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Neimark-Sacker Bifurcation of a Semi-Discrete Lasota-Waźewska Model

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Abstract In this paper, we derive and analyze a semi-discrete Lasota-Waźewska model. First, the existence, uniqueness, and local dynamical properties of the positive fixed point are systematically investigated. Subsequently, we explore the existence of Neimark-Sacker bifurcation and the stability of the bifurcated invariant curve. Finally, numerical simulations are provided to illustrate the theoretical findings.

Keywords Semi-discrete Lasota-Waźewska model, Neimark-Sacker bifurcation, invariant curve, numerical simulation

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1. Introduction

In 1976, M. Waźewska-Czyźewska and A. Lasota [16] proposed the following delayed differential equation:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\mu N(t) + \rho e^{-\gamma N(t-h)}, \tag{1.1}$$

which is a reduced system to describe the number of red blood cells (RBCs) in an animal, where $N(t)$ represents the population of RBCs at time t, $\mu \in (0,1)$ denotes the mortality rate for RBCs, $\rho, \gamma \in (0, +\infty)$ are constants characterizing the ability to generate new RBCs per unit time, and the delay $h > 0$ denotes the time required for RBCs to generate new cells. And system (1.1) is usually called the Lasota-Waźewska system.

The Lasota-Waźewska system and its generalized systems have been extensively studied since they were proposed. Significant progress has been made on the existence and stabilities of the positive equilibrium, positive periodic solutions, positive almost periodic solutions and positive pseudo-almost periodic solutions and so on [3–6, 8, 9, 15, 17].

These models are generally categorized into two types: continuous models defined by differential equations, and discrete models derived through discretization of continuous systems. The discrete framework is often regarded as more suitable for modeling real-world phenomena. Consequently, discretized models have garnered considerable research interest, with several discrete Lasota-Waźewska systems being proposed and analyzed in recent studies [1, 2, 7, 14]. Chen [1] discussed positive periodic solutions for a discrete Lasota-Waźewska model with impulse. Chen [2] et

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al. studied dynamic behaviors for a delay Lasota-Ważewska model with feedback control on time scales, which unified the continuous and discrete models.

In 2002, Tamas and Gabor [12] introduced a semi-discretization method for studying delayed systems, which provided a simple yet effective approach to handling delayed terms. Subsequently, this method has been applied to propose and analyze several new semi-discretized models. For instance, in [10], the authors proposed a semi-discrete hematopoiesis model and analyzed its dynamical behaviors, including the stabilities of the fixed points and the Neimark-Sacker bifurcation. Some other studies on semi-discrete models can be found in [11, 13] and related references. Additionally, Yao and Li [18] demonstrated differences in bifurcation behavior induced by distinct discretization methods within a discrete predator-prey model, highlighting the significance of methodological comparisons in discrete dynamical systems.

Inspired by the aforementioned studies, we shall investigate a semi-discrete version of system (1.1) in this paper. Our main aim is to find some new phenomena in the system. The existence, uniqueness and local dynamical behaviors of the positive fixed point are discussed. We also show that the system will undergo Neimark-Sacker bifurcation by both theoretical analysis and numerical simulations, and the stability of the bifurcated invariant curve is presented by computing the first Lyapunov coefficient. Notably, while the positive fixed point lacks an explicit analytical solution, we successfully analyze its local characteristics through the implicit function theorem and auxiliary analytical techniques.

The remainder of this paper is organized as follows. In Section 2, we derive a semi-discrete model for system (1.1), which is subsequently transformed into a discrete planar system through appropriate coordinate transformations. Section 3 presents the dynamical analysis of the proposed system, including the existence and local stability of the positive fixed point, conditions for Neimark-Sacker bifurcation occurrence, and the stability of the bifurcated invariant curve. Numerical simulations validating our theoretical results are provided in Section 4. Finally, we conclude the paper with a brief discussion in Section 5.

2. Problem and assumptions

We shall establish the semi-discrete model for system (1.1) using the method in [12] in this section. First, introduce the transformations $s = \frac{t}{h}$ and $N(t) = N(sh) = \eta(s)$, then (1.1) is changed to

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta(s) + pe^{-\gamma\eta(s-1)}, \quad s \geq 0, \tag{2.1}$$

where $\delta = \mu h$, $p = \rho h$, and the delay h is turned into 1, which makes the problem simple. Assume $[s]$ is the integer that not bigger than s, and consider the semi-discrete model of (2.1):

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta([s]) + pe^{-\gamma\eta([s-1])}, \quad s \neq 0, 1, 2, \cdots. \tag{2.2}$$

Obviously, equation (2.2) has piecewise constant arguments. We can directly have the following conclusion.

Lemma 2.1. *The solution $\eta(s)$ of equation (2.2) satisfies*

(i) $\eta(s)$ *is continuous on* $[0, +\infty)$*;*

(ii) $\frac{\mathrm{d}\eta}{\mathrm{d}s}$ *exists on* $\bigcup\limits_{s=0}^{+\infty}(s, s+1)$*;*

(iii) *equation* (2.2) *is true on every interval* $[k, k+1)$ *for* $k = 0, 1, 2, \ldots$.

So, we can integrate equation (2.2) from n to s for any $s \in [n, n+1)$, $n \in \{0, 1, 2, 3, \ldots\}$, and the following difference equation can be obtained as:

$$\eta(s) - \eta(n) = \Big(-\delta\eta(n) + pe^{-\gamma\eta(n-1)}\Big)(s-n).$$

Let $s \to (n+1)^-$ in the above equation, and then we derive the discrete system for (1.1) as

$$\eta(n+1) = (1-\delta)\eta(n) + pe^{-\gamma\eta(n-1)}.$$

Introducing the normal transformation

$$\begin{cases} x_n = \eta(n-1), \\ y_n = \eta(n), \end{cases}$$

we may achieve the semi-discrete planar dynamical system of (1.1) as follows:

$$\begin{cases} x_{n+1} = y_n, \\ y_{n+1} = (1-\delta)y_n + pe^{-\gamma x_n}, \end{cases} \tag{2.3}$$

where $0 < \delta < 1$, $\gamma, p > 0$, $x_0, y_0 \in (0, +\infty)$.

Next, we shall discuss the dynamics on system (2.3) in detail.

3. Local dynamics for the fixed point

In this section, we shall deal with the existence and local dynamical properties of the fixed point for system (2.3). First, we introduce the following result about the existence and uniqueness of the fixed point.

Theorem 3.1. *There is a unique positive fixed point* $E_*(x_*, y_*)$ *for system* (2.3) *satisfying*

$$\begin{cases} x_* = y_*, \\ \delta y_* = pe^{-\gamma x_*}, \end{cases} \tag{3.1}$$

and $x_* = y_* \in (0, \frac{p}{\delta})$.

Proof. To obtain the fixed point of system (2.3), we need to solve the following equation

$$\begin{cases} x = y, \\ \delta y = pe^{-\gamma x}. \end{cases}$$

That is to say, we should discuss the root of $\delta x = pe^{-\gamma x}$. It is obvious that $x > 0$.

Taking $f(x) = \delta x - pe^{-\gamma x}$, for all $x \in (0, +\infty)$, we have:

$$f'(x) = \delta + p\gamma e^{-\gamma x} > 0.$$

Notice that $f(0+0) = -p < 0$, $f(\frac{p}{\delta}) = p - pe^{-\frac{\gamma p}{\delta}} > 0$. So, there is a unique root $x_* \in (0, \frac{p}{\delta})$ satisfying $f(x_*) = 0$, which means system (2.3) has a unique positive fixed point (x_*, y_*) fulfilling

$$\begin{cases} x_* = y_*, \\ \delta y_* = pe^{-\gamma x_*}, \end{cases}$$

and $x_* = y_* \in (0, \frac{p}{\delta})$. □

Next, we investigate the stability of the fixed point $E_*(x_*, y_*)$.

Before that we need to present an important result first, which will be used in the following studies.

Lemma 3.1. [10] *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

- (i) *If $F(1) > 0$, then*
  - (i.1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;
  - (i.2) $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $B \neq 2$;
  - (i.3) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$;
  - (i.4) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;
  - (i.5) λ_1 *and* λ_2 *are a pair of conjugate complex roots and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < B < 2$ *and* $C = 1$;
  - (i.6) $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $B = 2$.
- (ii) *If $F(1) = 0$, namely, 1 is one root of $F(\lambda) = 0$, then the other root λ satisfies $|\lambda| > 1 (resp. |\lambda| < 1)$ if and only if $|C| > 1 (resp. |C| < 1)$.*
- (iii) *If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in $(1, +\infty)$. Moreover,*
  - (iii.1) *the other root* λ *satisfies* $\lambda < (=) - 1$ *if and only if* $F(-1) < (=) 0$;
  - (iii.2) *the other root* λ *satisfies* $-1 < \lambda < 1$ *if and only if* $F(-1) > 0$.

Since the Jacobian matrix of system (2.3) at the fixed point E_* is as follows

$$J(E_*) = \begin{pmatrix} 0 & 1 \\ -\gamma pe^{-\gamma x_*} & 1-\delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\gamma\delta x_* & 1-\delta \end{pmatrix},$$

the characteristic polynomial of (2.3) at E_* is

$$F(\lambda) = \lambda^2 - B\lambda + C,$$

where $B = 1 - \delta$, $C = \gamma\delta x_*$. Obviously,

$$F(1) = \delta + \gamma\delta x_* > 0,$$

and

$$F(-1) = 2 - \delta + \gamma\delta x_* > 0.$$

Notice that for $\gamma = \gamma_0 = \frac{1}{p}e^{\frac{1}{\delta}}$, equation (3.1) turns into

$$\begin{cases} x_* = y_*, \\ \delta y_* = pe^{-\frac{x_*}{p}e^{\frac{1}{\delta}}}. \end{cases}$$

It is easy to see $x_* = y_* = \frac{p}{\delta}e^{-\frac{1}{\delta}}$ is exactly the unique solution at this case.

So, for $\gamma = \gamma_0 = \frac{1}{p}e^{\frac{1}{\delta}}$, $C = \gamma\delta x_* = \frac{\delta}{p}e^{\frac{1}{\delta}} \times \frac{p}{\delta}e^{-\frac{1}{\delta}} = 1$ and $B = 1 - \delta \in (-2, 2)$. By Lemma 3.1, we may know the eigenvalues $\lambda_{1,2}$ of system (2.3) at E_* are a pair of conjugate complex satisfying $|\lambda_1| = |\lambda_2| = 1$, which means the fixed point E_* is non-hyperbolic and the system may undergo Neimark-Sacker bifurcation.

While $\gamma(x_*)$ satisfies the equation $F(x_*, \gamma) = \delta x_* - pe^{-\gamma x_*} = 0$, and

$$\frac{\partial F}{\partial \gamma} = px_*e^{-\gamma x_*} > 0,$$
$$\frac{\partial F}{\partial x_*} = \delta + \gamma pe^{-\gamma x_*} > 0,$$

using the implicit function theorem we know there exists an implicit function $x_* = x_*(\gamma)$ satisfing $D_{\gamma x_*} = \{(\gamma, x_*) : x_* \in (0, +\infty), \gamma > 0\}$, and

$$x'_*(\gamma) = -\frac{F_\gamma}{F_{x_*}} = -\frac{x_*pe^{-\gamma x_*}}{\delta + \gamma pe^{-\gamma x_*}} < 0.$$

Noticing the expression of C, we have

$$C'(\gamma) = \delta x_*(\gamma) + \delta\gamma x'_*(\gamma) = \frac{\delta^2 x_*}{\delta + \gamma pe^{-\gamma x_*}} > 0,$$

which implies $C(\gamma)$ is strictly increasing with γ. Combining the fact $C(\gamma_0) = 1$, one obtains $C < 1$ for $\gamma < \gamma_0$, and the fixed point E_* is then a sink by Lemma 3.1; while for $\gamma > \gamma_0$, we have $C > 1$, which means the fixed point E_* is a source. Summing up the results obtained above, we achieve the following theorem.

Theorem 3.2. *For $\gamma_0 = \frac{1}{p}e^{\frac{1}{\delta}}$, the following conclusions are true for the fixed point E_* of system* (2.3)*.*

(i) *The fixed point E_* is non-hyperbolic for $\gamma = \gamma_0$, where system* (2.3) *may undergo Neimark-Sacker bifurcation.*

(ii) *The fixed point E_* is a sink as $\gamma < \gamma_0$ and it is a source as $\gamma > \gamma_0$.*

Remark 3.1. The theorem shows that the positive fixed point E_* is stable for $\gamma < \gamma_0$ and is unstable for $\gamma > \gamma_0$.

3.1. Neimark-Sacker bifurcation

We know from the above analysis that when $\gamma = \gamma_0 = \frac{1}{p}e^{\frac{1}{\delta}}$, $x_* = y_* = \frac{p}{\delta}e^{-\frac{1}{\delta}}$, system (2.3) has a unique fixed point $E_*(x_*, y_*)$. Set

$$S_E = \{(\gamma, p, \delta) : \ 0 < \delta < 1, p > \delta, \gamma = \gamma_0 = \frac{1}{p}e^{\frac{1}{\delta}}\}.$$

Theorem 3.2 tells us that system (2.3) may undergo a Neimark-Sacker bifurcation when $(\gamma, p, \delta) \in S_E$ as γ varies in the neighborhood of γ_0. Next we shall discuss the existence of Neimark-Sacker bifurcation and the stability of the bifurcated invariant curve.

Firstly, choose γ as the bifurcation parameter, and introduce a perturbation γ_* of the parameter γ_0. Then we obtain the following perturbed system

$$\begin{cases} x \longmapsto y, \\ y \longmapsto (1-\delta)y + pe^{-(\gamma_0+\gamma_*)x}, \end{cases} \tag{3.2}$$

where $|\gamma_*| \ll 1$.

Next, take $u = x - x_*, v = y - y_*$. Then the fixed point can be transformed to the origin $O(0,0)$, and system (3.2) becomes

$$\begin{cases} u \longmapsto v, \\ v \longmapsto (1-\delta)v + pe^{-(\gamma_0+\gamma_*)(u+x_*)} - \delta y_*. \end{cases} \tag{3.3}$$

The characteristic function of the linearization of system (3.3) at the origin O is as follows

$$F(\lambda) = \lambda^2 - a(\gamma_*)\lambda + b(\gamma_*),$$

where $a(\gamma_*) = 1-\delta, b(\gamma_*) = p(\gamma_0+\gamma_*)e^{-(\gamma_0+\gamma_*)x_*} = \delta x_*(\gamma_0+\gamma_*)$. With simple computation, we have

$$\lambda_{1,2}(\gamma_*) = \frac{1}{2}[a(\gamma_*) \pm \mathrm{i}\sqrt{4b(\gamma_*) - a^2(\gamma_*)}].$$

As we know, to guarantee the existence of Neimark-Sacker bifurcation, the following conditions must be fulfilled:

$$(\mathbf{C.1}) \quad \frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}\gamma_*}\Big|_{\gamma_*=0} \neq 0;$$

$$(\mathbf{C.2}) \quad \lambda_{1,2}^k \neq 1, k = 1,2,3,4.$$

Notice that $a(\gamma_*)\big|_{\gamma_*=0} = 1-\delta$ and $b(\gamma_*)\big|_{\gamma_*=0} = 1$. Therefore

$$\lambda_{1,2} = \frac{1}{2}[(1-\delta) \pm \mathrm{i}\sqrt{(3-\delta)(1+\delta)}],$$

which means the condition (**C.2**) is true. Furthermore,

$$\frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}\gamma_*}\Big|_{\gamma_*=0} = \frac{b'(\gamma_*)}{2\sqrt{b(\gamma_*)}}\Big|_{\gamma_*=0} = \frac{p\delta}{2(\delta+1)}e^{-\frac{1}{\delta}} < 0,$$

which guarantees the condition (**C.1**) hold.

Therefore, system (2.3) undergoes Neimark-Sacker Bifurcation as γ_* varies in the neighborhood of γ_0 for $0 < \delta < 1$. And according to $\frac{\mathrm{d}|\lambda_{1,2}|}{\mathrm{d}\gamma_*}\Big|_{\gamma_*=0} < 0$, we know the fixed point O is a sink for $\gamma_* < 0$, and O is a source as $\gamma_* > 0$. Hence, a invariant curve will occur when $\gamma_* > 0$.

In the following, we shall discuss the stability of the invariant curve by three steps.

Step I. Performing a Taylor series expansion of the right-hand side of system (3.3) at the origin $(0,0)$, we have

$$\begin{cases} u \longmapsto a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 \\ \qquad +a_{12}uv^2 + a_{21}u^2v + a_{03}v^3 + O(\rho^4), \\ v \longmapsto b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 \\ \qquad +b_{12}uv^2 + b_{21}u^2v + b_{03}v^3 + O(\rho^4), \end{cases} \tag{3.4}$$

where $\rho=\sqrt{\|u\|^2+\|v\|^2}$,

$$a_{ij}=\frac{1}{i!j!}\frac{\partial^{i+j}f(u,v)}{\partial^i u\partial^j v}\Big|_{(u,v)=(0,0)},\quad i,j=0,1,2,3,\cdots,$$
$$b_{ij}=\frac{1}{i!j!}\frac{\partial^{i+j}g(u,v)}{\partial^i u\partial^j v}\Big|_{(u,v)=(0,0)},\quad i,j=0,1,2,3,\cdots.$$

Since $f(u,v)=v$, $g(u,v)=(1-\delta)v+pe^{-\gamma_0(u+x_*)}-\delta y_*$ in system (3.3), then we have

$$a_{01}=1,\ a_{10}=a_{20}=a_{11}=a_{02}=a_{30}=a_{21}=a_{12}=a_{03}=0,$$
$$b_{10}=-1,\ b_{01}=1-\delta, b_{11}=b_{02}=b_{12}=b_{21}=b_{03}=0, b_{20}=\frac{\gamma_0}{2},\ b_{30}=-\frac{\gamma_0^2}{6}.$$

Step II. Taking the invertible matrix

$$T=\begin{pmatrix} 0 & 1\\ \frac{\sqrt{3+2\delta-\delta^2}}{2} & \frac{1-\delta}{2}\end{pmatrix},$$

it is not difficult to see the inverse matrix of T is

$$T^{-1}=\begin{pmatrix} \frac{\delta-1}{\sqrt{3+2\delta-\delta^2}} & \frac{2}{\sqrt{3+2\delta-\delta^2}}\\ 1 & 0\end{pmatrix}.$$

Introduce the invertible transformation $(u,v)^T=T(X,Y)^T$, and the normal form of system (3.4) is arrived:

$$\begin{cases} X\longmapsto \frac{1-\delta}{2}X-\frac{\sqrt{3+2\delta-\delta^2}}{2}Y+F(X,Y)+O(\rho^4),\\ Y\longmapsto \frac{\sqrt{3+2\delta-\delta^2}}{2}X+\frac{1-\delta}{2}Y+G(X,Y)+O(\rho^4),\end{cases}\tag{3.5}$$

where $F(X,Y)=\frac{\gamma_0}{\sqrt{3+2\delta-\delta^2}}Y^2-\frac{\gamma_0^2}{3\sqrt{3+2\delta-\delta^2}}Y^3$, $G(X,Y)=0$, $\rho=\sqrt{\|X^2\|+\|Y^2\|}$, and by computation we get

$$\begin{aligned} &F_{XXX}\big|_{(0,0)}=F_{XXY}\big|_{(0,0)}=F_{XYY}\big|_{(0,0)}=0,\\ &F_{YY}\big|_{(0,0)}=\frac{2\gamma_0}{\sqrt{3+2\delta-\delta^2}}, F_{YYY}\big|_{(0,0)}=-\frac{2\gamma_0^2}{\sqrt{3+2\delta-\delta^2}},\\ &F_{XX}\big|_{(0,0)}=F_{XY}\big|_{(0,0)}=G_{XX}\big|_{(0,0)}=G_{XY}\big|_{(0,0)}=G_{YY}\big|_{(0,0)}=0,\\ &G_{XXX}\big|_{(0,0)}=G_{XXY}\big|_{(0,0)}=G_{XYY}\big|_{(0,0)}=G_{YYY}\big|_{(0,0)}=0.\end{aligned}\tag{3.6}$$

Step III. The first Lyapunov coefficient is

$$a^*=-\operatorname{Re}\left[\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}L_{11}L_{20}\right]-\frac{1}{2}\left|L_{11}\right|^2-\left|L_{02}\right|^2+\operatorname{Re}(\overline{\lambda}L_{21}),\tag{3.7}$$

where

$$L_{20}=\frac{1}{8}\left[(F_{XX}-F_{YY}+2G_{XY})+\mathrm{i}(G_{XX}-G_{YY}-2F_{XY})\right],$$

$$L_{11} = \frac{1}{4}\big[(F_{XX} + F_{YY}) + \mathrm{i}(G_{XX} + G_{XY})\big],$$
$$L_{02} = \frac{1}{8}\big[(F_{XX} - F_{YY} - 2G_{XY}) + \mathrm{i}(G_{XX} - G_{YY} + 2F_{XY})\big],$$
$$L_{21} = \frac{1}{16}\big[(F_{XXX}+F_{XYY}+G_{XXY}+G_{YYY})+\mathrm{i}(G_{XXX}+G_{XYY}-F_{XXY}-F_{YYY})\big].$$

Put the result obtained in (3.6) into the above equations and then we obtain

$$L_{20} = -\frac{\gamma_0}{4\sqrt{3+2\delta-\delta^2}}, \quad L_{11} = \frac{\gamma_0}{2\sqrt{3+2\delta-\delta^2}},$$
$$L_{02} = -\frac{\gamma_0}{4\sqrt{3+2\delta-\delta^2}}, \quad L_{21} = \frac{\gamma_0^2}{8\sqrt{3+2\delta-\delta^2}}\mathrm{i}.$$

Therefore by computation and simplification, one may achieve

$$a^* = \frac{\gamma_0^2(\delta-3)}{16(3+2\delta-\delta^2)} < 0 \quad \text{for } (\gamma, p, \delta) \in S_E,$$

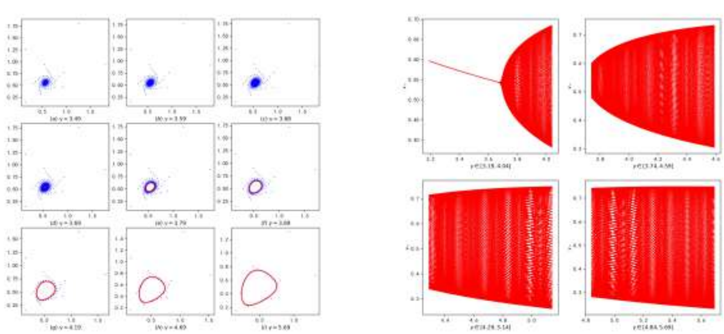
which shows that the bifurcated invariant curve is stable.

Consequently, we may achieve the following theorem by summing up the above results.

Theorem 3.3. *System* (2.3) *undergoes a supercritical Neimark-Sacker bifurcation at the fixed point E_* with γ varying in the neighborhood of γ_0, namely, an invariant curve will occour as $\gamma > \gamma_0$. Furthermore, the bifurcated invariant curve is stable.*

4. Numerical simulations

Select $\delta = 0.5$, $p = 2$, and the initial point $E_0 = (0.15, 0.15)$. By the theoretical analysis in Section 3.1, one has $\gamma_0 = 3.6945$, $a^* = -0.5687$. The according numerical simulations are shown in Figure 1.



(a) local (b) global

Figure 1. The Neimark-Sacker bifurcation as γ is varying when $p = 2$, $\delta = 0.5$, $E_0 = (0.15, 0.15)$

In Figure 1(a), figures (a)-(c) show that the fixed point E_* is a sink when $\gamma \leq 3.68$; while for $\gamma = 3.69$, a stable invariant curve occurs, as shown in figures (d)-

(f) in Figure 1(a), which can be also shown in Figure 1(b). Figure 1 demonstrates that the numerical simulation results validate Theorem 3.3.

Figure 2 shows the maximum Lyapunov exponent (MLE) is varying with the parameter γ, which displays the MLE is always less than zero when $\gamma < 3.6945$, while for $\gamma > 3.6945$, the MLE will fluctuate below 0, which implies the chaotic phenomenon does not occur in system (2.3) in this case.

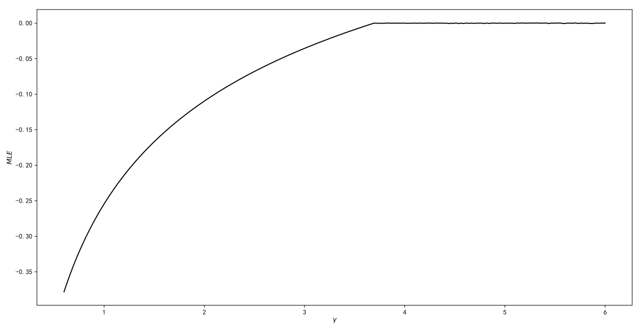


Figure 2. The Maximum Lyapunov Exponent as γ varying for $p = 2$, $\delta = 0.5$

The Neimark-Sacker bifurcation phenomenon describes the changes in hematopoietic behavior caused by the promotion of RBCs, which is induced by the changes of the constant γ denoting the abilities of generating new RBCs per unit time. The dynamical behaviors, including a stable invariant curve induced by the bifurcation and the absence of chaos, mean that metabolism is still normal.

In addition, compared to the continuous case, the semi-discrete model reduces the difficulties caused by the delay term without changing the existence of the fixed point in the system. At the same time, the discrete system exhibits more complex dynamic behaviors compared to one-dimensional delay differential equation (1.1), such as the Neimark-Sacker bifurcation and stable invariant curve.

5. Conclusions

The semi-discretization method is a simple but efficient method that is based on the discretization with respect to the delayed term, as demonstrated in this study. We propose and analyze a semi-discrete Lasota-Ważewska model in this paper. First, we rigorously establish the existence, uniqueness, and local stability of the fixed point E_*. Second, we prove that the system undergoes a Neimark-Sacker bifurcation, where an invariant curve emerges as parameters vary. Notably, the bifurcated invariant curve is shown to be stable. Finally, numerical simulations are presented to illustrate our theoretical findings, while computation of the maximal Lyapunov exponent confirms that there is no chaotic behavior in the proposed system under the specified parameter conditions.

While verifying the theoretical analysis results, the numerical simulation results also reflect the more complex and exciting dynamical behaviors of the semi-discrete model. The analysis indicates that when the constant γ which denotes the ability of generating new RBCs per unit time undergoes minor changes, the biological balance of the hematopoietic system is not easily disrupted.

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