac{1}{2}}^1(1-\zeta)\left|q\zeta-\frac{1}{3}\right|d_q\zeta=\begin{cases}\frac{q+q^2-20q^3}{24[2]_q[3]_q}, & 0<q\leq\frac{1}{3},\\ \frac{14-51q-51q^2+162q^3}{216[2]_q[3]_q}, & \frac{1}{3}<q\leq\frac{2}{3},\\ \frac{2-5q-5q^2+14q^3}{24[2]_q[3]_q}, & \frac{2}{3}<q<1.\end{cases}$$

Proof. By Lemma 3.1, we attain

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa+\int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\ \leq&\frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}^\rho D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|d_q\zeta+\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}_\sigma D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|d_q\zeta\right.\\ &\left.+\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left|{}^\rho D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|d_q\zeta+\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left|{}_\sigma D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|d_q\zeta\right].\end{aligned}\tag{3.9}$$

As $|{}^\rho D_q\mathsf{F}(\zeta)|$ and $|{}_\sigma D_q\mathsf{F}(\zeta)|$ are convex on $[\sigma,\rho]$, it yields

$$\left|{}^\rho D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|\leq\zeta\left|{}^\rho D_q\mathsf{F}(\sigma)\right|+(1-\zeta)\left|{}^\rho D_q\mathsf{F}(\rho)\right|,\tag{3.10}$$

and

$$\left|{}_\sigma D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|\leq\zeta\left|{}_\sigma D_q\mathsf{F}(\rho)\right|+(1-\zeta)\left|{}_\sigma D_q\mathsf{F}(\sigma)\right|.\tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we find

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho \mathsf{F}(\varkappa)^\rho d_q\varkappa+\int_\sigma^\rho \mathsf{F}(\varkappa)_\sigma d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\ \leq&\frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left[\left(\zeta\left|{}^\rho D_q\mathsf{F}(\sigma)\right|+(1-\zeta)\left|{}^\rho D_q\mathsf{F}(\rho)\right|\right)+\left(\zeta\left|{}_\sigma D_q\mathsf{F}(\rho)\right|\right.\right.\right.\\ &\left.\left.+(1-\zeta)\left|{}_\sigma D_q\mathsf{F}(\sigma)\right|\right)\right]d_q\zeta+\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left[\left(\zeta\left|{}^\rho D_q\mathsf{F}(\sigma)\right|+(1-\zeta)\left|{}^\rho D_q\mathsf{F}(\rho)\right|\right)\right.\\ &\left.\left.+\left(\zeta\left|{}_\sigma D_q\mathsf{F}(\rho)\right|+(1-\zeta)\left|{}_\sigma D_q\mathsf{F}(\sigma)\right|\right)\right]d_q\zeta\right]\\ =&\frac{\rho-\sigma}{2}\left[\left[\left|{}^\rho D_q\mathsf{F}(\sigma)\right|+\left|{}_\sigma D_q\mathsf{F}(\rho)\right|\right]\int_0^{\frac{1}{2}}\zeta\left|q\zeta-\frac{2}{3}\right|d_q\zeta\right.\\ &+\left[\left|{}^\rho D_q\mathsf{F}(\rho)\right|+\left|{}_\sigma D_q\mathsf{F}(\sigma)\right|\right]\int_0^{\frac{1}{2}}(1-\zeta)\left|q\zeta-\frac{2}{3}\right|d_q\zeta\\ &+\left[\left|{}^\rho D_q\mathsf{F}(\sigma)\right|+\left|{}_\sigma D_q\mathsf{F}(\rho)\right|\right]\int_{\frac{1}{2}}^1\zeta\left|q\zeta-\frac{1}{3}\right|d_q\zeta\end{aligned}$$

$$+\left[\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|\right]\int_{\frac{1}{2}}^{1}(1-\zeta)\left|q\zeta-\frac{1}{3}\right|d_q\zeta\Bigg].$$

By computing the quantum integrals, we acquire

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\&\leq\frac{\rho-\sigma}{2}\left[(\mathcal{A}_1(q)+\mathcal{A}_3(q))\left[\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|+\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|\right]\right.\\&\quad\left.+(\mathcal{A}_2(q)+\mathcal{A}_4(q))\left[\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|\right]\right].\end{aligned}$$

□

Remark 3.1. By setting $q\to 1^-$ in Theorem 3.1, we attain the subsequent inequality:

$$\begin{aligned}&\left|\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)d\varkappa-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\&\leq\frac{5(\rho-\sigma)}{24}\left(\left|\mathsf{F}'(\sigma)\right|+\left|\mathsf{F}'(\rho)\right|\right),\end{aligned}$$

which is obtained in [10].

Theorem 3.2. *Consider the conditions outlined in Lemma 3.1 hold. If* $\left|{}^{\rho}D_q\mathsf{F}(\zeta)\right|^s$ *and* $\left|{}_{\sigma}D_q\mathsf{F}(\zeta)\right|^s$ *are convex on* $[\sigma,\rho]$ *and* $\frac{1}{p}+\frac{1}{s}=1$ *with* $p,s>1$, *then the subsequent inequality is valid:*

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\&\leq\frac{\rho-\sigma}{2}\left[\left(\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\left\{\left(\frac{\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+(1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right.\right.\\&\quad\left.+\left(\frac{\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+(1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right\}+\left(\int_{\frac{1}{2}}^{1}\left|q\zeta-\frac{1}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\\&\quad\times\left\{\left(\frac{3\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+(-1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right.\\&\quad\left.\left.+\left(\frac{3\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+(-1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right\}\right].\end{aligned}\tag{3.12}$$

Proof. By utilizing q-Hölder's inequality in (3.9), it yields

$$\left|\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_q\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)_{\sigma}d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\tag{3.13}$$

$$\leq\frac{\rho-\sigma}{2}\left[\left(\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right.\right.$$

$$+\left(\int_0^{\frac{1}{2}}\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta\right)^{\frac{1}{s}}\Bigg\}+\left(\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\int_{\frac{1}{2}}^1\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\right)^{\frac{1}{s}}+\left(\int_{\frac{1}{2}}^1\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right\}\Bigg].$$

As $\left|{}^{\rho}D_q\mathsf{F}\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}\right|^s$ are convex on $[\sigma,\rho]$, it yields

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta &\le \int_0^{\frac{1}{2}}\left[\zeta\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+(1-\zeta)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right]d_q\zeta \\ &= \frac{1}{4[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\frac{1+2q}{4[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s.\end{aligned} \tag{3.14}$$

Similarly,

$$\int_0^{\frac{1}{2}}\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta \le \frac{1}{4[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+\frac{1+2q}{4[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s, \tag{3.15}$$

$$\int_{\frac{1}{2}}^1\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta \le \frac{3}{4[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\frac{-1+2q}{4[2]_q}\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s, \tag{3.16}$$

and

$$\int_{\frac{1}{2}}^1\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta \le \frac{3}{4[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+\frac{-1+2q}{4[2]_q}\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s. \tag{3.17}$$

By substituting (3.14)-(3.17) in (3.13), we achieve the desired result. □

Remark 3.2. By setting $q\to 1^-$ in Theorem 3.2, we attain the subsequent inequality:

$$\begin{aligned}&\left|\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)d\varkappa-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right| \\ \le&\frac{\rho-\sigma}{2}\left(\frac{4^{p+1}-1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathsf{F}'(\sigma)|^s+3|\mathsf{F}'(\rho)|^s}{8}\right)^{\frac{1}{s}}+\left(\frac{3|\mathsf{F}'(\sigma)|^s+|\mathsf{F}'(\rho)|^s}{8}\right)^{\frac{1}{s}}\right. \\ &\left.+\left(\frac{|\mathsf{F}'(\rho)|^s+3|\mathsf{F}'(\sigma)|^s}{8}\right)^{\frac{1}{s}}+\left(\frac{3|\mathsf{F}'(\rho)|^s+|\mathsf{F}'(\sigma)|^s}{8}\right)^{\frac{1}{s}}\right],\end{aligned}$$

which is obtained in [9, Corollary 3.1].

Theorem 3.3. *Consider the conditions outlined in Lemma 3.1 hold. If* $\left|{}^{\rho}D_q\mathsf{F}(\zeta)\right|^s$ *and* $\left|{}_{\sigma}D_q\mathsf{F}(\zeta)\right|^s$ *are convex on* $[\sigma,\rho]$ *for* $s\ge 1$*, then the subsequent inequality is valid:*

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)\,{}^{\rho}d_q\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)\,{}_{\sigma}d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right| \\ \le&\frac{\rho-\sigma}{2}\left[(\mathcal{A}_5(q))^{1-\frac{1}{s}}\left\{\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}+\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right.\right.\right. \\ &\left.\left.+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\right\}+(\mathcal{A}_6(q))^{1-\frac{1}{s}}\left\{\left(\mathcal{A}_3(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_4(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}\right.\end{aligned}$$

$$+\left(\mathcal{A}_3(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+\mathcal{A}_4(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\Big\}\Big], \tag{3.18}$$

where

$$\mathcal{A}_5(q)=\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|d_q\zeta=\frac{4+q}{12[2]_q},$$

$$\mathcal{A}_6(q)=\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|d_q\zeta=\begin{cases}\frac{2-7q}{12[2]_q}, & 0<q\le\frac{1}{3},\\ \frac{-10+27q}{36[2]_q}, & \frac{1}{3}<q\le\frac{2}{3},\\ \frac{-2+7q}{12[2]_q}, & \frac{2}{3}<q<1.\end{cases}$$

$\mathcal{A}_1(q)-\mathcal{A}_4(q)$ are defined as in Theorem 3.1.

Proof. By employing q-power mean inequality in (3.9), we acquire

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_\sigma^\rho\mathsf{F}(\varkappa)^\rho d_q\varkappa+\int_\sigma^\rho\mathsf{F}(\varkappa)_\sigma d_q\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\
\le&\frac{\rho-\sigma}{2}\left[\left(\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|d_q\zeta\right)^{1-\frac{1}{s}}\left\{\left(\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right\}+\left(\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|d_q\zeta\right)^{1-\frac{1}{s}}\\
&\times\left\{\left(\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta\right)^{\frac{1}{s}}\right\}\right].\end{aligned} \tag{3.19}$$

As $\left|{}^{\rho}D_q\mathsf{F}\right|^s$ and $\left|{}_{\sigma}D_q\mathsf{F}\right|^s$ are convex on $[\sigma,\rho]$ and the equalities attained in the proof of Theorem 3.1, it yields

$$\begin{aligned}&\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\\
\le&\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left[\zeta\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+(1-\zeta)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right]d_q\zeta\\
=&\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s.\end{aligned} \tag{3.20}$$

Similarly,

$$\int_0^{\frac{1}{2}}\left|q\zeta-\frac{2}{3}\right|\left|{}_{\sigma}D_q\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^s d_q\zeta\le\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s, \tag{3.21}$$

$$\int_{\frac{1}{2}}^1\left|q\zeta-\frac{1}{3}\right|\left|{}^{\rho}D_q\mathsf{F}(\rho+\zeta(\sigma-\rho))\right|^s d_q\zeta\le\mathcal{A}_3(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_4(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s, \tag{3.22}$$

and

$$\int_{\frac{1}{2}}^{1}\left|q\zeta-\frac{1}{3}\right|\left|{}_{\sigma}D_{q}\mathsf{F}(\sigma+\zeta(\rho-\sigma))\right|^{s}d_{q}\zeta\leq\mathcal{A}_{3}(q)\left|{}_{\sigma}D_{q}\mathsf{F}(\rho)\right|^{s}+\mathcal{A}_{4}(q)\left|{}_{\sigma}D_{q}\mathsf{F}(\sigma)\right|^{s}. \tag{3.23}$$

By substituting (3.20)-(3.23) in (3.19), we achieve the desired result. □

Remark 3.3. By setting $q\to 1^{-}$ in Theorem 3.3, we attain the subsequent inequality:

$$\begin{aligned}&\left|\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)d\varkappa-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\&\leq\frac{5(\rho-\sigma)}{24}\left[\left(\frac{|\mathsf{F}'(\sigma)|^{s}+4|\mathsf{F}'(\rho)|^{s}}{5}\right)^{\frac{1}{s}}+\left(\frac{4|\mathsf{F}'(\sigma)|^{s}+|\mathsf{F}'(\rho)|^{s}}{5}\right)^{\frac{1}{s}}\right],\end{aligned}$$

which is obtained in [10, Remark 2].

4. Examples

In this section, we provide examples that validate our theorems.

Example 4.1. Consider the function $\mathsf{F}:[0,1]\to\mathcal{R}$ defined as $\mathsf{F}(\varkappa)=\varkappa^{3}$. Then F is q-differentiable. Under these assumptions, we have

$${}^{\rho}D_{q}\mathsf{F}(\varkappa)={}^{1}D_{q}\mathsf{F}(\varkappa)=1+\varkappa+\varkappa^{2}+q(\varkappa^{2}+\varkappa-2)+q^{2}(x-1)^{2},$$

and

$${}_{\sigma}D_{q}\mathsf{F}(\varkappa)={}_{0}D_{q}\mathsf{F}(\varkappa)=[3]_{q}\varkappa^{2}.$$

These functions are convex on $[0,1]$. By Theorem 3.1 to the function $\mathsf{F}(\varkappa)=\varkappa^{3}$, we attain

$$\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]=0.625,$$

and

$$\begin{aligned}\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_{q}\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)_{\sigma}d_{q}\varkappa\right]&=\frac{1}{2}\left[\int_{0}^{1}\varkappa^{3}\ {}^{1}d_{q}\varkappa+\int_{0}^{1}\varkappa^{3}\ {}_{0}d_{q}\varkappa\right]\\&=\frac{1}{2}\left[1-\frac{3}{[2]_{q}}+\frac{3}{[3]_{q}}\right].\end{aligned}$$

Thus, the left-hand side of (3.8) is

$$\begin{aligned}&\left|\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)^{\rho}d_{q}\varkappa+\int_{\sigma}^{\rho}\mathsf{F}(\varkappa)_{\sigma}d_{q}\varkappa\right]-\frac{1}{3}\left[2\mathsf{F}(\sigma)-\mathsf{F}\left(\frac{\sigma+\rho}{2}\right)+2\mathsf{F}(\rho)\right]\right|\\&=\left|\frac{1}{2}\left[1-\frac{3}{[2]_{q}}+\frac{3}{[3]_{q}}\right]-0.625\right|.\end{aligned} \tag{4.1}$$

Now, we let

$$\left|{}^{\rho}D_{q}\mathsf{F}(\sigma)\right|=\left|{}^{1}D_{q}\mathsf{F}(0)\right|=(1-q)^{2},\quad\left|{}_{\sigma}D_{q}\mathsf{F}(\rho)\right|=\left|{}_{0}D_{q}\mathsf{F}(1)\right|=[3]_{q},$$

$$\left|{}^{\rho}D_q\mathsf{F}(\rho)\right| = \left|{}^{1}D_q\mathsf{F}(1)\right| = 3, \qquad \left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right| = \left|{}_{0}D_q\mathsf{F}(0)\right| = 0.$$

Hence, the right-hand side of (3.8) is

$$\begin{aligned}&\frac{\rho-\sigma}{2}\left[(\mathcal{A}_1(q)+\mathcal{A}_3(q))\left[\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|+\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|\right]\right.\\&\left.\quad+(\mathcal{A}_2(q)+\mathcal{A}_4(q))\left[\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|+\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|\right]\right]\\&=\frac{1}{2}\left[\left(\frac{4+q+q^2}{24[2]_q[3]_q}+\mathcal{A}_3(q)\right)\left[(1-q)^2+[3]_q\right]+3\left(\frac{4+9q+9q^2+2q^3}{24[2]_q[3]_q}+\mathcal{A}_4(q)\right)\right].\end{aligned}$$

By inequality (3.8), we have

$$\begin{aligned}\left|\frac{1}{2}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}\right]-0.625\right| \le &\frac{1}{2}\left[\left(\frac{4+q+q^2}{24[2]_q[3]_q}+\mathcal{A}_3(q)\right)\left[(1-q)^2+[3]_q\right]\right.\\&\left.+3\left(\frac{4+9q+9q^2+2q^3}{24[2]_q[3]_q}+\mathcal{A}_4(q)\right)\right].\end{aligned} \tag{4.2}$$

Figures 1-3 demonstrate the validity of inequality (4.2).

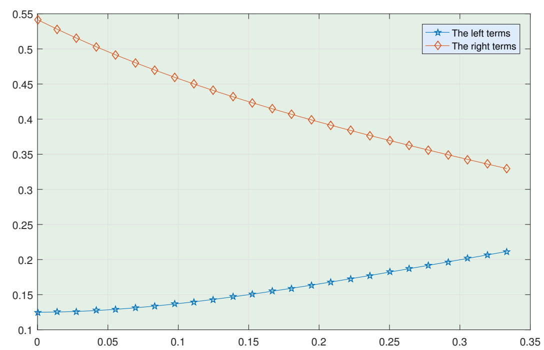

Figure 1. The graphs of left-hand side and right-hand side of the inequality (4.2) for $q \in \left(0, \frac{1}{3}\right]$

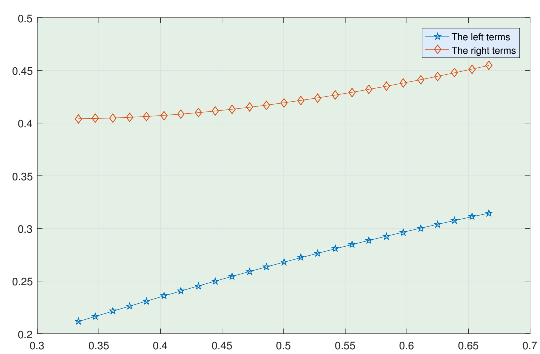

Figure 2. The graphs of left-hand side and right-hand side of the inequality (4.2) for $q \in \left(\frac{1}{3}, \frac{2}{3}\right]$

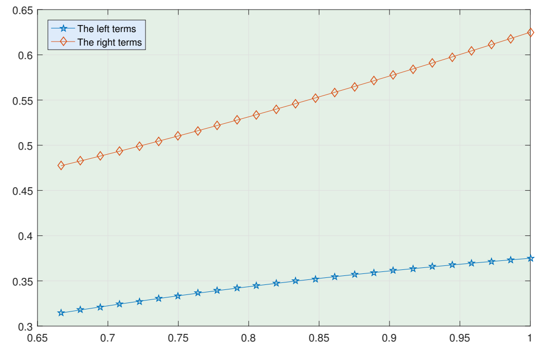

Figure 3. The graphs of left-hand side and right-hand side of the inequality (4.2) for $q \in \left(\frac{2}{3}, 1\right)$

Example 4.2. Consider the function $\mathsf{F} : [0,1] \to \mathcal{R}$ defined as $\mathsf{F}(\varkappa) = \varkappa^3$ and $p = s = 2$. Then F is q-differentiable. Under these assumptions, we have

$$\left|{}^{\rho}D_q\mathsf{F}(\varkappa)\right|^s = \left|{}^{1}D_q\mathsf{F}(\varkappa)\right|^2 = \left(1 + \varkappa + \varkappa^2 + q(\varkappa^2 + \varkappa - 2) + q^2(x-1)^2\right)^2,$$

and

$$\left|{}_{\sigma}D_q\mathsf{F}(\varkappa)\right|^s = \left|{}_{0}D_q\mathsf{F}(\varkappa)\right|^2 = [3]_q^2\varkappa^4.$$

These functions are convex on $[0,1]$. By utilizing Theorem 3.2, the left-hand side of inequality (3.12) is similar to (4.1).

On the other hand, by (3.12), we have

$$\begin{aligned} \left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s &= \left|{}^{1}D_q\mathsf{F}(0)\right|^2 = (1-q)^4, \\ \left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s &= \left|{}_{0}D_q\mathsf{F}(1)\right|^2 = [3]_q^2, \\ \left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s &= \left|{}^{1}D_q\mathsf{F}(1)\right|^2 = 9, \end{aligned}$$

and

$$\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s = \left|{}_{0}D_q\mathsf{F}(0)\right|^2 = 0.$$

Hence, the right-hand side of (3.12) is

$$\begin{aligned} &\frac{\rho-\sigma}{2}\left[\left(\int_0^{\frac{1}{2}}\left|q\zeta - \frac{2}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\left\{\left(\frac{\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s + (1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right.\right. \\ &\left.+\left(\frac{\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s + (1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right\} \\ &+\left(\int_{\frac{1}{2}}^{1}\left|q\zeta - \frac{1}{3}\right|^p d_q\zeta\right)^{\frac{1}{p}}\left\{\left(\frac{3\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s + (-1+2q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right. \\ &\left.\left.+\left(\frac{3\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s + (-1+2q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s}{4[2]_q}\right)^{\frac{1}{s}}\right\}\right] \end{aligned}$$

$$=\frac{1}{2}\left[\left(\frac{16+8q+17q^2+q^3}{72[2]_q[3]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{(1-q)^4+9(1+2q)}{4[2]_q}\right)^{\frac{1}{2}}+\left(\frac{[3]_q^2}{4[2]_q}\right)^{\frac{1}{2}}\right\}\right.$$
$$\left.+\left(\frac{4-28q+35q^2+31q^3}{72[2]_q[3]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{3(1-q)^4+9(-1+2q)}{4[2]_q}\right)^{\frac{1}{2}}+\left(\frac{3[3]_q^2}{4[2]_q}\right)^{\frac{1}{2}}\right\}\right].$$

By inequality (3.12), we have

$$\left|\frac{1}{2}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}\right]-0.625\right| \tag{4.3}$$
$$\leq\frac{1}{2}\left[\left(\frac{16+8q+17q^2+q^3}{72[2]_q[3]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{(1-q)^4+9(1+2q)}{4[2]_q}\right)^{\frac{1}{2}}+\left(\frac{[3]_q^2}{4[2]_q}\right)^{\frac{1}{2}}\right\}\right.$$
$$\left.+\left(\frac{4-28q+35q^2+31q^3}{72[2]_q[3]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{3(1-q)^4+9(-1+2q)}{4[2]_q}\right)^{\frac{1}{2}}+\left(\frac{3[3]_q^2}{4[2]_q}\right)^{\frac{1}{2}}\right\}\right].$$

Figure 4 demonstrates the validity of inequality (4.3).

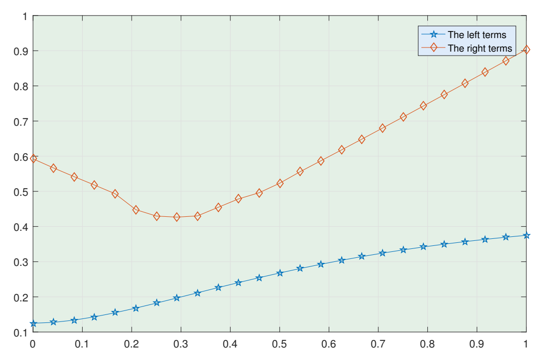

Figure 4. In Example 4.2, depending on $q \in [0,1]$, MATLAB has been used to compute and plot the graph of both sides of (4.3). Therefore, the validity of inequality (4.3) has been verified.

Example 4.3. Consider the function $\mathsf{F} : [0,1] \to \mathcal{R}$ defined as $\mathsf{F}(\varkappa) = \varkappa^3$ and $s = 2$. Then F is q-differentiable. Under these assumptions, we have

$$|{}^{\rho}D_q\mathsf{F}(\varkappa)|^s = \left|{}^{1}D_q\mathsf{F}(\varkappa)\right|^2 = \left(1+\varkappa+\varkappa^2+q(\varkappa^2+\varkappa-2)+q^2(x-1)^2\right)^2,$$

and

$$|{}_{\sigma}D_q\mathsf{F}(\varkappa)|^s = |{}_{0}D_q\mathsf{F}(\varkappa)|^2 = [3]_q^2\varkappa^4.$$

These functions are convex on $[0,1]$. By utilizing Theorem 3.3, the left-hand side of inequality (3.18) is similar to (4.1).

On the other hand, by (3.18), we have

$$|{}^{\rho}D_q\mathsf{F}(\sigma)|^s = \left|{}^{1}D_q\mathsf{F}(0)\right|^2 = (1-q)^4,$$
$$|{}_{\sigma}D_q\mathsf{F}(\rho)|^s = |{}_{0}D_q\mathsf{F}(1)|^2 = [3]_q^2,$$

$$\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s = \left|{}^{1}D_q\mathsf{F}(1)\right|^2 = 9,$$

and

$$\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s = \left|{}_{0}D_q\mathsf{F}(0)\right|^2 = 0.$$

Hence, the right-hand side of (3.18) is

$$\begin{aligned}
&\frac{\rho-\sigma}{2}\Bigg[(\mathcal{A}_5(q))^{1-\frac{1}{s}}\Big\{\left(\mathcal{A}_1(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_2(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}+\left(\mathcal{A}_1(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s\right.\\
&\left.+\mathcal{A}_2(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\Big\}+(\mathcal{A}_6(q))^{1-\frac{1}{s}}\Big\{\left(\mathcal{A}_3(q)\left|{}^{\rho}D_q\mathsf{F}(\sigma)\right|^s+\mathcal{A}_4(q)\left|{}^{\rho}D_q\mathsf{F}(\rho)\right|^s\right)^{\frac{1}{s}}\\
&+\left(\mathcal{A}_3(q)\left|{}_{\sigma}D_q\mathsf{F}(\rho)\right|^s+\mathcal{A}_4(q)\left|{}_{\sigma}D_q\mathsf{F}(\sigma)\right|^s\right)^{\frac{1}{s}}\Big\}\Bigg]\\
=&\frac{1}{2}\left[\left(\frac{4+q}{12[2]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{(4+q+q^2)(1-q)^4}{24[2]_q[3]_q}+\frac{9(4+9q+9q^2+2q^3)}{24[2]_q[3]_q}\right)^{\frac{1}{2}}\right.\right.\\
&\left.+\left(\frac{(4+q+q^2)[3]_q^2}{24[2]_q[3]_q}\right)^{\frac{1}{2}}\right\}\\
&\left.+(\mathcal{A}_6(q))^{\frac{1}{2}}\left\{\left(\mathcal{A}_3(q)(1-q)^4+\mathcal{A}_4(q)(9)\right)^{\frac{1}{2}}+\left(\mathcal{A}_3(q)[3]_q^2\right)^{\frac{1}{2}}\right\}\right].
\end{aligned}$$

By inequality (3.18), we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}\right]-0.625\right|\\
\leq&\frac{1}{2}\left[\left(\frac{4+q}{12[2]_q}\right)^{\frac{1}{2}}\left\{\left(\frac{(4+q+q^2)(1-q)^4}{24[2]_q[3]_q}+\frac{9(4+9q+9q^2+2q^3)}{24[2]_q[3]_q}\right)^{\frac{1}{2}}\right.\right.\\
&\left.+\left(\frac{(4+q+q^2)[3]_q^2}{24[2]_q[3]_q}\right)^{\frac{1}{2}}\right\}\\
&\left.+(\mathcal{A}_6(q))^{\frac{1}{2}}\left\{\left(\mathcal{A}_3(q)(1-q)^4+\mathcal{A}_4(q)(9)\right)^{\frac{1}{2}}+\left(\mathcal{A}_3(q)[3]_q^2\right)^{\frac{1}{2}}\right\}\right].
\end{aligned}\tag{4.4}$$

Figures 5-7 demonstrate the validity of inequality (4.4).

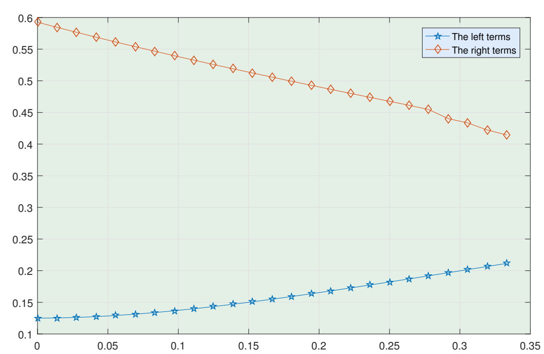

Figure 5. The graphs of left-hand side and right-hand side of the inequality (4.4) for $q \in \left(0, \frac{1}{3}\right]$

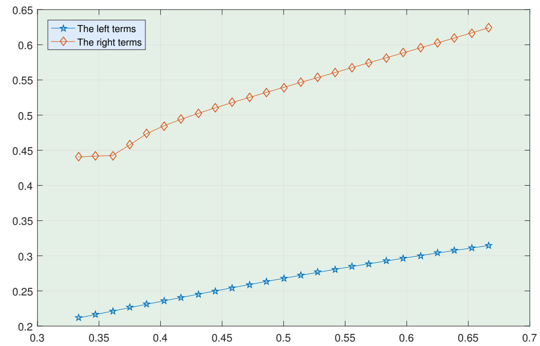

Figure 6. The graphs of left-hand side and right-hand side of the inequality (4.4) for $q \in \left(\frac{1}{3}, \frac{2}{3}\right]$

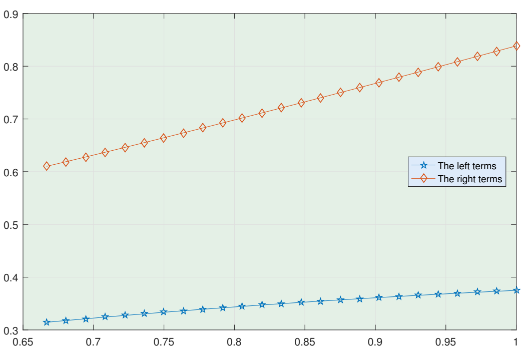

Figure 7. The graphs of left-hand side and right-hand side of the inequality (4.4) for $q \in \left(\frac{2}{3}, 1\right)$

5. Conclusion

In this study, we reported novel inequalities for determining error bounds in Milne's rule within classical and quantum calculus. We derived these inequalities by employing a quantum integral identity and leveraging convexity properties, which are crucial for open Newton-Cotes formulas. This research opens avenues for future exploration into similar inequalities for different integral operators. Future research endeavours may focus on expanding the scope of these inequalities and exploring their implications in diverse mathematical disciplines. Our approach is more straightforward and less conditional compared to existing methods.

References

[1] A. Abdeldaim and M. Yakout, *On some new integral inequalities of Gronwall-Bellman-Pachpatte type*, Applied Mathematics and Computation, 2011, 217(20), 7887–7899.

[2] M. A. Ali, H. Budak, F. Michal and K. Sundas, *A new version of q-Hermite-*

Hadamard's midpoint and trapezoid type inequalities for convex functions, Mathematica Slovaca, 2023, 73(2), 369–386.

[3] N. Alp, M. Z. Sarıkaya, M. Kunt and İ. İşcan, *q-Hermite-Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University-Science, 2018, 30(2), 193–203.

[4] M. A. Ali, H. Budak, K. Nanlaopon and Z. Abdullah, *Simpson's and Newton's inequalities for* (α, m)*-convex functions via quantum calculus*, Symmetry, 2021, 14(4), 736.

[5] M. A. Ali, M. Fečkan and A. Mateen, *Study of Quantum Ostrowski-Type Inequalities for Differentiable Convex Functions*, Ukrainian Mathematical Journal, 2023, 75(1), 5–28.

[6] S. Bermudo, P. Kórus and J. E. Nápoles Valdés, *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Mathematica Hungarica, 2020, 162, 364–374.

[7] S. I. Butt, H. Budak and K. Nonlaopon, *New quantum Mercer estimates of Simpson-Newton-like inequalities via convexity*, Symmetry, 2022, 14(9), 1935.

[8] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones (Antofagasta), 2021, 40(1), 199–215.

[9] H. Budak and A. A. Hyder, *Enhanced bounds for Riemann-Liouville fractional integrals: Novel variations of Milne inequalities*, AIMS Mathematics, 2023, 8(12), 30760–30776.

[10] H. Budak, P. Kösem and H. Kara, *On new Milne-type inequalities for fractional integrals*, Journal of Inequalities and Applications, 2023, 2023(1), 1–15.

[11] H. Budak, M. A. Ali and M. Tarhanaci, *Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions*, Journal of Optimization Theory and Applications, 2020, 186, 899–910.

[12] B. Celik, H. Budak and E. Set, *On generalized Milne type inequalities for new conformable fractional integrals*, Filomat, 2024, 38(5), 1807–1823.

[13] R. J. Dwilewicz, *A short history of Convexity*, Differential Geometry-Dynamical Systems, 2009.

[14] S. S. Dragomir and R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied Mathematics Letters, 1998, 11(5), 91–95.

[15] İ. Demir, *A new approach of Milne-type inequalities based on proportional Caputo-Hybrid operator: A new approach for Milne-type inequalities*, Journal of Advances in Applied and Computational Mathematics, 2023, 10, 102–119.

[16] H. D. Desta, H. Budak and H. Kara, *New perspectives on fractional Milne-type inequalities: Insights from twice-differentiable functions*, Universal Journal of Mathematics and Applications, 2024, 7(1), 30–37.

[17] A. El Farissi, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Ineq., 2010, 4(3), 365–369.

[18] T. Ernst, *A comprehensive treatment of q-calculus*, Springer Science & Business Media, 2012.

[19] J. W. Green, *Recent applications of convex functions*, The American Mathematical Monthly, 1954, 61(7P1), 449–454.

[20] H. Gauchman, *Integral inequalities in q-calculus*, Computers and Mathematics with Applications, 2004, 47(2-3), 281–300.

[21] C. Hermite, *Sur deux limites d'une intégrale définie*, Mathesis, 1883, 3(1), 1–82.

[22] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, Journal de mathématiques pures et appliquées, 1893, 9, 171–215.

[23] F. Hezenci, H. Budak, H. Kara and U. Baş, *Novel results of Milne-type inequalities involving tempered fractional integrals*, Boundary Value Problems, 2024, 2024(1), 12.

[24] W. Haider, H. Budak, A. Shehzadi, F. Hezenci and H. Chen, *Analysing Milne-type inequalities by using tempered fractional integrals*, Analysis and Mathematical Physics, 2024, 14(5), 101.

[25] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Applied Mathematics and Computation, 2004, 147(1), 137–146.

[26] M. A. Khan, M. Noor, E. R. Nwaeze and Y. M. Chu, *Quantum Hermite–Hadamard inequality by means of a Green function*, Advances in Differential Equations, 2020, 2020, 1–20.

[27] H. Kalsoom, M. A. Ali, M. Abbas, H. Budak and G. Murtaza, *Generalized quantum Montgomery identity and Ostrowski type inequalities for preinvex functions*, TWMS Journal Of Pure And Applied Mathematics, 2022, 13(1), 72–90.

[28] M. A. Noor, K. I. Noor and M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Applied Mathematics and Computation, 2015, 269, 242–251.

[29] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon and Z. Abdullah, *Simpson's and Newton's type inequalities for (α, m)-convex functions via quantum calculus*, Symmetry, 2022, 14(4), 736.

[30] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 2013, 57(9-10), 2403–2407.

[31] I. B. Sial, S. Mei, M. A. Ali and K. Nonlaopon, *On some generalized Simpson's and Newton's inequalities for (α, m)-convex functions in q-calculus*, Mathematics, 2021, 9(24), 3266.

[32] I. B. Sial, H. Budak and M. A. Ali, *Some Milne's rule type inequalities in quantum calculus*, Filomat, 2023, 37(27), 9119–9134.

[33] A. Shehzadi, H. Budak, W. Haider and H. Chen, *Milne-type Inequalities for co-ordinated Convex Functions*, Filomat, 2024, 38(23).

[34] J. Tariboon and S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Advances in Difference Equations, 2013, 1–19.

[35] M. J. Vivas-Cortez, M. A. Ali, S. Qaisar, I. B. Sial, S. Jansem and A. Mateen, *On some new Simpson's formula type inequalities for convex functions in post-quantum calculus*, Symmetry, 2021, 13(12), 2419.

[36] B. Y. Xi, J. Hua and F. Qi, *Hermite-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle*, Journal of Applied Analysis, 2014, 20(1), 29–39.