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Modeling Divorce Dynamics Due to Extramarital Affairs with Piecewise Classical-Fractional Operator

R.P. Chauhan[1,†] and Mehar Chand[2]

Abstract In recent decades, various methodologies have been proposed to model the complexities of challenging global problems across different domains. One such challenge involves understanding multi-step behaviors observed in certain situations. Newly proposed piecewise derivatives are known to address these issues. This study utilizes a mathematical model to examine the spread of a social issue of divorce among married couples resulting from extramarital affairs, using piecewise derivatives. Initially, we develop the model with Caputo fractional derivative and conduct some basic mathematical computations. Furthermore, the model is explored within the framework of the piecewise operator, incorporating both classical and Caputo operators. Within this framework, the study presents the existence and uniqueness of the solution using the fixed-point results. To analyze the behavior of the considered model, the Newton polynomial interpolation method is employed. The findings are subsequently illustrated through graphical representations, considering various values of fractional order.

Keywords Divorce, piecewise derivative, Caputo derivative, existence and uniqueness, numerical simulation

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1. Introduction

Over the past twenty years, fractional calculus (FC) has gained popularity among researchers and scientists [7, 24, 25, 28, 44, 48]. FC applications cover a wide range of scientific and engineering fields, including theoretical, numerical, and experimental aspects [14, 32, 45]. Fractional differential equations (FDEs) are fundamentally characterized by their ability to incorporate memory effects. Classical models fall short compared with fractional models due to the inherent genetic properties, nonlocalities, and memory effects of fractional derivatives. The application of FDEs enhances our understanding of real-world phenomena, offering greater flexibility and precision in modeling. The literature extensively explores various fractional operators, each characterized by different kernels. Among these, the Caputo fractional derivative having a power-law kernel stands out as particularly significant

[†]the corresponding author.
Email address: rampratapc23@gmail.com(R.P. Chauhan)
[1]Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Amaravati, Andhra Pradesh, 522503, India
[2]Department of Mathematics, Faculty of Computational and Mathematical Sciences, Baba Farid College, Bathinda, 151001, Punjab, India

due to its ability to account for memory effects and long-range dependence. The Caputo derivative integrates initial conditions naturally and is especially well-suited for applied fields. This derivative is widely used by researchers to solve various real-world problems and effectively captures how past interactions and events influence the current state of a system [2, 8, 9, 27, 30, 33, 34, 46, 47, 49–51]. Supporting the applicability of fractional derivatives, Atede et al. [6] analyzed the effectiveness of the Pfizer vaccination program against COVID-19 considering real data from Nigeria. They observed that varying fractional-order values exert different influences on each compartment of the model. Iwa et al. [19] proposed a Caputo-type new mathematical model to study co-infection dynamics of malaria and COVID-19. M. O. Olayiwola et al. [31] investigated a fractional model for COVID-19 that integrates high-risk quarantine measures and vaccination. Joshi et al. [20] proposed a four-dimensional system to estimate the effects of burned plastic and recycled plastic on air pollution.

Some real-world problems exhibit multi-step behaviors, where different processes occur in distinct phases. Such problems cannot be fully replicated using traditional fractional derivatives because the crossover behaviors between these phases remain inadequately addressed. To overcome this, Atangana and Araz [5] introduced a novel concept of piecewise derivatives and integrals. This new approach provides substantial advantages by allowing the analysis of mathematical models utilizing both classical and fractional operators inside a unified interval that can be subdivided. This significant development provides researchers with a powerful tool for investigating crossover behaviors in their studies. Piecewise derivatives have proven to be the most effective approach for illustrating this crossover event, as opposed to utilizing continuous derivatives in a range of real-world challenges [1,4,16,22,23,39–42].

Marriage is an officially and socially acknowledged tradition that is upheld by every religion to establish a relationship between a man and a woman (usually). Individuals enter marriage for numerous reasons, including the fulfilment of emotional and physical desires, familial and societal expectations, and raising children. From arranged or love marriages to now legally permitted same-sex marriages, several matrimonial trends have been noted over the years. Nowadays, a growing number of individuals are choosing divorce or seeking extramarital pleasures (infidelity), which leads to the deterioration of family structures. A successful marriage often requires effort and intentionality from both partners to maintain and nurture the relationship over time. This can be ascribed to a lack of patience and perseverance. Divorce is an endemic problem that, like any disease, has a significant impact on the social and economic structure of modern society. As a result, a structured methodical data analysis is required to tackle this socially complicated subject. In this world, individuals are generally expected to eventually encounter their life partner, whether through love or an arranged marriage. Additionally, some may seek emotional connections outside their primary relationship. As a result, they may face separation, which may lead to divorce in some situations. Certain cases of negotiations have also been recorded where spouses rejoin and give their marriage a second opportunity. To study this scenario, a mathematical model was formulated by Shah et al. [43] using ordinary differential equations.

The significance and dynamic nature of the divorce phenomenon have drawn attention from both biological and sociological perspectives, making it a subject of growing importance in diverse studies. Employing mathematical models, proven effective in controlling epidemics, can offer a valuable approach to proactively address and potentially prevent the spread of divorce in marital contexts. Various efforts have been made in the past to explore the complex phenomena of marriage, divorce, and remarriage. Becker [10] examined and explained marriage theory, suggesting that the benefits of getting married instead of staying single are linked to factors such as income, expenses, and the difference in wage rates between partners. Becker [11] extended their work by incorporating various aspects of marriage, separation, divorce, and remarriage. In 1995, Bramlett and Mosher [12] performed an analysis of first and second marriages among women aged 15–44 in the United States, examining the probabilities of divorce and remarriage using various approaches. Cherlin [13] wrote a book explaining marriage, divorce, and remarriage. A mathematical model was developed by Duato and Jodar [15] to describe a social epidemiology model of divorce initiated by women in Spain. Oppenheimer [35] examined the trends and disparities in the timing of marriage, highlighting an imbalance that can occasionally result in divorce. Gottman [17] studies an argument on what predicts divorce and the relation between marital processes and marital outcomes. Owolabi [36] investigated love dynamics in terms of memory effect. Gweryina et al. [18] construct a divorce epidemic model with anti-divorce therapy. Karaagac and Owolabi [21] analyzed a marriage divorce model incorporating fractal-fractional derivative. Given that love is a state constantly stimulated by historical facts, fractional differential equations prove useful for modeling marital relations.

Marital relationships often exhibit nonlinear behaviors influenced by factors such as financial stability, communication, cultural values, and external pressures. These relationships evolve across different phases, including initial bonding, conflict, and separation. Each phase may operate on varying time scales, making a piecewise differentiation approach essential for accurately modeling these dynamics. The use of piecewise differential equations allows the model to transition between different rates of change and mechanisms governing these phases, reflecting the unique characteristics of each stage. Since memory plays a crucial role in relationships and interactions, when we meet someone, the effects of that interaction, positive or negative, persist in our memory. This memory-bound nature significantly shapes marital dynamics over time. FDEs are ideal for modeling such phenomena, as they are inherently useful for memory effects and long-range dependence. The fractional derivative can be particularly effective in assessing the impact of extramarital affairs on divorce, which are inherently tied to memory and long-term consequences. FDEs allow for a more comprehensive understanding of how past events influence present behaviors. The Caputo derivative has the ability to accommodate local initial conditions, and it is compatible with biological and physical principles. By utilizing piecewise hybrid operators, we can better understand and predict the evolving dynamics of marital relationships, especially under complex, nonlinear conditions. Its importance lies in providing more accurate approximations for systems that experience sudden or step changes that are often observed in real-life problems. The dynamic behavior of divorce in society is intriguing from both biological and sociological perspectives, and the study of this subject is becoming increasingly important. To the best of our knowledge, this is the first instance where this model has been solved using a piecewise fractional approach. While existing models typically use either classical or fractional derivatives to analyze complex systems, they often fail to capture the transitional behavior between different phases of a process. The novelty of this work lies in the following points:

- This study improves the classical divorce model that considers extramarital affairs [43] by employing the Caputo fractional derivative and the piecewise Caputo derivative, offering a more precise representation of extramarital relationship dynamics.

- The existence and uniqueness of the solution are thoroughly investigated for the piecewise Caputo model.
- Through numerical simulations, we illustrate the practical applicability of our model and provide new insights into the dynamics of divorce and relationship dissolution.

The article is organized as follows: Section 2 includes essential definitions. Section 3 presents mathematical findings related to the considered model utilizing the fractional Caputo derivative. Section 4 introduces the model in the piecewise derivative along with qualitative analysis. Section 5 outlines the numerical scheme used to obtain the solution. Section 6 discusses the simulation results. Finally, Section 7 presents the conclusions.

2. Basic preliminaries

Here, we present some basic preliminaries related to this study [5,38].

Definition 2.1. The Riemann-Liouville fractional integral of order $\vartheta \in [0,1)$ is defined as:

$$\mathfrak{I}_t^{\vartheta}\aleph(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-z)^{\vartheta-1}\aleph(z)dz. \tag{2.1}$$

Definition 2.2. The Caputo fractional derivative of order $\vartheta \in [0,1)$ is defined as:

$${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\aleph(t) = \frac{1}{\Gamma(1-\vartheta)}\int_0^t (t-z)^{-\vartheta}\aleph'(z)dz. \tag{2.2}$$

Definition 2.3. The Laplace transform in the Caputo sense is defined as:

$$\mathcal{L}[{}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\aleph(t)] = s^{\vartheta}\mathcal{L}[\aleph(t)] - \sum_{j=0}^{n-1}\aleph^{(j)}(0)s^{\vartheta-j-1}, \;\; n-1<\vartheta\leq n, \; n\in\mathbb{N}. \tag{2.3}$$

Definition 2.4. Taking $\aleph(t)$ as differentiable and $g(t)$ as an increasing function, then classical piecewise integration can be defined as:

$${}_0^{\mathcal{PF}}\mathfrak{I}_t\aleph(t) = \begin{cases} \int_0^t \aleph(z)dz, & 0<t\leq t_1, \\ \int_{t_1}^t \aleph(z)g'(z)dz, & t_1<t\leq t_2=T, \end{cases} \tag{2.4}$$

where ${}_0^{\mathcal{PF}}\mathfrak{I}_t$ represents classical integration on $0<t\leq t_1$ and global integration on $t_1<t\leq t_2=T$.

Definition 2.5. Let $\aleph(t)$ be a differentiable function. Then the piecewise derivative in classical and fractional-order Caputo sense is defined as:

$${}_0^{\mathcal{PF}}\mathfrak{D}_t^{\vartheta}\aleph(t) = \begin{cases} \aleph'(z), & 0<t\leq t_1, \\ {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\aleph(z), & t_1<t\leq t_2=T, \end{cases} \tag{2.5}$$

where ${}_0^{\mathcal{PF}}\mathfrak{D}_t^{\vartheta}$ represents classical derivative on $0<t\leq t_1$ and fractional derivative on $t_1<t\leq t_2=T$.

Definition 2.6. Consider $\aleph(t)$ be a differentiable function, then the piecewise integration of classical and Caputo integral is defined as:

$$ {}_0^{\mathcal{PF}}\mathfrak{I}_t \aleph(t) = \begin{cases} \int_0^t \aleph(z)dz, & 0 < t \leq t_1, \\ \frac{1}{\Gamma(\vartheta)} \int_{t_1}^t (t-z)^{\vartheta-1} \aleph(z)dz, & t_1 < t \leq t_2 = T. \end{cases} \tag{2.6} $$

Lemma 2.1. *The solution of piecewise derivable equation*

$$ {}_0^{\mathcal{PFC}}\mathfrak{D}_t^{\vartheta} \aleph(t) = F(t, \aleph(t)), \qquad 0 < \vartheta \leq 1, \tag{2.7} $$

is

$$ \aleph(t) = \begin{cases} \aleph_0 + \int_0^t \aleph(z)dz, & 0 < t \leq t_1, \\ \aleph(t_1) + \frac{1}{\Gamma(\vartheta)} \int_{t_1}^t (t-z)^{\vartheta-1} \aleph(z)dz, & t_1 < t \leq t_2 = T. \end{cases} \tag{2.8} $$

3. Model in Caputo sense

Here, we consider a mathematical model for divorce in marriage due to extramarital affairs [43]. The assumption is that extramarital affairs are a major factor contributing to the breakdown of both arranged and love marriages, often leading to separation and divorce. Additionally, divorced individuals re-enter the susceptible class, reflecting their willingness to remarry, regardless of the type of marriage. The model divides the population into six compartments: susceptible individuals ($\mathrm{S_I}$) for marriage who either choose arranged marriage ($\mathrm{A_M}$) or love marriage ($\mathrm{L_M}$). Some of them may have extramarital affairs ($\mathrm{E_M}$) which sometimes leads to partition (P) or in some situations, divorce (D). The description of parameters of the divorce model is described in Table 1. The model is determined by the following system of differential equations:

$$ \begin{aligned} \frac{d\mathrm{S_I}}{dt} &= \Upsilon - \beta_1 \mathrm{S_I}\mathrm{A_M} - \beta_2 \mathrm{S_I}\mathrm{L_M} + \beta_{13}\mathrm{D} - \mu \mathrm{S_I}, \\ \frac{d\mathrm{A_M}}{dt} &= \beta_1 \mathrm{S_I}\mathrm{A_M} + \beta_6 \mathrm{P} - y_1 \mathrm{A_M}, \\ \frac{d\mathrm{E_M}}{dt} &= \beta_3 \mathrm{A_M} + \beta_4 \mathrm{L_M} - y_2 \mathrm{E_M}, \\ \frac{d\mathrm{L_M}}{dt} &= \beta_2 \mathrm{S_I}\mathrm{L_M} + \beta_{10}\mathrm{P} - y_3 \mathrm{L_M}, \\ \frac{d\mathrm{P}}{dt} &= \beta_5 \mathrm{A_M} + \beta_8 \mathrm{E_M} + \beta_9 \mathrm{L_M} - y_4 \mathrm{P}, \\ \frac{d\mathrm{D}}{dt} &= \beta_7 \mathrm{A_M} + \beta_{11}\mathrm{L_M} + \beta_{12}\mathrm{P} - y_5 \mathrm{D}, \end{aligned} \tag{3.1} $$

where $y_1 = (\beta_3 + \beta_5 + \beta_7 + \mu)$, $y_2 = (\beta_8 + \mu)$, $y_3 = (\beta_4 + \beta_9 + \beta_{11} + \mu)$, $y_4 = (\beta_6 + \beta_{10} + \beta_{12} + \mu)$, $y_5 = (\beta_{13} + \mu)$, and $\mathrm{S_I} + \mathrm{A_M} + \mathrm{E_M} + \mathrm{L_M} + \mathrm{P} + \mathrm{D} \leq \mathrm{N}$, and the initial conditions $\mathrm{S_I}(0) \geq 0, \mathrm{A_M}(0) \geq 0, \mathrm{E_M}(0) \geq 0, \mathrm{L_M}(0) \geq 0, \mathrm{P}(0) \geq 0, \mathrm{D}(0) \geq 0$.

To incorporate memory effects into the classical model represented by (3.1), we extend it using the Caputo derivative. The extended model is expressed as follows:

$$\begin{aligned}
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{S}_{\mathtt{I}} &= \Upsilon - \beta_1 \mathrm{S}_{\mathtt{I}}\mathrm{A}_{\mathtt{M}} - \beta_2 \mathrm{S}_{\mathtt{I}}\mathrm{L}_{\mathtt{M}} + \beta_{13}\mathrm{D} - \mu \mathrm{S}_{\mathtt{I}},\\
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{A}_{\mathtt{M}} &= \beta_1 \mathrm{S}_{\mathtt{I}}\mathrm{A}_{\mathtt{M}} + \beta_6 \mathrm{P} - y_1 \mathrm{A}_{\mathtt{M}},\\
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{E}_{\mathtt{M}} &= \beta_3 \mathrm{A}_{\mathtt{M}} + \beta_4 \mathrm{L}_{\mathtt{M}} - y_2 \mathrm{E}_{\mathtt{M}},\\
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{L}_{\mathtt{M}} &= \beta_2 \mathrm{S}_{\mathtt{I}}\mathrm{L}_{\mathtt{M}} + \beta_{10} \mathrm{P} - y_3 \mathrm{L}_{\mathtt{M}},\\
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{P} &= \beta_5 \mathrm{A}_{\mathtt{M}} + \beta_8 \mathrm{E}_{\mathtt{M}} + \beta_9 \mathrm{L}_{\mathtt{M}} - y_4 \mathrm{P},\\
{}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{D} &= \beta_7 \mathrm{A}_{\mathtt{M}} + \beta_{11} \mathrm{L}_{\mathtt{M}} + \beta_{12} \mathrm{P} - y_5 \mathrm{D},
\end{aligned} \tag{3.2}$$

with the initial conditions $\mathrm{S}_{\mathtt{I}}(0) \geq 0, \mathrm{A}_{\mathtt{M}}(0) \geq 0, \mathrm{E}_{\mathtt{M}}(0) \geq 0, \mathrm{L}_{\mathtt{M}}(0) \geq 0, \mathrm{P}(0) \geq 0$, and $\mathrm{D}(0) \geq 0$.

Table 1. Parameter description [43]

Parameters	Description	Values
Υ	Recruitment rate of individuals	0.4
β_1	Arranged marriage rate of individuals	0.6
β_2	Love marriage rate of individuals	0.05
β_3	Rate of infidelity among arranged married couples	0.01
β_4	Rate of infidelity among love married couples	0.01
β_5	Rate at which individuals initiate separation	0.1
β_6	Rate at which arranged married couples reunite	0.02
β_7	Divorce rate of arranged married couples	0.05
β_8	Separation rate of individuals involved in extramarital affairs	0.3
β_9	Separation rate of love married couples	0.1
β_{10}	Rate at which love married couples reunite	0.02
β_{11}	Divorce rate of love married couples	0.15
β_{12}	Divorce rate of separated individuals	0.3
β_{13}	Rate at which divorced individuals again become susceptible for marriage	0.2
μ	Natural death rate	0.1

3.1. Positivity and boundedness

To demonstrate the positivity of solution to the fractional model (3.2), we define $\mathbb{R}_+^6 = \{y \in \mathbb{R}^6 | y \geq 0\}$ and $y(t) = (\mathrm{S}_{\mathtt{I}}(t), \mathrm{A}_{\mathtt{M}}(t), \mathrm{E}_{\mathrm{M}}(t), \mathrm{L}_{\mathrm{M}}(t), \mathrm{P}(t), \mathrm{D}(t))^T$.

To prove the desired results, we recall the generalized mean values theorem [29].

Lemma 3.1. *[29]. Let $\vartheta \in (0, 1]$, $G(t) \in C[b_1, b_2]$ and ${}_{b_1}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} G(t) \in C(b_1, b_2]$. Then*

$$G(t) = G(b_1) + \frac{1}{\Gamma(\vartheta)} ({}_{b_1}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} G)(\zeta)(t - b_1)^{\vartheta},$$

with $0 \leq \zeta \leq t, \forall t \in (b_1, b_2]$.

Corollary 3.1. *Assume that* $G(t) \in C[b_1, b_2]$ *and* ${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} G(t) \in C(b_1, b_2]$ *where* $\vartheta \in (0, 1]$. *Then if*

i. ${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} G(t) \geq 0$, $\forall t \in (b_1, b_2)$, *then* $G(t)$ *is non-decreasing.*

ii. ${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} G(t) \leq 0$, $\forall t \in (b_1, b_2)$, *then* $G(t)$ *is non-increasing.*

Theorem 3.1. *There exists a solution, say* $y(t)$, *for the fractional model* (3.2), *which is unique and remains in* $\mathbb{R}_+^6$ *and non-negative.*

Proof. Following the Theorem 3.1 described in [26], one can simply show the existence of the Caputo model (3.2). Furthermore, the uniqueness of the solution can be determined by using the Remark 3.2 in [26] for all $t \in (0, \infty)$. Moreover, to ensure non-negativity of the solution, it is essential to show that on each hyperplane bounding the non-negative orthant, the vector field points to $\mathbb{R}_+^6$. From the system (3.2), we have

$$\begin{aligned}
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{S_I}(t)|_{\mathrm{S_I}=0} &= \Upsilon + \beta_{13}\mathrm{D} \geq 0, \\
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{A_M}(t)|_{\mathrm{A_M}=0} &= \beta_6 \mathrm{P} \geq 0, \\
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{E_M}(t)|_{\mathrm{E_M}=0} &= \beta_3 \mathrm{A_M} + \beta_4 \mathrm{L_M} \geq 0, \\
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{L_M}(t)|_{\mathrm{L_M}=0} &= \beta_{10} \mathrm{P} \geq 0, \\
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{P}(t)|_{\mathrm{P}=0} &= \beta_5 \mathrm{A_M} + \beta_8 \mathrm{E_M} + \beta_9 \mathrm{L_M} \geq 0, \\
{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{D}(t)|_{\mathrm{D}=0} &= \beta_7 \mathrm{A_M} + \beta_{11} \mathrm{L_M} + \beta_{12} \mathrm{P} \geq 0.
\end{aligned}$$

Therefore, in view of the aforesaid corollary, we conclude that the solution remains in the region $\mathbb{R}_+^6$ for all $t \geq 0$.

Next, we will show the boundedness of the model (3.2). Here, we construct the following biologically feasible region given by:

$$\Omega \subset \mathbb{R}_+^6,$$

such that

$$\Omega = \left\{ (\mathrm{S_I}, \mathrm{A_M}, \mathrm{E_M}, \mathrm{L_M}, \mathrm{P}, \mathrm{D}) \in \mathbb{R}_+^6 : \mathrm{N} \leq \frac{\Upsilon}{\mu} \right\}.$$

Lemma 3.2. *The region given by* Ω, *is positively invariant for the system* (3.2) *with the initial conditions in* $\mathbb{R}_+^6$

Proof. By summing all the equations in the model (3.2), we obtain

$${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{N}(t) = {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{S_I}(t) + {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{A_M}(t) + {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{E_M}(t) + {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{L_M}(t) + {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{P}(t) + {}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{D}(t).$$

Hence,

$${}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{N}(t) = \Upsilon - \mu \mathrm{N}(t). \tag{3.3}$$

Applying the Laplace transform gives

$$\mathcal{L}[{}_0^{\mathcal{C}}\mathfrak{D}_t^{\vartheta} \mathrm{N}(t) + \mu \mathrm{N}(t)] = \mathcal{L}[\Upsilon],$$

$$s^{\vartheta} \mathrm{N}(s) - s^{\vartheta - 1} \mathrm{N}(0) + \mu \mathrm{N}(s) = \frac{\Upsilon}{s},$$

$$\mathrm{N}(s) = \frac{s^{-1}}{(s^{\vartheta} + \mu)}\Upsilon + \mathrm{N}(0)\frac{s^{\vartheta-1}}{s^{\vartheta} + \mu}.$$

Using inverse Laplace transform, we have

$$\mathrm{N}(t) = \mathrm{N}(0)E_{\vartheta,1}(-\mu t^{\vartheta}) + \Upsilon t^{\vartheta} E_{\vartheta,\vartheta+1}(-\mu t^{\vartheta}),$$

where the Mittag-Leffler (ML) function is given by

$$E_{\vartheta,\gamma}(z) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\Gamma(\ell\vartheta + \gamma)},$$

and the Laplace transform of the function $t^{\gamma-1}E_{\vartheta,\gamma}(\pm\lambda t^{\vartheta})$ is defined as:

$$\mathcal{L}[t^{\gamma-1}E_{\vartheta,\gamma}(\pm\lambda t^{\vartheta})] = \frac{s^{\vartheta-\gamma}}{s^{\vartheta} \mp \lambda}.$$

The asymptotic characteristics of ML function [37] lead to the conclusion that $\mathrm{N}(t)$ converges to $\frac{\Upsilon}{\mu}$ whenever $t \to \infty$. Hence, for every $t > 0$ all the solutions of the model with initial conditions in Ω remain in Ω. Therefore, the region Ω is positively invariant and attracts all solutions in $\mathbb{R}^6_+$.

4. Model in piecewise sense

In this section, we will explore the behavior of the model using a piecewise differential operator. The introduction of this novel operator offers a new avenue for studying crossover behavior in real-world problems. The classical model (3.1) can be written in piecewise classical and Caputo derivative as follows:

$$\begin{aligned}
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{S_I}(t) &= \Upsilon - \beta_1\mathrm{S_I}\mathrm{A_M} - \beta_2\mathrm{S_I}\mathrm{L_M} + \beta_{13}\mathrm{D} - \mu\mathrm{S_I},\\
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{A_M}(t) &= \beta_1\mathrm{S_I}\mathrm{A_M} + \beta_6\mathrm{P} - y_1\mathrm{A_M},\\
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{E_M}(t) &= \beta_3\mathrm{A_M} + \beta_4\mathrm{L_M} - y_2\mathrm{E_M},\\
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{L_M}(t) &= \beta_2\mathrm{S_I}\mathrm{L_M} + \beta_{10}\mathrm{P} - y_3\mathrm{L_M},\\
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{P}(t) &= \beta_5\mathrm{A_M} + \beta_8\mathrm{E_M} + \beta_9\mathrm{L_M} - y_4\mathrm{P},\\
{}^{\mathcal{PCC}}_{0}\mathfrak{D}^{\vartheta}_{t}\mathrm{D}(t) &= \beta_7\mathrm{A_M} + \beta_{11}\mathrm{L_M} + \beta_{12}\mathrm{P} - y_5\mathrm{D},
\end{aligned} \tag{4.1}$$

where $\mathcal{PCC}$ denotes piecewise classical and Caputo derivatives having two subintervals in $[0, T]$. In a more simplified representation, Eq. (4.1) can be expressed as follows:

$$
\begin{aligned}
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{S}_{\mathtt{I}}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{S}_{\mathtt{I}}(t) = \dfrac{d}{dt}\mathbb{Z}_1(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{S}_{\mathtt{I}}(t) = {}^{\mathcal{C}}\mathbb{Z}_1(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T,\end{cases}\\
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{A}_{\mathtt{M}}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{A}_{\mathtt{M}}(t) = \dfrac{d}{dt}\mathbb{Z}_2(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{A}_{\mathtt{M}}(t) = {}^{\mathcal{C}}\mathbb{Z}_2(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T,\end{cases}\\
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{E}_{\mathtt{M}}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{E}_{\mathtt{M}}(t) = \dfrac{d}{dt}\mathbb{Z}_3(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{E}_{\mathtt{M}}(t) = {}^{\mathcal{C}}\mathbb{Z}_3(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T,\end{cases}\\
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{L}_{\mathtt{M}}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{L}_{\mathtt{M}}(t) = \dfrac{d}{dt}\mathbb{Z}_4(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{L}_{\mathtt{M}}(t) = {}^{\mathcal{C}}\mathbb{Z}_4(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T,\end{cases}\\
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{P}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{P}(t) = \dfrac{d}{dt}\mathbb{Z}_5(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{P}(t) = {}^{\mathcal{C}}\mathbb{Z}_5(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T,\end{cases}\\
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\mathrm{D}(t) &= \begin{cases} \mathfrak{D}_t\mathrm{D}(t) = \dfrac{d}{dt}\mathbb{Z}_6(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & 0<t\le t_1,\\ {}_{0}^{\mathcal{C}}\mathfrak{D}_t^{\vartheta}\mathrm{D}(t) = {}^{\mathcal{C}}\mathbb{Z}_6(t,\mathrm{S}_{\mathtt{I}},\mathrm{A}_{\mathtt{M}},\mathrm{E}_{\mathtt{M}},\mathrm{L}_{\mathtt{M}},\mathrm{P},\mathrm{D}), & t_1<t\le t_2=T.\end{cases}
\end{aligned}
\tag{4.2}
$$

4.1. Theoretical analysis

Here, we will examine the existence and uniqueness of solution for the suggested piecewise model. To achieve this, we express the system (4.2) as outlined in Lemma (3.3). Further elaborating on this description, we express it as follows:

$$
{}_{0}^{\mathcal{PCC}}\mathfrak{D}_t^{\vartheta}\aleph(t) = \mathbb{Z}(t,\aleph(t)), \qquad 0<\vartheta\le 1, \tag{4.3}
$$

$$
\aleph(t) = \begin{cases} \aleph(0) + \displaystyle\int_0^{t_1}\mathbb{Z}(z,\aleph(z))dz, & 0<t\le t_1,\\ \aleph(t_1) + \dfrac{1}{\Gamma(\vartheta)}\displaystyle\int_{t_1}^{t_2}(t-z)^{\vartheta-1}\mathbb{Z}(z,\aleph(z))dz, & t_1<t\le t_2=T,\end{cases} \tag{4.4}
$$

where

$$
\aleph(t) = \begin{cases} \mathrm{S}_{\mathtt{I}}(t)\\ \mathrm{A}_{\mathtt{M}}(t)\\ \mathrm{E}_{\mathtt{M}}(t)\\ \mathrm{L}_{\mathtt{M}}(t)\\ \mathrm{P}(t)\\ \mathrm{D}(t)\end{cases},\ \aleph(0) = \begin{cases} \mathrm{S}_{\mathtt{I}}(0)\\ \mathrm{A}_{\mathtt{M}}(0)\\ \mathrm{E}_{\mathtt{M}}(0)\\ \mathrm{L}_{\mathtt{M}}(0)\\ \mathrm{P}(0)\\ \mathrm{D}(0)\end{cases},\ \aleph(t_1) = \begin{cases} \mathrm{S}_{\mathtt{I}}(t_1)\\ \mathrm{A}_{\mathtt{M}}(t_1)\\ \mathrm{E}_{\mathtt{M}}(t_1)\\ \mathrm{L}_{\mathtt{M}}(t_1)\\ \mathrm{P}(t_1)\\ \mathrm{D}(t_1)\end{cases}, \tag{4.5}
$$

$$\mathbb{Z}(t,\aleph(t)) = \begin{cases} \mathbb{Z}_1 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_1(t,\mathrm{S}_{\mathtt{I}}) \\ {}^{C}\mathbb{Z}_1(t,\mathrm{S}_{\mathtt{I}}) \end{cases}, \\ \mathbb{Z}_2 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_2(t,\mathrm{A}_{\mathtt{M}}) \\ {}^{C}\mathbb{Z}_2(t,\mathrm{A}_{\mathtt{M}}) \end{cases}, \\ \mathbb{Z}_3 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_3(t,\mathrm{E}_{\mathtt{M}}) \\ {}^{C}\mathbb{Z}_3(t,\mathrm{E}_{\mathtt{M}}) \end{cases}, \\ \mathbb{Z}_4 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_4(t,\mathrm{L}_{\mathtt{M}}) \\ {}^{C}\mathbb{Z}_4(t,\mathrm{L}_{\mathtt{M}}) \end{cases}, \\ \mathbb{Z}_5 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_5(t,\mathrm{P}) \\ {}^{C}\mathbb{Z}_5(t,\mathrm{P}) \end{cases}, \\ \mathbb{Z}_6 = \begin{cases} \dfrac{d}{dt}\mathbb{Z}_6(t,\mathrm{D}) \\ {}^{C}\mathbb{Z}_6(t,\mathrm{D}) \end{cases}. \end{cases} \tag{4.6}$$

Let $0 < t_1 < t \le t_2 < \infty$ with Banach space $B^* = C[0,T]$ having the norm $\|\aleph\| = \max_{t\in[0,T]}|\aleph(t)|$. We take the following growth condition on the nonlinear operator to achieve the desired result.

- (A_1) $\exists\ \mathcal{L}_{\aleph} > 0,\ \forall \mathbb{Z}, \bar{\aleph} \in B^*$, we have

$$|\mathbb{Z}(t,\aleph) - \mathbb{Z}(t,\bar{\aleph})| \le \mathcal{L}_{\mathbb{Z}}|\aleph - \bar{\aleph}|.$$

- (A_2) $\exists\ C_{\mathbb{Z}} > 0$ and $M_{\mathbb{Z}} > 0$, we have

$$|\mathbb{Z}(t,\aleph(t))| \le C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}. \tag{4.7}$$

Theorem 4.1. *Let $\mathbb{Z}$ be a piecewise continuous on subinterval $(0,t_1]$ and $(t_1,T]$ on $[0,T]$ also satisfying (A_2). Then the piecewise problem (4.1) has a solution.*

Proof. By using Schauder's theorem, let us define a closed subset B of B^* in both subintervals of $[0,T]$ as

$$B = \{\aleph \in B^* : \|\aleph\| \le R_{1,2}, R_{1,2} > 0\}.$$

Let an operator $\mathbb{T} : B \to B$ and using (4.4) as:

$$\mathbb{T}(\aleph) = \begin{cases} \aleph_0 + \displaystyle\int_0^{t_1} \mathbb{Z}(z,\aleph(z))dz, & 0 < t \le t_1, \\ \aleph(t_1) + \dfrac{1}{\Gamma(\vartheta)}\displaystyle\int_{t_1}^{t_2}(t-z)^{\vartheta-1}\mathbb{Z}(z,\aleph(z))dz, & t_1 < t \le t_2 = T. \end{cases}$$

On any $\aleph \in B$, we have

$$
\begin{aligned}
|\mathbb{T}(\aleph)| &\leq \begin{cases} |\aleph_0| + \displaystyle\int_0^{t_1} |\mathbb{Z}(z,\aleph(z))|dz, & 0 < t \leq t_1, \\ |\aleph(t_1)| + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} |\mathbb{Z}(z,\aleph(z))|dz, & t_1 < t \leq T, \end{cases} \\
&\leq \begin{cases} |\aleph_0| + \displaystyle\int_0^{t_1} [C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}]dz, & 0 < t \leq t_1, \\ |\aleph(t_1)| + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} [C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}]dz, & t_1 < t \leq T, \end{cases} \\
&\leq \begin{cases} |\aleph_0| + t[C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}] = R_1, & 0 < t \leq t_1, \\ |\aleph(t_1)| + \dfrac{(t_2-t_1)^2}{\Gamma(\vartheta+1)} [C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}] = R_2, & t_1 < t \leq T, \end{cases} \\
&\leq \begin{cases} R_1, & 0 < t \leq t_1, \\ R_2, & t_1 < t \leq T. \end{cases}
\end{aligned} \tag{4.8}
$$

Thus, it implies $\mathbb{T}(B) \subset B$. Hence, it shows that $\mathbb{T}$ is closed and complete. Next, for complete continuity of $\mathbb{T}$, we consider $t_k < t_l \in [0, t_1]$, which gives

$$
\begin{aligned}
|\mathbb{T}(\aleph)(t_l) - \mathbb{T}(\aleph)(t_k)| &= \left| \int_0^{t_l} \mathbb{Z}(z,\aleph(z))dz - \int_0^{t_k} \mathbb{Z}(z,\aleph(z))dz \right| \\
&\leq \int_0^{t_l} |\mathbb{Z}(z,\aleph(z))|dz - \int_0^{t_k} |\mathbb{Z}(z,\aleph(z))|dz \\
&\leq \left[\int_0^{t_l} (C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}) - \int_0^{t_k} (C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}) \right] \\
&\leq (C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}})[t_l - t_k].
\end{aligned} \tag{4.9}
$$

From the above equation, when $t_k \to t_l$, then

$$
|\mathbb{T}(\aleph)(t_l) - \mathbb{T}(\aleph)(t_k)| \to 0. \tag{4.10}
$$

So, $\mathbb{T}$ is equicontinuous on the interval $[0, t_1]$. Next, we take $t_k, t_l \in [t_1, T]$ in the Caputo sense as:

$$
\begin{aligned}
&|\mathbb{T}(\aleph)(t_l) - \mathbb{T}(\aleph)(t_k)| \\
=& \Big| \frac{1}{\Gamma(\vartheta)} \int_0^{t_l} (t_l - z)^{\vartheta-1} \mathbb{Z}(z,\aleph(z))dz \\
&- \frac{1}{\Gamma(\vartheta)} \int_0^{t_k} (t_k - z)^{\vartheta-1} \mathbb{Z}(z,\aleph(z))dz \Big| \\
\leq& \frac{1}{\Gamma(\vartheta)} \int_0^{t_k} [(t_k - z)^{\vartheta-1} - (t_l - z)^{\vartheta-1}] |\mathbb{Z}(z,\aleph(z))|dz \\
&+ \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{t_l} (t_l - z)^{\vartheta-1} |\mathbb{Z}(z,\aleph(z))|dz \\
\leq& \frac{1}{\Gamma(\vartheta)} \left[\int_0^{t_k} [(t_k - z)^{\vartheta-1} - (t_l - z)^{\vartheta-1}]dz + \int_{t_k}^{t_l} (t_l - z)^{\vartheta-1} dz \right] (C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}}) \\
\leq& \frac{(C_{\mathbb{Z}}|\aleph| + M_{\mathbb{Z}})}{\Gamma(\vartheta+1)} [t_l^{\vartheta} - t_k^{\vartheta} + 2(t_l - t_k)^{\vartheta}].
\end{aligned} \tag{4.11}
$$

If $t_k \to t_l$, then

$$
|\mathbb{T}(\aleph)(t_l) - \mathbb{T}(\aleph)(t_k)| \to 0. \tag{4.12}
$$

So, $\mathbb{T}$ is equicontinuous on the interval $[t_1, t_2]$. Hence, $\mathbb{T}$ is an equicontinuous mapping. According to the Arzel'a-Ascoli theorem, the operator $\mathbb{T}$ is completely and uniformly continuous and bounded. Therefore, by Schauder's fixed point theorem, piecewise problem (4.2) has at least one solution on subintervals.

Theorem 4.2. *If $\mathbb{T}$ is a contraction operator with condition (A_1), the considered piecewise model has a unique solution.*

Proof. As we have mapping $\mathbb{T} : B \to B$ piecewise continuous from the above proof, take $\aleph$ and $\bar{\aleph} \in B$ on $[0, t_1]$ in classical sense as

$$\begin{aligned} \|\mathbb{T}(\aleph) - \mathbb{T}(\bar{\aleph})\| &= \max_{t\in[0,t_1]} \left| \int_0^{t_1} \mathbb{Z}(z, \aleph)dz - \int_0^{t_1} \mathbb{Z}(z, \bar{\aleph})dz \right| \\ &\le t_1 \mathcal{L}_{\mathbb{Z}} \|\aleph - \bar{\aleph}\|. \end{aligned} \tag{4.13}$$

From (4.13), we have

$$\|\mathbb{T}(\aleph) - \mathbb{T}(\bar{\aleph})\| \le t_1 \mathcal{L}_{\mathbb{Z}} \|\aleph - \bar{\aleph}\|. \tag{4.14}$$

Therefore, $\mathbb{T}$ is a contraction. By Banach theorem, the considered problem has a unique solution on the given subinterval. Moreover, for the second subinterval $t \in [t_1, t_2]$, we have

$$\begin{aligned} \|\mathbb{T}(\aleph) - \mathbb{T}(\bar{\aleph})\| &= \max_{t\in[t_1,t_2]} \Big| \int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}(z, \aleph(z))dz \\ &\quad - \tfrac{1}{\Gamma(\vartheta)} \int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}(z, \bar{\aleph}(z))dz \Big| \\ &\le \tfrac{(t_2-t_1)^{\vartheta}}{\Gamma(\vartheta+1)} \mathcal{L}_{\mathbb{Z}} \|\aleph - \bar{\aleph}\|. \end{aligned} \tag{4.15}$$

From the above equation, we have

$$\|\mathbb{T}(\aleph) - \mathbb{T}(\bar{\aleph})\| \le \frac{(t_2-t_1)^{\vartheta}}{\Gamma(\vartheta+1)} \mathcal{L}_{\mathbb{Z}} \|\aleph - \bar{\aleph}\|. \tag{4.16}$$

As a result, T is a contraction. Hence, within the context of Banach contraction theorem, it can be concluded that the problem under consideration possesses a unique solution on given subinterval. Thus, by Eqs. (4.14) and (4.16), the suggested piecewise model has a unique solution on each subintervals.

5. Numerical technique

Here, we aim to devise a numerical scheme for the considered system. We adopt the Newton polynomial interpolation numerical scheme proposed by Atangana et al. [5]. The piecewise integral employing to Eq. (4.2) is delineated as follows:

$$\mathrm{S}_{\mathtt{I}}(t) = \begin{cases} \mathrm{S}_{\mathtt{I}}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_1(z, \mathrm{S}_{\mathtt{I}})dz, & 0 < t \le t_1, \\ \mathrm{S}_{\mathtt{I}}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_1(z, \mathrm{S}_{\mathtt{I}})dz, & t_1 < t \le t_2, \end{cases}$$

$$\mathrm{A}_{\mathtt{M}}(t) = \begin{cases} \mathrm{A}_{\mathtt{M}}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_2(z, \mathrm{A}_{\mathtt{M}})dz, & 0 < t \le t_1, \\ \mathrm{A}_{\mathtt{M}}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_2(z, \mathrm{A}_{\mathtt{M}})dz, & t_1 < t \le t_2, \end{cases}$$

$$\mathrm{E_{\mathtt{M}}}(t) = \begin{cases} \mathrm{E_{\mathtt{M}}}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_3(z, \mathrm{E_{\mathtt{M}}})dz, & 0 < t \le t_1, \\ \mathrm{E_{\mathtt{M}}}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_3(z, \mathrm{E_{\mathtt{M}}})dz, & t_1 < t \le t_2, \end{cases}$$

$$\mathrm{L_{\mathtt{M}}}(t) = \begin{cases} \mathrm{L_{\mathtt{M}}}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_4(z, \mathrm{L_{\mathtt{M}}})dz, & 0 < t \le t_1, \\ \mathrm{L_{\mathtt{M}}}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_4(z, \mathrm{L_{\mathtt{M}}})dz, & t_1 < t \le t_2, \end{cases}$$

$$\mathrm{P}(t) = \begin{cases} \mathrm{P}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_5(z, \mathrm{P})dz, & 0 < t \le t_1, \\ \mathrm{P}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_5(z, \mathrm{P})dz, & t_1 < t \le t_2, \end{cases}$$

$$\mathrm{D}(t) = \begin{cases} \mathrm{D}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_6(z, \mathrm{D})dz, & 0 < t \le t_1, \\ \mathrm{D}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_2} (t-z)^{\vartheta-1} \mathbb{Z}_6(z, \mathrm{D})dz, & t_1 < t \le t_2. \end{cases} \tag{5.1}$$

For the first equation of system (5.1), at $t = t_{n+1}$

$$\mathrm{S_{\mathtt{I}}}(t) = \begin{cases} \mathrm{S_{\mathtt{I}}}(t_0) + \displaystyle\int_0^{t_1} \mathbb{Z}_1(z, \mathrm{S_{\mathtt{I}}})dz, & 0 < t \le t_1, \\ \mathrm{S_{\mathtt{I}}}(t_1) + \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_{t_1}^{t_{n+1}} (t-z)^{\vartheta-1} \mathbb{Z}_1(z, \mathrm{S_{\mathtt{I}}})dz, & t_1 < t \le t_2. \end{cases}$$

Similarly, we can express the rest of equations of (5.1). Using the Newton polynomials and some calculation given in [5], we have

$$\mathrm{S_{\mathtt{I}}}(t_{n+1}) = \begin{cases} \mathrm{S_{\mathtt{I}}}(0) + \begin{cases} \displaystyle\sum_{k=2}^{n} \Bigg[\frac{5}{12}\mathbb{Z}_1(t_{k-2}, \mathrm{S}_{\mathtt{I}}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_1(t_{k-1}, \mathrm{S}_{\mathtt{I}}^{k-1})\Delta t \\ \qquad + \dfrac{23}{12}\mathbb{Z}_1(t_k, \mathrm{S}_{\mathtt{I}}^{k})\Delta t \Bigg], \end{cases} \\ \mathrm{S_{\mathtt{I}}}(t_1) + \begin{cases} \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_1(t_{k-2}, \mathrm{S}_{\mathtt{I}}^{k-2})]\Pi \\ + \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_1(t_{k-1}, \mathrm{S}_{\mathtt{I}}^{k-1}) - \mathbb{Z}_1(t_{k-2}, \mathrm{S}_{\mathtt{I}}^{k-2})]\beth \\ \dfrac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_1(t_k, \mathrm{S}_{\mathtt{I}}^{k}) - 2\mathbb{Z}_1(t_{k-1}, \mathrm{S}_{\mathtt{I}}^{k-1}) + \mathbb{Z}_1(t_{k-2}, \mathrm{S}_{\mathtt{I}}^{k-2})]\Lambda, \end{cases} \end{cases}$$

$$\mathrm{A}_{\mathtt{M}}(t_{n+1}) = \begin{cases} \mathrm{A}_{\mathtt{M}}(0) + \begin{cases} \displaystyle\sum_{k=2}^{n} \left[\frac{5}{12}\mathbb{Z}_2(t_{k-2}, \mathrm{A}_{\mathtt{M}}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_2(t_{k-1}, \mathrm{A}_{\mathtt{M}}^{k-1})\Delta t \right. \\ \qquad \left. + \frac{23}{12}\mathbb{Z}_2(t_k, \mathrm{A}_{\mathtt{M}}^{k})\Delta t \right], \end{cases} \\ \mathrm{A}_{\mathtt{M}}(t_1) + \begin{cases} \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_2(t_{k-2}, \mathrm{A}_{\mathtt{M}}^{k-2})]\Pi \\ + \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_2(t_{k-1}, \mathrm{A}_{\mathtt{M}}^{k-1}) - \mathbb{Z}_2(t_{k-2}, \mathrm{A}_{\mathtt{M}}^{k-2})]\beth \\ \dfrac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_2(t_k, \mathrm{A}_{\mathtt{M}}^{k}) - 2\mathbb{Z}_2(t_{k-1}, \mathrm{A}_{\mathtt{M}}^{k-1}) + \mathbb{Z}_2(t_{k-2}, \mathrm{A}_{\mathtt{M}}^{k-2})]\Lambda, \end{cases} \end{cases}$$

$$\mathrm{E}_{\mathtt{M}}(t_{n+1}) = \begin{cases} \mathrm{E}_{\mathtt{M}}(0) + \begin{cases} \displaystyle\sum_{k=2}^{n} \left[\frac{5}{12}\mathbb{Z}_3(t_{k-2}, \mathrm{E}_{\mathtt{M}}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_3(t_{k-1}, \mathrm{E}_{\mathtt{M}}^{k-1})\Delta t \right. \\ \qquad \left. + \frac{23}{12}\mathbb{Z}_3(t_k, \mathrm{E}_{\mathtt{M}}^{k})\Delta t \right], \end{cases} \\ \mathrm{E}_{\mathtt{M}}(t_1) + \begin{cases} \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_3(t_{k-2}, \mathrm{E}_{\mathtt{M}}^{k-2})]\Pi \\ + \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_3(t_{k-1}, \mathrm{E}_{\mathtt{M}}^{k-1}) - \mathbb{Z}_3(t_{k-2}, \mathrm{E}_{\mathtt{M}}^{k-2})]\beth \\ \dfrac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_3(t_k, \mathrm{E}_{\mathtt{M}}^{k}) - 2\mathbb{Z}_3(t_{k-1}, \mathrm{E}_{\mathtt{M}}^{k-1}) + \mathbb{Z}_3(t_{k-2}, \mathrm{E}_{\mathtt{M}}^{k-2})]\Lambda, \end{cases} \end{cases}$$

$$\mathrm{L}_{\mathtt{M}}(t_{n+1}) = \begin{cases} \mathrm{L}_{\mathtt{M}}(0) + \begin{cases} \displaystyle\sum_{k=2}^{n} \left[\frac{5}{12}\mathbb{Z}_4(t_{k-2}, \mathrm{L}_{\mathtt{M}}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_4(t_{k-1}, \mathrm{L}_{\mathtt{M}}^{k-1})\Delta t \right. \\ \qquad \left. + \frac{23}{12}\mathbb{Z}_4(t_k, \mathrm{L}_{\mathtt{M}}^{k})\Delta t \right], \end{cases} \\ \mathrm{L}_{\mathtt{M}}(t_1) + \begin{cases} \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_4(t_{k-2}, \mathrm{L}_{\mathtt{M}}^{k-2})]\Pi \\ + \dfrac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_4(t_{k-1}, \mathrm{L}_{\mathtt{M}}^{k-1}) - \mathbb{Z}_4(t_{k-2}, \mathrm{L}_{\mathtt{M}}^{k-2})]\beth \\ \dfrac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \displaystyle\sum_{k=i+3}^{n} [\mathbb{Z}_4(t_k, \mathrm{L}_{\mathtt{M}}^{k}) - 2\mathbb{Z}_4(t_{k-1}, \mathrm{L}_{\mathtt{M}}^{k-1}) + \mathbb{Z}_4(t_{k-2}, \mathrm{L}_{\mathtt{M}}^{k-2})]\Lambda, \end{cases} \end{cases}$$

$$
\mathrm{P}(t_{n+1}) = \begin{cases} \mathrm{P}(0) + \begin{cases} \sum_{k=2}^{n} \left[\frac{5}{12}\mathbb{Z}_5(t_{k-2}, \mathrm{P}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_5(t_{k-1}, \mathrm{P}^{k-1})\Delta t \right. \\ \left. \qquad + \frac{23}{12}\mathbb{Z}_5(t_k, \mathrm{P}^k)\Delta t \right], \end{cases} \\ \mathrm{P}(t_1) + \begin{cases} \frac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \sum_{k=i+3}^{n} [\mathbb{Z}_5(t_{k-2}, \mathrm{P}^{k-2})]\Pi \\ + \frac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \sum_{k=i+3}^{n} [\mathbb{Z}_5(t_{k-1}, \mathrm{P}^{k-1}) - \mathbb{Z}_5(t_{k-2}, \mathrm{P}^{k-2})]\sqsupset \\ \frac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \sum_{k=i+3}^{n} [\mathbb{Z}_5(t_k, \mathrm{P}^k) - 2\mathbb{Z}_5(t_{k-1}, \mathrm{P}^{k-1}) + \mathbb{Z}_5(t_{k-2}, \mathrm{P}^{k-2})]\Lambda, \end{cases} \end{cases}
$$

$$
\mathrm{D}(t_{n+1}) = \begin{cases} \mathrm{D}(0) + \begin{cases} \sum_{k=2}^{i} \left[\frac{5}{12}\mathbb{Z}_6(t_{k-2}, \mathrm{D}^{k-2})\Delta t - \frac{4}{3}\mathbb{Z}_6(t_{k-1}, \mathrm{D}^{k-1})\Delta t \right. \\ \left. \qquad + \frac{23}{12}\mathbb{Z}_6(t_k, \mathrm{D}^k)\Delta t \right], \end{cases} \\ \mathrm{D}(t_1) + \begin{cases} \frac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+1)} \sum_{k=i+3}^{n} [\mathbb{Z}_6(t_{k-2}, \mathrm{D}^{k-2})]\Pi \\ + \frac{(\Delta t)^{\vartheta-1}}{\Gamma(\vartheta+2)} \sum_{k=i+3}^{n} [\mathbb{Z}_6(t_{k-1}, \mathrm{D}^{k-1}) - \mathbb{Z}_6(t_{k-2}, \mathrm{D}^{k-2})]\sqsupset \\ \frac{\vartheta(\Delta t)^{\vartheta-1}}{2\Gamma(\vartheta+3)} \sum_{k=i+3}^{n} [\mathbb{Z}_6(t_k, \mathrm{D}^k) - 2\mathbb{Z}_6(t_{k-1}, \mathrm{D}^{k-1}) + \mathbb{Z}_6(t_{k-2}, \mathrm{D}^{k-2})]\Lambda, \end{cases} \end{cases} \tag{5.2}
$$

where

$$
\begin{aligned}
\Lambda &= \Big[(1+n-k)^{\vartheta}\Big(2(n-k)^2 + (n-k)(3\vartheta+10) + 2\vartheta^2 + 9\vartheta + 12\Big) \\
&\quad -(n-k)\Big(2(n-k)^2 + (n-k)(5\vartheta+10) + 6\vartheta^2 + 18\vartheta + 12\Big)\Big], \\
\sqsupset &= \left[(1+n-k)^{\vartheta}(3+2\vartheta+n-k) - (n-k)^{\vartheta}(n-k+3+3\vartheta)\right], \\
\Pi &= [(1+n-k)^{\vartheta} - (n-k)^{\vartheta}].
\end{aligned} \tag{5.3}
$$

6. Numerical results and discussion

This section presents the simulations of numerical approximations performed using the fractional and piecewise approaches, implemented through the Newton polynomial interpolation scheme [3, 5]. We take the parameter values from Table 1 for graphical results with some new values $\beta_3 = 0.3, \beta_4 = 0.03, \beta_5 = 0.4, \beta_{10} = 0.2$. We assume the initial conditions as $\mathrm{S_I}(0) = 20, \mathrm{A_M} = 10, \mathrm{E_M}(0) = 8, \mathrm{L_M}(0) = 6, \mathrm{P}(0) = 4$, and $\mathrm{D}(0) = 2$. The simulation results for the Caputo fractional model (3.2) at various choices of fractional order ϑ are represented in Fig. 1.

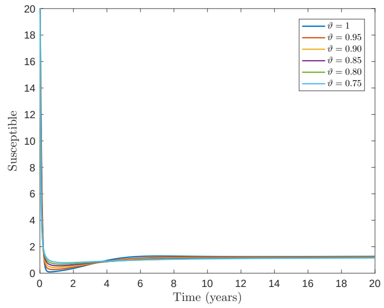
Fig. 1(a) illustrates how the fractional order influences the decline in the susceptible population. The decline occurs as individuals go for either arranged or love marriages. After 2 years, a slight increase is observed, likely due to individuals re-entering the susceptible class following divorce. This is followed by stabilization across different values of ϑ. Fig. 1(b) shows the number of individuals opting for arranged marriages, with the curves plotted for different fractional order ϑ. Initially, individuals in arranged marriages increase as more people transition from the susceptible class. However, after 1 year, the population begins to decline due to the influence of extramarital affairs, which lead to separation or divorce. The rate of decline varies with the choice of fractional order, where smaller values of ϑ result in a slower rate of decline, suggesting that memory effects and past interactions have a more prolonged impact on sustaining marriages in these cases.

Fig. 1(c) illustrates the dynamics of individuals engaged in extramarital affairs, with the population initially increasing as some individuals from both arranged and love marriages transition into this class. This initial rise suggests that a portion of the married population explores extramarital relationships over time, eventually reaching a peak. However, as these affairs contribute to marital discord, many individuals either transition to the separation or divorce class, causing the population in the extramarital affairs compartment to decline. The rate of this decline is influenced by the fractional order shown in the figure. Fig. 1(d) illustrates the dynamics of individuals who choose love marriages. These individuals may also engage in extramarital affairs, leading to a gradual decline in their numbers as they transition to the extramarital affairs or separation or divorce compartments. The decline occurs slowly at smaller values of ϑ. This indicates that individuals in love marriages are less affected by immediate consequences, allowing relationships to persist for a longer period before transitioning to separation or divorce. This behavior highlights the role of memory effects. Fig. 1(e) illustrates the individuals in the separation compartment. Various factors, such as family pressure or emotional dissatisfaction, may lead individuals in arranged marriages to engage in extramarital affairs, which can result in separation. Similarly, individuals in love marriages may also experience infidelity, ultimately contributing to marital separation. The population of individuals in the separated compartment initially rises to a peak before gradually declining over time. This peak is lower for smaller values of ϑ, indicating memory effects. Fig. 1(f) shows the dynamics of divorced individuals. Divorce cases increase as individuals in arranged or love marriages or separation lead to divorce. The number of individuals entering the divorced compartment rises steadily, leading to a peak in the divorced population. As time progresses, the number of new entrants into the divorced compartment declines due to the exhaustion of individuals in other compartments.

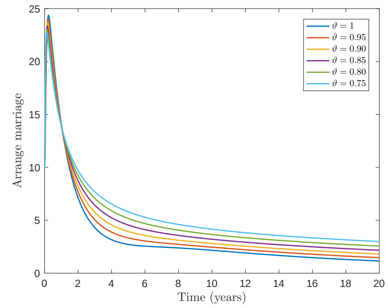
The graphical representation of the solution for the piecewise model (4.2) is provided in Figs. 2 and 3. In Fig. 2, the two subintervals are defined as $\mathcal{I}_1 = [0, 5]$ and $\mathcal{I}_2 = [5, 20]$, forming the entire interval $\mathcal{I} = [0, 20]$. In Fig. 3, the subintervals are $\mathcal{I}_1 = [0, 3]$ and $\mathcal{I}_2 = [3, 20]$, again covering the entire interval $[0, 20]$. In both cases, the first interval $\mathcal{I}_1$ is modeled using the classical approach, while the second interval $\mathcal{I}_2$ employs a fractional approach. The classical derivative, applied in the first subinterval, captures the standard rate of change, whereas the fractional derivative in the second subinterval incorporates memory effects. In Fig. 2, where the transition occurs at $t_{=}5$, there is a noticeable shift in the rate of change, which is influenced by the memory effects embedded in the fractional operator. Similarly, in Fig. 3, where the crossover occurs at $t_1 = 3$, the impact of the fractional derivative is felt sooner, resulting in a slight change in trajectory compared with Fig. 2. The earlier transition introduces a more immediate change in the system's behavior. Fewer occurrences of divorce are observed when $t_1 = 5$ compared with $t_1 = 3$. The difference in trajectories is evident when the Caputo fractional derivative is applied, as depicted in Fig. 1, and the piecewise Caputo fractional derivative, as illustrated in Figs. 2 and 3. When the outputs of these two approaches are analyzed, the variations in the timing and magnitude of peaks and troughs become clear. These differences highlight the distinct computational strategies inherent in each approach and their respective impacts on the model dynamics. Our analysis demonstrates that the piecewise fractional approach captures behavioral changes during different phases of the process and accounts for long-term memory effects observed in real-life scenarios more effectively.

7. Conclusion

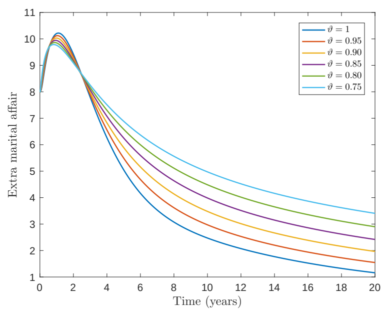
In this study, we examined a model addressing divorce among married couples resulting from extramarital affairs, utilizing both the Caputo fractional derivative and the piecewise Caputo derivative. The piecewise framework plays a crucial role in enhancing the numerical analysis of the model, as it effectively captures dynamic shifts in marital relationships during transitional phases. We scrutinized the existence of a unique solution for the model by applying fixed-point theory. The model solution was obtained using a piecewise Newton polynomial interpolation approximation. The results obtained were visually represented using MATLAB-R2022b. We have plotted the results for both the fractional model and the piecewise model, which combines the classical and fractional approaches. These plots allowed us to compare the trajectories of the model under both scenarios, highlighting the changes in behavior with and without the piecewise approach. This comparison provides valuable insight into how the integration of piecewise derivatives influences the model's dynamics. Our graphical results highlight the significant impact of the choice of fractional orders on the dynamics of the considered model. The results demonstrated that lower fractional orders slow down the transitions between compartments. In contrast, higher fractional orders expedite these transitions, resulting in quicker peaks and faster declines. The piecewise model effectively captures sudden changes in relationship dynamics, accurately reflecting shifts from one phase to another compared with the standard fractional model. In conclusion, the graphical representations emphasized the efficacy of the piecewise approach in capturing crossover behaviors at different times for different fractional orders. The variations in the model trajectory for different fractional order values illustrate the change in the underlying physical phenomenon. Thus, the role of fractional operators in this social issue is very critical. In the future, the study can be extended by considering stochastic piecewise derivatives.



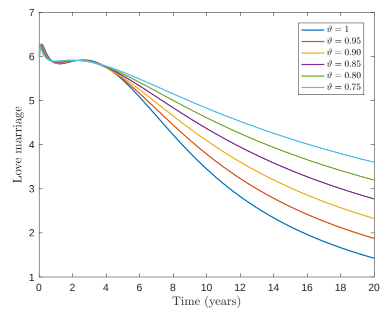
(a) Plot of susceptible individual



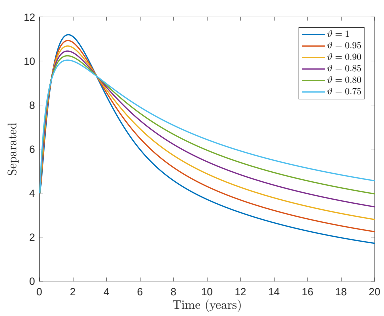
(b) Plot of arrange marriage



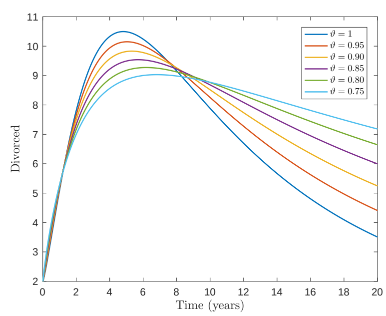
(c) Plot of extra marital affair



(d) Plot of love marriage

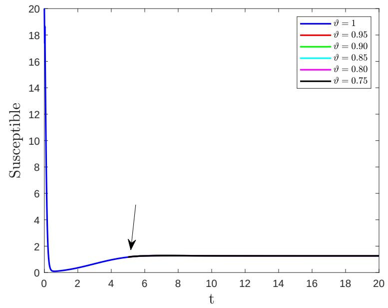


(e) Plot of separated individual

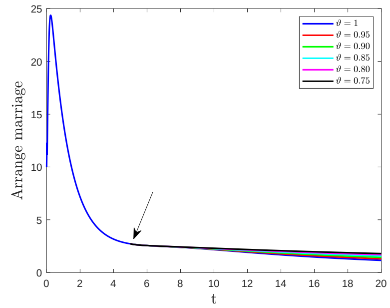


(f) Plot of divorced individual

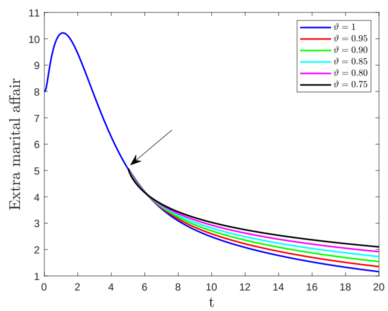
Figure 1. Simulation results of fractional model (3.2) for arbitrary values of ϑ.



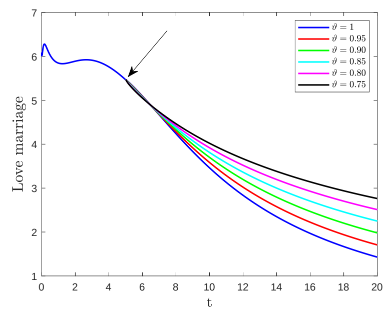
(a) Plot of susceptible individual



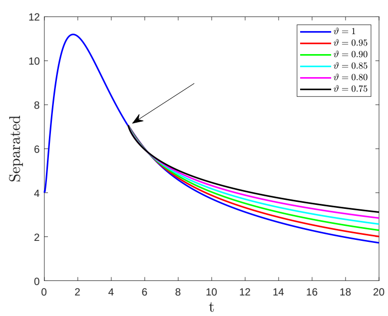
(b) Plot of arrange marriage



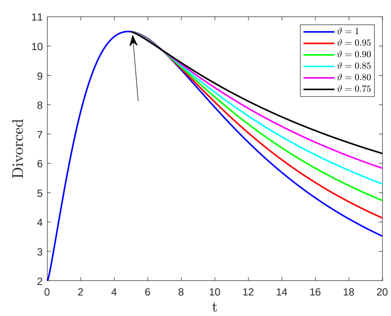
(c) Plot of extra marital affair



(d) Plot of love marriage

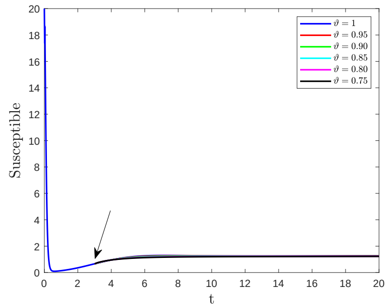


(e) Plot of separated individual

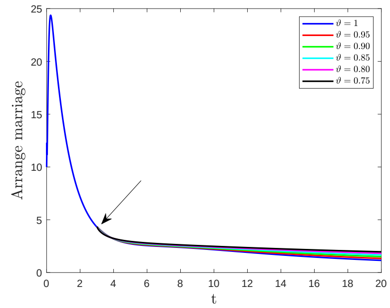


(f) Plot of divorced individual

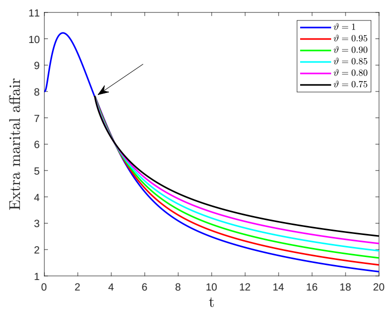
Figure 2. Simulation results of piecewise model (4.2) with $t_1 = 5$.



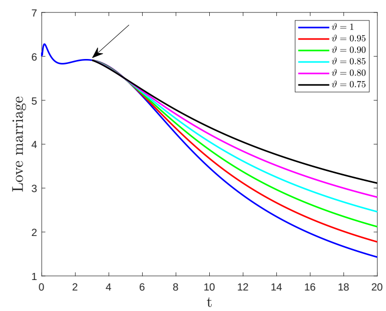
(a) Plot of susceptible individual



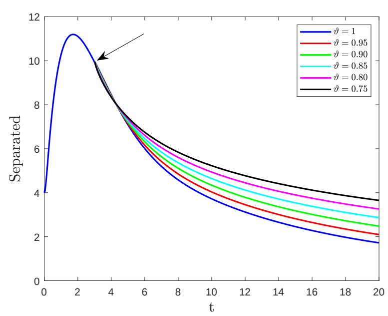
(b) Plot of arrange marriage



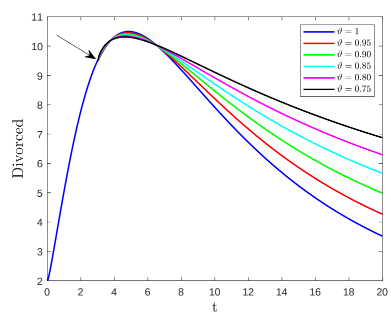
(c) Plot of extra marital affair



(d) Plot of love marriage



(e) Plot of separated individual



(f) Plot of divorced individual

Figure 3. Simulation results of piecewise model (4.2) with $t_1 = 3$.

Declaration of competing interest

The authors declare that they have no financial or non-financial competing interests. We confirm that the manuscript has been read and approved by all named authors.

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