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Double-Phase Elliptic Equations with Nonlinear Sources Existence and Uniqueness of the Weak Solution

El Mehdi Hassoune[1,†], Ahmed Jamea[2], Mohamed Laghdir[1] and Adnane Kaddiri[1]

Abstract In our work, our objective is to prove the existence and uniqueness of a weak solution to a class of nonlinear degenerate elliptic (p,q)-Laplacian problem with Dirichlet-type boundary condition by giving L^∞ data. The principal technique utilized here is the variational method alongside the theory of Orlicz spaces and Minty-Browder theorem.

Keywords Weak solution, uniqueness solutions, double phase operator, nonlinear elliptic equations

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1. Introduction

The study of unbalanced integral functionals and double phase problems experienced a revolution for the Italian school under the hands of Marcellini, Mingione, Colombo, Baroni, et al, their study and remark is based entirely on the work of Zikhov in order to describe the behavior of phenomena arising in nonlinear elasticity. In reality, Zhikov aimed to offer prototypes for highly anisotropic materials within the framework of homogenization. Specifically, Zhikov explored three distinct functional models in connection with the Lavrentiev phenomenon. These models are:

$$\begin{aligned} \mathcal{M}(u) &:= \int_\Omega c(x)|\nabla u|^2 dx, \\ \mathcal{N}(u) &:= \int_\Omega |\nabla u|^{p(x)} dx, \\ \mathcal{T}_{p,q}(u) &:= \int_\Omega c(x)|u|^p dx + a(x)|\nabla u|^q dx, \end{aligned} \tag{1.1}$$

where $0 < 1/c(.) \in L^1(\Omega), 1 < p(.) < \infty, 1 < p < q, 0 \le a(.) \le L$. The functional $\mathcal{M}$ has been extensively investigated in the context of equations that incorporate

[†]the corresponding author.
Email address: mahdihassoune@gmail.com(E. Hassoune), a.jamea77@gmail.com(A. Jamea), laghdirm@gmail.com(M. Laghdir), adnanekaddiri@gmail.com(A. Kaddiri)
[1]Laboratoire LITE, Faculté des Sciences, Université Chouaib Doukkali El Jadida, Morocco
[2]CRMEF Casablanca-Settat, S.P. El Jadida,El Jadida, Morocco

Muckenhoupt weights. The functional $\mathcal{N}$ has recently garnered significant attention, leading to a substantial body of literature devoted to its study. The functional $\mathcal{T}_{p,q}$ in (1.1) is presented as an enhanced form of $\mathcal{N}$. Here as well, the coefficient $a(.)$ influences the geometry of the composite consisting of two differential materials with hardening exponents p and q, respectively. The functionals depicted in (1.1) belong to the category of functionals with nonstandard growth conditions of type (p, q), as classified by Marcellini.

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be an open bounded domain and let $p, q \in (1, \infty)$. Our aim in this work is to study the existence and uniqueness of the weak solution to the following nonlinear degenerate elliptic problem

$$\begin{cases} -div\left(\mathcal{A}_{p,q}(x, \nabla u, \theta(u))\right) + g(u) = f \ \ in \ \ \Omega \\ u = 0 \ \ on \ \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{A}_{p,q}(x, \nabla u, \theta(u)) = |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u)) + a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))$ and $g(u) = |u|^{p-2}u$ and θ are continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in L^∞ and $a : \Omega \to \mathbb{R}$ is an Lipschitz continuous map, $a(x) \geq 0$ for all $x \in \overline{\Omega}$. These types of PDEs model several physical phenomena, including elastic mechanics, electrorheological fluid dynamics, and image processing, among others, for example.

Reaction-Diffusion Systems are mathematical models used to describe the spatial and temporal evolution of multiple substances or populations under the influence of two key processes: reaction and diffusion. These systems have widespread applications across various fields such as chemistry, biology, ecology, and physics, due to their ability to model phenomena involving pattern formation, wave propagation, and concentration dynamics.

Reaction refers to the interactions between substances (e.g., chemical reactions or biological processes like predator-prey dynamics), where the concentration of one or more substances changes over time due to chemical or biological reactions.

Diffusion describes the spreading or movement of substances from regions of high concentration to low concentration, driven by concentration gradients.

A general mathematical model for a Reaction-Diffusion System is described by a system of partial differential equations (PDEs).

$$-div(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) + R(u) = 0$$

where a(x) is the modulating coefficient that switches between p and q-phase behaviors, and R is the reaction term.

This paper primarily focuses on the study of unbalanced double-phase problems. We recall some basic properties of the Musielak-Orlicz and Sobolev spaces and introduce a new non-homogeneous differential operator, which will be utilized in this work. Using the Minty-Browder theorem and suitable assumptions, we prove the existence of a weak solution to our problem. Additionally, we employ fundamental lemmas to establish the uniqueness of the solution.

Recently, when $a(x) \equiv 0$, the existence and uniqueness of entropy solutions for the p-Laplace equation were proved by A. Sabri and A. Jamea. When $\theta \equiv 0$, the study of existence and uniqueness of entropy solutions for the problem has been further investigated. Moreover, R. Arora and S. Shmarev proved the existence and uniqueness of strong solutions when $\theta \equiv 0$, and for p non constant and A. Sabri,

A. Jamea, H. T. Alaoui proved the existence of entropy solutions to nonlinear degenerate parabolic problems in [16].

This paper is organized as follows: In Section 2, we introduce the basic assumptions and recall some definitions and properties of generalized Sobolev spaces. Section 3 is dedicated to proving the existence and uniqueness of the weak solution to the problem by verifying the conditions of the Minty-Browder theorem (bounded, hemi-continuous, coercive, and monotone).

2. Preliminaries and notations

In the present section we give some definitions, notations and results which will be used in this work.

Let φ function from $\Omega \times \mathbb{R}^+$ to $\mathbb{R}^+$ be defined by

$$\varphi(x,y) = y^p + a(x)\, y^q,$$

where a and p, q verify condition (H_1).

(H_1) $a : \Omega \to \mathbb{R}$ is a Lipschitz continuous and $p > \frac{Nq}{N+q-1}$ i.e ($\frac{q}{p} < 1 + \frac{q-1}{N}$).

The function φ is a generalized N-function and

$$\varphi(x, 2y) = 2^p \varphi(x,y).$$

Now we define the Musielak-Orlicz space $L^\varphi(\Omega)$ by

$$L^\varphi(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{g is measurable and } \int_\Omega \varphi(x, |v|)dx < \infty \right\}$$

endowed with the Luxemburg norm

$$\|v\|_\varphi = \inf \left\{ \lambda > 0, \int_\Omega \varphi\left(x, \frac{|v|}{\lambda}\right) dx \le 1 \right\}.$$

The Sobolev space corresponding to the L^φ space is

$$W^{1,\varphi}(\Omega) = \{ v \in L^\varphi(\Omega) \text{ such that } \nabla v \in L^\varphi(\Omega) \}$$

with the norm

$$\|v\|_{1,\varphi} = \|v\|_\varphi + \|\nabla v\|_\varphi.$$

Theorem 2.1. *i) If $q \ne N$. For all $r \in [1, q*]$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$ with*

$$q* = \begin{cases} \frac{Nq}{N-q} \; if\, q \text{ ¡ } N; \\ +\infty \;\; if\, q \ge N. \end{cases}$$

ii) If $q = N$. For all $r \in [1, +\infty[$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$.

iii) If $q \le N$. For all $r \in [1, q*]$ we have $W^{1,\varphi}(\Omega) \hookrightarrow L^r(\Omega)$ compactly.

iv) If $q > N$. $W^{1,\varphi}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.

v) $W^{1,\varphi}(\Omega) \hookrightarrow L^q(\Omega)$.

We define now the weighted Lebesgue space $L^q_a(\Omega)$ by

$$L^q_a(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \text{ is mesurable and } \|v\|_{q,a} = \int_\Omega a(x)|v|^q dx < \infty \right\}.$$

On the space $L^\varphi(\Omega)$ we consider the function $\varrho_\varphi : L^\varphi(\Omega) \to \mathbb{R}^+$ defined by

$$\varrho_\varphi(v) = \int_\Omega \varphi\left(x, \frac{|v|}{\lambda}\right) dx = \int_\Omega [|v|^p + a(x)|v|^q] dx.$$

The connection between ϱ_φ and $\|.\|_\varphi$ is established by the next result.

Proposition 2.1 ([4]). *Let u be an element of $L^\varphi(\Omega)$. The following assertions hold:*

i) $\|u\|_\varphi < 1(\textit{respectively} >, = 1) \Leftrightarrow \varrho_\varphi(u) < 1(\textit{respectively} >, = 1)$,

ii) If $\|u\|_\varphi < 1$ *then* $\|u\|^p_\varphi \leq \varrho_\varphi(u) \leq \|u\|^q_\varphi$,

iii) If $\|u\|_\varphi > 1$ *then* $\|u\|^q_\varphi \leq \varrho_\varphi(u) \leq \|u\|^p_\varphi$,

iv) $\|u\|_\varphi \to 0 \Leftrightarrow \varrho_\varphi(u) \to 0$ *and* $\|u\|_\varphi \to \infty \Leftrightarrow \varrho_\varphi(u) \to \infty$.

Definition 2.1. Let $\varphi : [0, \infty) \to [0, \infty]$. We denote by $\varphi^*(u)$ the conjugate function of φ which is defined, for $u \geq 0$, by

$$\varphi^*(u) := \sup_{t \geq 0}(tu - \varphi(t)).$$

Proposition 2.2 ([4]). *For any functions $u \in L^\varphi(\Omega)$, $v \in L^{\varphi^*}(\Omega)$, and under the assumption that hypothesis (H_1) be satisfied, we have:*

$$\int_\Omega |uv| dx \leq 2\|u\|_\varphi \|v\|_{\varphi^*}.$$

In the following of this paper, the space $W^{1,\varphi}_0(\Omega)$ denotes the closure of C^∞_0 in $W^{1,\varphi}(\Omega)$ with respect the norm $\|.\|_{1,\varphi}$ (see [10]).

Proposition 2.3 ([4]). *The spaces $(L^\varphi(\Omega), \|.\|_\varphi)$ and $(W^{1,\varphi}(\Omega), \|.\|_{1,\varphi})$ are separable and uniformly convex (hence reflexive) Banach space.*

We have

$$L^p(\Omega) \hookrightarrow L^\varphi(\Omega) \hookrightarrow L^p(\Omega) \bigcap L^q_a(\Omega).$$

Proposition 2.4 ([3]). *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded set, and let $u \in W^{1,\varphi}(\Omega)$*

$$\|u\|_\varphi \leq C_0 \|\nabla v\|_\varphi.$$

C_0 is a strictly positive constant depending on the exponent diam(Ω) and the dimension N.

Proposition 2.5 ([7]). *Let $1 < p < +\infty$, there exist two positive constants μ_p and ρ_p such that for every $x, y \in \mathbb{R}^N$, it holds that*

$$\mu_p(|x| + |y|)^{p-2}|x - y|^2 \leq \langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \leq \rho_p(|x| + |y|)^{p-2}|x - y|^2.$$

Definition 2.2. Given a constant $k > 0$. we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| < k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

Lemma 2.1 ([2]). *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have:*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\eta - \xi),$$

where a dot denotes the Euclidean scalar product in $\mathbb{R}^N$.

Lemma 2.2 ([2]). *For $a > 0, b > 0$ and $1 \leq p < \infty$ we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

Lemma 2.3 ([6]). *Let p and p' be two real numbers such that $p > 1$, and $\frac{1}{p} + \frac{1}{p'} = 1$. There exists a positive constant m such that*

$$|(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)|^{p'} \leq m\{(\xi - \eta)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\}^{\frac{\beta}{2}}\{\xi^p + \eta^{p'}\}^{1-\frac{\beta}{2}}.$$

For all $\xi, \eta \in \mathbb{R}^N$, $\beta = 2$ if $1 < p \leq 2$, and $\beta = p'$ if $p > 2$.

Definition 2.3 ([9]). Let Y be a reflexive Banach space and let A be an operator from Y to its dual Y'. We say that A is monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \ \ \forall u, v \in Y.$$

Theorem 2.2 ([9]). *Let Y be a reflexive real Banach space and $A : Y \to Y'$ be a bounded, hemi-continuous, coercive and monotone operator on space Y . Then the equation $Au = v$ has at least one solution $u \in Y$ for each $v \in Y'$.*

3. Assumptions and main result

In this section, we introduce the concept of weak solution to problem (1.2) and we state the existence results for this type of solutions. Firstly and in addition to hypotheses (H_1) listed earlier, we suppose the following assumptions:

(H_2) θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0) = 0$ and for all real numbers x, y we have $|\theta(x) - \theta(y)| < \lambda_0|x - y|$ where λ_0 is a real constant such that $0 < \lambda_0 < \frac{1}{2C_0}$.

(H_3) $f \in L^\infty(\Omega)$.

Definition 3.1. A function $u \in W_0^{1,\varphi}(\Omega)$ is a weak solution of degenerate elliptic problem (1.2) if and only if

$$\int_\Omega \Phi(\nabla u - \theta(u))\nabla\psi dx + \int_\Omega F(\nabla u - \theta(u))\nabla\psi dx + \int_\Omega g(u)\psi dx = \int_\Omega f\psi dx \quad (3.1)$$

for all $\psi \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$, where

$$\Phi(\xi) = |\xi|^{p-2}\xi,$$
$$F(\xi) = a(x)|\xi|^{q-2}\xi.$$

$\forall (x, \xi) \in \Omega \times \mathbb{R}^N$.

Our main result of this work is the following Theorem.

Theorem 3.1. *Let hypotheses* (H_1), (H_2) *and* (H_3) *be satisfied. Then the problem* (1.2) *has a unique weak solution.*

4. Proof of theorem

Let the operator $T : W_0^{1,\varphi}(\Omega) \to (W_0^{1,\varphi}(\Omega))'$, where $(W_0^{1,\varphi}(\Omega))'$ is the dual space of $W_0^{1,\varphi}(\Omega)$.

$$T(u) = A(u) + A^3(u) - L, \ \textit{and}\ A(u) = A^1(u) + A^2(u)$$

where for $u, v \in W_0^{1,\varphi}(\Omega,\omega)$

$$\begin{aligned}
\langle A^1 u, v\rangle &= \int_\Omega \omega\Phi(\nabla u - \theta(u))\nabla v dx,\\
\langle A^2 u, v\rangle &= \int_\Omega F(\nabla u - \theta(u))\nabla v dx,\\
\langle A^3 u, v\rangle &= \int_\Omega g(u) v dx,\\
\langle L, v\rangle &= \int_\Omega f v dx.
\end{aligned}$$

We must use Theorem 2.2 to prove the existence of the weak solution. For that, it is necessary to show that the operator T is bounded, monotone coercive and hemi continuous.

Step 1. The operator T is bounded.

We use Hölder's inequality, Lemma 2.2 and hypothesis (H_3). For any $u, v \in W_0^{1,\varphi}(\Omega)$ we have

$$\begin{aligned}
|\langle Au, v\rangle| \le& \int_\Omega |\nabla u - \theta(u)|^{p-1}|\nabla v| dx + \int_\Omega a(x)|\nabla u - \theta(u)|^{q-1}|\nabla v| dx\\
\le& \int_\Omega 2^{p-2}(|\nabla u|^{p-1} + |\theta(u)|^{p-1})|\nabla v| dx\\
&+ \int_\Omega a(x) 2^{q-2}(|\nabla u|^{q-1} + |\theta(u)|^{q-1})|\nabla v| dx\\
\le& 2^{p-2}\int_\Omega (|\nabla u|^{p-1} + |\theta(u)|^{p-1})|\nabla v| dx\\
&+ 2^{q-2}\int_\Omega a(x)(|\nabla u|^{q-1} + |\theta(u)|^{q-1})|\nabla v| dx\\
\le& 2^{p-2}\left(\int_\Omega (|\nabla u|^{p-1}|\nabla v| dx + \int_\Omega \lambda_0^{p-1}|u|^{p-1}|\nabla v| dx\right)\\
&+ 2^{q-2}\left(\int_\Omega a(x)(|\nabla u|^{q-1}|\nabla v| dx + \int_\Omega a(x)\lambda_0^{q-1}|u|^{q-1}|\nabla v| dx\right)\\
\le& 2^{p-2}\left(2\|\nabla u\|_p^{p-1}\|\nabla v\|_{p'} + 2\lambda_0^{p-1}\|u\|_p^{p-1}\|\nabla v\|_{p'}\right)\\
&+ 2^{q-2}\left(2\|\nabla u\|_{q,a}^{q-1}\|\nabla v\|_{q',a} + 2\lambda_0^{q-1}\|u\|_{q,a}^{q-1}\|\nabla v\|_{q',a}\right)\\
\le& 2^{p-1}(\|\nabla u\|_p^{p-1}\|\nabla v\|_{p'} + \lambda_0^{p-1}{C_0}^p\|\nabla u\|_p^{p-1}\|\nabla v\|_{p'})
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{q-1}(\|\nabla u\|_{q,a}^{q-1}\|\nabla v\|_{q,a} + \lambda_0^{p-1}{C_0}^q\|\nabla u\|_{q,a}^{q-1}\|\nabla v\|_{q',a}) \\
\leq& 2^{p-1}(1+{C_0}^p)\|\nabla u\|_p^{p-1}\|\nabla v\|_{p'} + 2^{q-1}(1+{C_0}^q)\|\nabla u\|_{q,a}^{q-1}\|\nabla v\|_{q',a} \\
\leq& C\|\nabla u\|_{\varphi}^{p-1}\|\nabla v\|_{\varphi^*} \\
\leq& C\|u\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi^*},
\end{aligned}
$$

where

$$C = 2^p(1+{C_0}^p),$$

and we have

$$
\begin{aligned}
|\langle A^3u, v\rangle| &\leq \int_\Omega |u|^{p-1}|v|dx \\
&\leq \|u\|_{\varphi}^{p-1}\|v\|_{\varphi^*} \\
&\leq \|u\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi^*}.
\end{aligned}
$$

We get immediately the boundedness of L and A^3. Hence, T is bounded.

Step 2. The operator T is coercive.

For any $u \in W_0^{1,\varphi}(\Omega)$. remark that by application hypothesis (H_3), there exists a positive constant C_3 such that

$$\int_\Omega f u dx \leq \|f\|_\infty \|u\|_{W_0^{1,\varphi}}.$$

On the other hand for u large enough and by Lemma 2.2, we get

$$
\begin{aligned}
\langle Au, u\rangle =& \int_\Omega |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u))\nabla u dx \\
&+ \int_\Omega a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))\nabla u dx \\
\geq& \int_\Omega \frac{1}{p}|\nabla u - \theta(u)|^p dx - \int_\Omega \frac{1}{p}|\theta(u)|^p dx \\
&+ \int_\Omega a(x)\frac{1}{q}|\nabla u - \theta(u)|^q dx - \int_\Omega a(x)\frac{1}{q}|\theta(u)|^q dx
\end{aligned}
$$

By Lemma 2.3 we find

$$\tfrac{1}{2^{p-1}}|\nabla u|^p - |\theta(u)|^p \leq |\nabla u - \theta(u)|^p,$$

$$
\begin{aligned}
\langle Au, u\rangle \geq& \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}\int_\Omega |\theta(u)|^p dx \\
&+ \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx - \frac{2}{q}\int_\Omega a(x)|\theta(u)|^q dx.
\end{aligned}
$$

By (H_3) we have $|\theta(u)| \leq \lambda_0|u|$

$$
\begin{aligned}
\langle Au, u\rangle \geq& \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}\int_\Omega \lambda_0^p|u|^p dx + \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx \\
&- \frac{2}{q}\int_\Omega a(x)\lambda_0^q|u|^q dx.
\end{aligned}
$$

Then by Proposition 2.4 we have

$$\begin{aligned}\langle Au,u\rangle \geq &\frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega |\nabla u|^p dx - \frac{2}{p}{C_0}^p\lambda_0^p\int_\Omega |\nabla u|^p dx + \frac{1}{q}\frac{1}{2^{q-1}}\int_\Omega a(x)|\nabla u|^q dx\\ &- \frac{2}{q}{C_0}^q\lambda_0^q\int_\Omega a(x)|\nabla u|^q dx\\ \geq &\left(\frac{1}{p}\frac{1}{2^{p-1}} - \frac{2}{p}{C_0}^p\lambda_0^p\right)\int_\Omega |\nabla u|^p dx + \left(\frac{1}{q}\frac{1}{2^{q-1}} - \frac{2}{q}{C_0}^q\lambda_0^q\right)\int_\Omega a(x)|\nabla u|^q dx\\ \geq &M\int_\Omega |\nabla u|^p + a(x)|\nabla u|^q dx\\ \geq &M\|u\|_{1,\varphi}^p,\end{aligned}$$

where

$$M = \sup\left(\frac{1}{p}\frac{1}{2^{p-1}} - \frac{2}{p}{C_0}^p\lambda_0^p, \frac{1}{q}\frac{1}{2^{q-1}} - \frac{2}{q}{C_0}^q\lambda_0^q\right).$$

Then

$$\frac{\langle Au,u\rangle}{\|u\|_{1,\varphi}} \longrightarrow +\infty \text{ as } \|u\|_{1,\varphi} \longrightarrow +\infty,$$

$$\begin{aligned}|\langle A^3u,u\rangle| &\leq \textstyle\int_\Omega |u|^p dx\\ &\leq \|u\|_{1,\varphi}^p.\end{aligned}$$

Then A and A^3 is coercive.

Finally the operator T is coercive.

Step.3 The operator T is monotone.

Firstly, we have that T is bounded and coercive, so there exist $M_1 > 0$ and C' such that

$$\langle Tu,u\rangle \geq M_1\|u\|_{1,\varphi}^p$$

and

$$\langle Tu,v\rangle \leq C'\|u\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi}.$$

So

$$\begin{aligned}\langle Tu-Tv,u-v\rangle =&\langle Tu,u\rangle + \langle Tv,v\rangle - \langle Tu,v\rangle - \langle Tv,u\rangle\\ \geq &M_1(\|u\|_{1,\varphi}^p + \|v\|_{1,\varphi}^p)\\ &- C'(\|u\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi} + \|v\|_{1,\varphi}^{p-1}\|u\|_{1,\varphi})\\ \geq &M_2[\|u\|_{1,\varphi}^p + \|v\|_{1,\varphi}^p\\ &- \|u\|_{1,\varphi}^{p-1}\|v\|_{1,\varphi} - \|v\|_{1,\varphi}^{p-1}\|u\|_{1,\varphi}],\end{aligned}$$

$$\langle Tu-Tv,u-v\rangle \geq M_2(\|u\|_{1,\varphi}^{p-1} - \|v\|_{1,\varphi}^{p-1})(\|u\|_{1,\varphi} - \|v\|_{1,\varphi}) \geq 0$$

with

$$M_2 = \min(M_1, C').$$

Finally, the operator is monotone.

Step 4. The operator T is hemi continues.

Let $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,\varphi}(\Omega)$ and $u \in W_0^{1,\varphi}(\Omega)$ such that $u_n \to u$ strongly in $W_0^{1,\varphi}(\Omega)$.

Firstly, we will prove that A_1 is continuous on $W_0^{1,\varphi}(\Omega)$. Indeed, we have for $\psi \in W_0^{1,\varphi^*}(\Omega)$

$$|\langle A_1 u_n - A_1 u, \psi\rangle| = \Big| \int_\Omega |\nabla u_n - \theta(u_n)|^{p-2}(\nabla u_n - \theta(u_n))\nabla\psi dx - \int_\Omega |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u))\nabla\psi dx\Big|,$$

$$|\langle A_2 u_n - A_2 u, \psi\rangle| = \Big| \int_\Omega a(x)|\nabla u_n - \theta(u_n)|^{q-2}(\nabla u_n - \theta(u_n))\nabla\psi dx - \int_\Omega a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))\nabla\psi dx\Big|.$$

We denote that

$$\begin{aligned} F_n &= |\nabla u_n - \theta(u_n)|^{p-2}(\nabla u_n - \theta(u_n)). \\ F &= |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u)). \\ G_n &= a(x)|\nabla u_n - \theta(u_n)|^{q-2}(\nabla u_n - \theta(u_n)). \\ G &= a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u)). \end{aligned}$$

This implies that

$$\langle Au_n - Au, \psi\rangle \le \int_\Omega |F_n - F||\nabla\psi| dx + \int_\Omega |G_n - G||\nabla\psi| dx.$$

We have $u_n \to u$ strongly in $W_0^{1,\varphi}(\Omega)$, so $\theta(u_n)$ and ∇u_n are bounded. Finally we have F_n and G_n bounded.

$$\langle Au_n - Au, \psi\rangle \le \|F_n - F\|_{W^{1,\varphi}}\|\psi\|_{W^{1,\varphi^*}} + \|G_n - G\|_{W^{1,\varphi}}\|\psi\|_{W^{1,\varphi^*}}.$$

Since $u_n \to u$ strongly in $W_0^{1,\varphi}(\Omega,\omega)$ then

$$F_n \to F \quad \textit{strongly in } (W^{1,\varphi}(\Omega))^N,$$

$$G_n \to G \quad \textit{strongly in } (W^{1,\varphi}(\Omega))^N,$$

and

$$Au_n \to Au \quad \textit{strongly in } W^{1,\varphi^*}(\Omega).$$

This implies that A is continuous on $W^{1,\varphi^*}(\Omega)$ and we can verify immediately that A^3 is continuous on $W^{1,\varphi^*}(\Omega)$, while L is linear bounded, hence continuous. Therefore, T is hemi-continuous on $W^{1,\varphi^*}(\Omega)$. Finally, by Theorem 3.1, there exists a weak solution to problem (1.2).

Uniqueness:

Let u and v be two weak solutions of the problem (1.2). For the solution u , we take $\psi = u - v$ as test function and for the solution v we take $\psi = v - u$ as test function in Equation (3.1). Then we get

$$\langle Au - Av, u - v \rangle + \int_\Omega (g(u) - g(v))(u - v)dx = 0.$$

The operator A is monotone, so

$$\langle Au - Av, u - v \rangle \geq 0.$$

Then

$$\int_\Omega (g(u) - g(v))(u - v)dx \leq 0$$

and by Proposition 2.5, we find

$$\int_\Omega \mu_p(|u| + |v|)^{p-2}|u - v|^2 dx \leq \int_\Omega (g(u) - g(v))(u - v)dx \leq 0.$$

Then

$$|u - v|^2 = 0 \qquad a.e \quad in \quad \Omega.$$

Finally, we have

$$u = v \qquad a.e \quad in \quad \Omega.$$

□

5. Conclusion

In this paper, we first prove the existence of a weak solution to the double-phase elliptic equations by defining the Orlicz space corresponding to our variational problem and verifying the conditions of Theorem 2.2. Secondly, we prove the uniqueness of this weak solution.

References

[1] R. Arora and S. Shmarev, *Double-phase parabolic equations with variable growth and nonlinear sources*, Advances in Nonlinear Analysis, 12, 304–335, 2023. https://doi.org/10.1515/anona-2022-0271.

[2] F. Behboudi and A. Razani, *Two weak solutions for a singular (p,q)-Laplacian problem*, Filomat, 33(11) (2019), 3399–3407.

[3] Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, and J. L. Vazquez, *An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 22(2), 241–273, 1995.

[4] Ph. Bénilan and H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation*, Journal of Evolution Equations, 3(4), 673–770, 2003.

[5] L. Boccardo, T. Gallouet, and F. Murat, *Unicité de la solution de certaines équations elliptiques non linéaires*, C. R. Acad. Sci. Paris, t. 315, Série I, pp. 1159–1164, 1992.

[6] L. Boccardo, T. Gallouet, and L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. Henri Poincaré, Vol. 13(5), pp. 539–551, 1992.

[7] A. C. Cavalheiro, *Weighted Sobolev spaces and degenerate elliptic equations*, Bol. Soc. Paran. Mat., 26, 117–132, 2008.

[8] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhauser, Berlin, 2009.

[9] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., Vol. XXVIII, pp. 741–808, 1999.

[10] N. Ivanchuk, P. Kogut, and O. Kupenko, *Generalized active contour model in an anisotropic variable exponent Sobolev space*, Journal of Optimization, Differential Equations and Their Applications, 2024, pp. 32:1, 1–32. https://doi.org/10.15421/142401.

[11] Y. Jiao, K. Saibi, and C. Zhang, *Weighted $L^{p(\cdot)}$-estimates for the nonlinear parabolic equations with non-standard growth*, J. Math Anal. and App., vol. 489, no. 1, 2020, Art. No. 124145.
https://doi.org/10.1016/j.jmaa.2020.124145.

[12] B. Laadjal, K. Saibi, O. Melkemi, and Z. Mokhtari, *Weighted variable Hardy spaces associated with operators satisfying Davies-Gaffney estimates*, Probl. Anal. Issues Anal., 2022. https://doi.org/10.15393/j3.art.2022.11130.

[13] M. Makvand Chaharlang and A. Razani, *Two weak solutions for some Kirchhoff-type problem with Neumann boundary condition*, Georgian Math. J., (2020), 110.

[14] V. Mustonen, P. Drabek, and A. Kufner, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Austral. Math. Soc., 58, 213–221, 1998.

[15] A. Sabri and A. Jamea, *Weak solution for nonlinear degenerate elliptic problem with Dirichlet-type boundary condition in weighted Sobolev spaces*, Mathematica Bohemica, 26(3):4–18, 2021.
https://doi.org/10.21136/MB.2021.0004-20.

[16] A. Sabri, A. Jamea, and H. T. Alaoui, *Existence of entropy solutions to nonlinear degenerate parabolic problems with variable exponent and L^1-data*, 25(1973), 565–590.

[17] A. Sabri and A. Jamea, *Entropy solution for nonlinear degenerate elliptic problem with Dirichlet-type boundary condition in weighted Sobolev spaces*, LE MATEMATICHE, Vol. LXXVI, pp. 109–131, 2021.
https://doi.org/10.4418/2021.76.1.7.

[18] M. Sanchón and J. M. Urbano, *Entropy solutions for the p(x)-Laplace equation*, Transactions of the American Mathematical Society, 361(12), 6387–6405, 2009.

[19] C. Zhang, *Entropy solutions for nonlinear elliptic equations with variable exponents*, Electronic Journal of Differential Equations, 2014(92), 1–14.