Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1792–1810
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1792

On a Class of Eigenvalue Elliptic Systems in Fractional Sobolev Spaces with Variable Exponents

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Abstract In this work, we consider a class of eigenvalue elliptic systems involving the fractional $(p(x,.), q(x,.))$-Laplacian operators. Our main tools are based on Mountain Pass Theorem and Fountain Theorem.

Keywords Elliptic systems, generalized fractional Sobolev spaces, variational methods, variational principle, fractional $p(x,.)$-Laplacian, mountain pass Theorem, Fountain Theorem

MSC(2010) 35J48, 35J50, 46E35.

1. Introduction

The birth of fractional calculus was in a letter dated in 1695 when L'Hôpital wrote to Leibniz and asked him about the nth-derivative of the function $f(x) = x$, $\dfrac{D^n x}{Dx^n}$. What could be the result if $n = \dfrac{1}{2}$? Leibniz responded: "An apparent paradox from which, one day, useful consequences will be drawn."

After this first discussion between L'Hôpital and Leibniz, fractional calculus became above all for big mathematicians and can be traced back to L'Hôpital (1695), Wallis (1697), Euler (1738), Laplace (1812), Lacroix (1820), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Leibniz (1853), Grunwald (1867), Letnikov (1868) and many others. We refer the reader to [17,27,30] and the references therein for a detailed exposition about the history of the classical fractional calculus.

Thanks to these classical definitions, fractional calculus becomes a venerable branch of mathematics in the last century, and fractional operators give more development to the fields of Potential theory, Probability, Hyper singular integrals, Harmonic analysis, Functional analysis, Pseudo-differential operators, Semigroup theory etc. In particular, the fractional Laplacian operator $(-\Delta)^s$, $s \in (0,1)$ is a pseudo-differential operator which has various definitions in different fields: Fourier transform, distributional definition, Bochner's definition, Balakrishnan's definition,

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singular integral definition, quadratic definition, semigroup definition, definition as harmonic extension, and definition as the inverse od Riez potential. We refer to Kwasnicki [21] who collected all these definitions and established the equivalence between them.

In the last decade, great attention has been given to the study of nonlinear nonlocal case. More precisely, the problems involving the fractional p-Laplacian operator $(-\Delta)_p^s$. We refer to Di Nezza et al. [26] for a comprehensive introduction to the study of nonlocal problems. Basic properties, embedding theorems and regularity results are established. The paper opens the door for many mathematicians to deal with general problems in the field of partial differential equations. In the context of nonhomogeneous materials (such as electrorheological fluids and smart fluids), the use of Lebesgue and Sobolev spaces L^p and $W^{s,p}$ seems to be inadequate, which leads to the study of variable exponent functional spaces. The study of problems which involves the $p(.)$-Laplacian and the corresponding elliptic equations constitutes promising a domain of research. The interest in studying such problems was stimulated by their applications in many physical phenomena such as conservation laws, ultra-materials and water waves, optimization, population dynamics, soft thin films, mathematical finance, phases transitions, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flames propagation, ultra-relativistic limits of quantum mechanics, electrorheological fluids. We refer the reader to [7–9, 13, 26, 28, 29, 33] for details.

Now, what results can be recovered if the $p(.)$-Laplace operator is replaced by the fractional $p(x,.)$-Laplacian of the form $(-\Delta)^s_{p(x,.)}$? In 2017, Kaufmann et al. in [19] introduced the fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$. They established continuous and compact embedding theorems of these spaces into variable exponent Lebesgue spaces with the restriction $p(x,x) < q(x)$, and as applications, they also proved an existence result for nonlocal problems involving the fractional $p(x,y)$-Laplacian. In [6], Bahrouni et al. presented some further qualitative properties of both on this function space and the related $p(x,.)$-Laplacian operator $\mathcal{L}_{p(x,.)}$.

Under the restricted condition $p(x,x) < q(x)$, the space $W^{s,q(x),p(x,y)}(\Omega)$ is in fact not a generalization of the typical fractional Sobolev space $W^{s,p}(\Omega)$ with a constant exponent. However, Ho et al. [18] and Azroul et al. [2] provided some fundamental embeddings for the fractional Sobolev space with variable exponent to cover the case $p(x,x) = q(x)$ and their applications such as a priori bounds and multiplicity of solutions of the fractional $p(x,.)$-Laplacian problems. We also refer to [12] in which the authors proved a trace theorem in fractional Sobolev spaces with variable exponents. Indeed, fractional Sobolev spaces with variable exponents have been studied in depth during the last decade. We refer the interested reader to [2, 5, 6, 10, 22, 23, 25] and the references therein for some recent existence results for fractional type problems driven by a $p(x,.)$-Laplacian operator.

The purpose of this paper is to study the following fractional elliptic eigenvalue system

$$(\mathcal{P}_s)\begin{cases} (-\Delta)^s_{p(x,.)}(u)+|u|^{\bar{p}(x)-2}u=\lambda\alpha(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} \ in \quad \Omega, \\ \\ (-\Delta)^s_{q(x,.)}(v)+|u|^{\bar{q}(x)-2}u=\lambda\beta(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v \ in \quad \Omega, \\ \\ \qquad\qquad u=v=0 \qquad\qquad in\ \mathbb{R}^N\backslash\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$; $\lambda>0$ is a real number, $s\in(0,1)$ and $p,\,q:\,\overline{Q}\longrightarrow(1,+\infty)$, $\alpha,\,\beta:\,\overline{\Omega}\longrightarrow(1,+\infty)$ are continuous bounded functions such that

$$\begin{aligned} &1<p^-\leq p^+<\alpha^-\leq\alpha^+<2\alpha^+<p_s^{*-} \\ \text{and} \\ &1<q^-\leq q^+<\beta^-\leq\beta^+<2\beta^+<q_s^{*-}, \end{aligned} \tag{1.1}$$

where the critical fractional Sobolev exponent is given by

$$p_s^*(x)=\begin{cases} \dfrac{Np(x,x)}{N-sp(x,x)} \ if\ N>sp(x,x), \\ \quad +\infty \qquad if\ N\leqslant sp(x,x), \end{cases}$$

and $Q:=\mathbb{R}^{2N}\setminus(C\Omega\times C\Omega)$ and $C\Omega=\mathbb{R}^N\setminus\Omega$, while

$$p^-=\inf_{(x,y)\in\overline{Q}}p(x,y),\qquad p^+=\sup_{(x,y)\in\overline{Q}}p(x,y), \tag{1.2}$$

$$q^-=\inf_{(x,y)\in\overline{Q}}q(x,y),\qquad q^+=\sup_{(x,y)\in\overline{Q}}q(x,y) \tag{1.3}$$

and $\bar{p}(x)=p(x,x)$ for all $x\in\overline{\Omega}$.

The operator $(-\Delta)^s_{p(x)}$ is called the fractional $p(x,.)$-Laplacian, and defined as

$$(-\Delta)^s_{p(x,.)}u(x)=p.v.\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy \quad \text{for all } x\in\mathbb{R}^N, \tag{1.4}$$

where $p.v.$ represents the Cauchy's principal value. This nonlocal operator is a generalization of the fractional p-Laplacian operator $(-\Delta)^s_p$ in the constant exponent case, and it is the fractional version of the so-called $p(x)$-Laplacian operator which is given by $\Delta_{p(x)}u=div(|\nabla u|^{p(x)-2}\nabla u)$.

Eigenvalue problems have recently been studied in many papers. In our context we refer to Mihăilescu et al. [24] which considered the local case and established that any $\lambda>0$ sufficiently small is an eigenvalue of the following nonhomogeneous quasilinear problem

$$(\mathcal{P}_1)\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\lambda|u|^{q(x)-2}u, & \text{for } x\in\Omega, \\ u=0, & \text{for } x\in\partial\Omega. \end{cases}$$

More precisely, they showed that there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem $(\mathcal{P}_1)$. The proof is based on Ekeland's variational principle. Note that the authors considered the subcritical case i.e.

$$\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x).$$

Going further, Azroul et al. [2] generalized the above problem to include the fractional case. The authors considered the following eigenvalue problem

$$(\mathcal{P}_2)\begin{cases} \left(-\Delta_{p(x)}\right)^s u(x) + |u(x)|^{\bar{p}(x)-2}u(x) = \lambda |u(x)|^{q(x)-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

Using adequate variational techniques, mainly based on Ekeland's variational principle, they established the existence of a continuous family of eigenvalues lying in a neighborhood at the right of the origin. We also refer to Chung [10] who considered an eigenvalue problem for fractional $p(x,.)$-Laplacian equations with indefinite weight.

While scalar fractional equations have been extensively studied, systems remain less explored despite their relevance in modeling interactions between multiple components. A notable example lies in biology, where the dynamics of competing species in heterogeneous environments require models that account for variable interactions. Similarly, in physics, the diffusion of particles in anisotropic or nonhomogeneous materials can be effectively captured by fractional operators with variable exponents. These scenarios highlight the importance of extending the theory of fractional systems to better understand multi-component processes in such settings.

In [16], the authors have established stability inequalities for minimal solutions and they have examined regularity of the extremal solution of nonlinear nonlocal eigenvalue problem of the form

$$\begin{cases} \mathcal{L}u = \lambda F(u,v) & \text{in } \Omega, \\ \mathcal{L}v = \gamma G(u,v) & \text{in } \Omega, \\ u, v = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases}$$

with an integro-differential operator, including the fractional Laplacian, of the form

$$\mathcal{L}(u(x)) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \backslash B_\epsilon(x)} [u(x) - u(z)] J(z-x) dz,$$

when J is a nonnegative measurable even jump kernel.

In [11], the authors have studied the following fractional eigenvalue system

$$\begin{cases} (-\Delta_p)^r u = \lambda \frac{\alpha}{p} |u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ (-\Delta_p)^s u = \lambda \frac{\beta}{p} |u|^\alpha |v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \Omega^c = \mathbb{R}^N \backslash \Omega. \end{cases}$$

They showed that there is a sequence of eigenvalues λ_n such that $\lambda_n \to \infty$ as $n \to \infty$.

In our paper, we are concerned with the supercritical case and we aim to prove that any $\lambda > 0$ sufficiently small is an eigenvalue of the system $(\mathcal{P}_s)$ using Mountain Pass Theorem. Moreover, using Fountain Theorem, we show that $(\mathcal{P}_s)$ has infinitely many eigenvalues.

Our study makes several key contributions to the field. First, it extends eigenvalue analysis to fractional systems governed by the $(p(x,.), q(x,.))$-Laplacian, accounting for spatially dependent growth rates. Unlike systems with constant exponents, which assume uniform growth conditions, those with variable exponents reflect the heterogeneity found in many real-world scenarios. This adaptability is crucial for accurately modeling systems where local dynamics vary with spatial location, such as anisotropic diffusion or heterogeneous biological environments. Second, it leverages advanced techniques in fractional Sobolev spaces with variable exponents to address the challenges posed by nonlocality and coupling. Finally, it situates these eigenvalue problems within the broader context of spectral theory, demonstrating their relevance to multi-component and heterogeneous systems.

Our existence results are formulated as follows.

Theorem 1.1. *Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$ and let $s \in (0,1)$. Let $p, q : \overline{Q} \longrightarrow (1, +\infty)$ be two continuous variable exponents with $sp(x,y) < N$ and $sq(x,y) < N$ for all $(x,y) \in \overline{Q}$ satisfying* (1.1). *Then, there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of system $(\mathcal{P}_s)$.*

Theorem 1.2. *Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$ and let $s \in (0,1)$. Let $p, q : \overline{Q} \longrightarrow (1, +\infty)$ be two continuous variable exponents with $sp(x,y) < N$ and $sq(x,y) < N$ for all $(x,y) \in \overline{Q}$ satisfying* (1.1). *Then, system $(\mathcal{P}_s)$ has infinitely many eigenvalues.*

The rest of our paper is organized as follows. In section 2 we briefly review some notations and basic properties about Lebesgue and Sobolev spaces with variable exponents and their generalisation to the fractional case. In Section 3 we prove Theorem 1.1 by showing that the energy functional associated with the problem satisfies the Palais-Smale condition and two Mountain Pass geometric conditions. Finally, in Section 4 we prove Theorem 1.2 using Fountain Theorem which assures the existence of infinitely many eigenvalues of system $(\mathcal{P}_s)$.

2. Preliminaries and basic assumptions

2.1. Lebesgue spaces with variable exponents

Here we recall the definition and some important properties of the Lebesgue spaces with variable exponents. For more details regarding these spaces, one can refer to [15, 20] and the references therein.

Consider the set

$$C_+(\overline{\Omega}) = \left\{\gamma \in C(\overline{\Omega}) : \gamma(x) > 1, \forall x \in \overline{\Omega}\right\}.$$

For any $\gamma \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{\gamma(x)}(\Omega)$ as

$$L^{\gamma(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \;\; \text{measurable} : \int_\Omega |u(x)|^{\gamma(x)} dx < +\infty\right\}.$$

This space equipped with the *Luxemburg* norm

$$\|u\|_{\gamma(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{\gamma(x)} dx \leqslant 1\right\},$$

is a separable reflexive Banach space.

Let $\gamma' \in C_+(\overline{\Omega})$ be the conjugate exponent of γ, i.e. $\dfrac{1}{\gamma(x)} + \dfrac{1}{\gamma'(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1. *(Hölder inequality). If $u \in L^{\gamma(x)}(\Omega)$ and $v \in L^{\gamma'(x)}(\Omega)$, then*

$$\left|\int_\Omega uv dx\right| \leqslant \left(\frac{1}{\gamma^-} + \frac{1}{\gamma'^-}\right)\|u\|_{\gamma(x)}\|v\|_{\gamma'(x)} \leqslant 2\|u\|_{\gamma(x)}\|v\|_{\gamma'(x)}.$$

The modular of $L^{\gamma(x)}(\Omega)$ is defined by

$$\begin{aligned}\rho_{\gamma(.)} : L^{\gamma(x)}(\Omega) &\longrightarrow \mathbb{R}\\ u &\longrightarrow \rho_{\gamma(.)}(u) = \int_\Omega |u(x)|^{\gamma(x)} dx.\end{aligned}$$

Proposition 2.1. *[14, 20] Let $u \in L^{\gamma(x)}(\Omega)$. Then we have*

1. $\|u\|_{\gamma(x)} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho_{\gamma(.)}(u) < 1$ *(resp* $= 1, > 1$*).*
2. $\|u\|_{\gamma(x)} < 1 \Rightarrow \|u\|^{\gamma+}_{\gamma(x)} \leqslant \rho_{\gamma(.)}(u) \leqslant \|u\|^{\gamma-}_{\gamma(x)}$.
3. $\|u\|_{\gamma(x)} > 1 \Rightarrow \|u\|^{\gamma-}_{\gamma(x)} \leqslant \rho_{\gamma(.)}(u) \leqslant \|u\|^{\gamma+}_{\gamma(x)}$.

Proposition 2.2. *If $u, u_k \in L^{\gamma(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

1. $\lim\limits_{k\to+\infty} \|u_k - u\|_{\gamma(x)} = 0$.
2. $\lim\limits_{k\to+\infty} \rho_{\gamma(.)}(u_k - u) = 0$.
3. $u_k \longrightarrow u$ *in measure in* Ω *and* $\lim\limits_{k\to+\infty} \rho_{\gamma(.)}(u_k) = \rho_{\gamma(.)}(u)$.

Proposition 2.3. *[14] Let Ω be a bounded open subset of $\mathbb{R}^N$, $\gamma \in C(\bar{\Omega})$. Then $\left(L^{\gamma(x)}(\Omega), \|u\|_{\gamma(x)}\right)$ is a reflexive uniformly convex and separable Banach space.*

2.2. Fractional Sobolev spaces with variable exponents

In this part, we discuss the properties of the fractional Sobolev spaces with variable exponents. These spaces have been introduced for the first time in [19]. Also, in [2,12,18], the authors have established some important properties of these spaces. Let Ω be an open bounded set in $\mathbb{R}^N$ and $p(x, .)$ satisfying (1.1)-(1.2). For any $x \in \mathbb{R}^N$, we denote

$$\bar{p}(x) = p(x, x).$$

We define the usual fractional Sobolev space with variable exponent as:
$W = W^{s,p(x,y)}(\Omega)$

$$= \{u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < \infty,\ for\ some\ \mu > 0\},$$

which we endow with the Luxemberg norm

$$\|u\|_W = \|u\|_{\bar{p}(x)} + [u]_{s,p(x,y)},$$

where $[u]_{s,p(x,y)}$ is a Gagliardo semi-norm with variable exponent defined by

$$[u]_{s,p(x,y)} = \inf\left\{\mu > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \leqslant 1\right\}.$$

Then $(W; ||.||_W)$ is a separable reflexive Banach space.

For studying nonlocal problems involving the operator $(-\Delta)^s_{p(x,.)}$ with Dirichlet boundary datum via variational methods, we define another fractional type Sobolev spaces with variable exponents.

We set $Q = \mathbb{R}^{2n}\backslash(C\Omega \times C\Omega)$ and define the new fractional Sobolev space with variable exponent as

$$W^{s,p(x,y)}(Q) = \left\{\begin{array}{l} u : \mathbb{R}^n \to \mathbb{R} : u_{|\Omega} \in L^{\bar{p}(x)}(\Omega), \\ \displaystyle\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy < \infty,\ for\ some\ \mu > 0 \end{array}\right\}.$$

The space $W^{s,p(x,y)}(Q)$ is equipped with the norm

$$\|u\|_{s,p(x,y)} = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{W^{s,p(x,y)}(Q)},$$

where $[u]_{W^{s,p(x,y)}(Q)}$ is the seminorm

$$[u]_{W^{s,p(x,y)}(Q)} = \inf\left\{\mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \leq 1\right\}.$$

Then $(W^{s,p(x,y)}(Q), \|.\|_{s,p(x,y)})$ is a separable reflexive Banach space. The modular on $W^{s,p(x,y)}(Q)$ is the mapping $\rho_{p(.,.)} : W^{s,p(x,y)}(Q) \to \mathbb{R}$ defined as follows

$$\rho_{p(.,.)}(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx.$$

Next, let us denote by $X_0^{s,p(x,y)}$ the following linear subspace of $W^{s,p(x,y)}(Q)$

$$X_0^{s,p(x,y)} = \left\{u \in W^{s,p(x,y)}(Q) : u = 0\ a.e.\ x \in C\Omega\right\},$$

with the norm

$$\|u\|_{0,s,p(x,y)} = [u]_{X_0^{s,p(x,y)}} = \inf\left\{\mu > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \leq 1\right\}.$$

The space $(X_0^{s,p(x,y)}, \|u\|_{0,s,p(x,y)})$ is a separable reflexive Banach space (see [3, Lemma 2.3]). We define the modular $\rho^0_{p(.,.)} : X_0^{s,p(x,y)} \to \mathbb{R}$, by

$$\rho^0_{p(.,.)}(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

Consequently, $\|u\|_{\rho^0_{p(.,.)}} = \inf\left\{\mu > 0 : \rho^0_{p(.,.)}(\frac{u}{\mu}) \leq 1\right\} = \|u\|_{0,s,p(x,y)}$.

Similar to Propositions 2.1, $\rho^0_{p(.,.)}$ and $\|u\|_{0,s,p(x,y)}$ satisfy the following assertions.

Proposition 2.4. *([34], Lemma 2.1) Let $u \in X_0^{s,p(x,y)}$ and $\{u_k\} \subset X_0^{s,p(x,y)}$. We have*

1. $\|u\|_{0,s,p(x,y)} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho^0_{p(.,.)}(u) < 1$ *(resp* $= 1, > 1$*).*
2. $\|u\|_{0,s,p(x,y)} < 1 \Rightarrow \|u\|^{p^+}_{0,s,p(x,y)} \leqslant \rho^0_{p(.,.)}(u) \leqslant \|u\|^{p^-}_{0,s,p(x,y)}$.
3. $\|u\|_{0,s,p(x,y)} > 1 \Rightarrow \|u\|^{p^-}_{0,s,p(x,y)} \leqslant \rho^0_{p(.,.)}(u) \leqslant \|u\|^{p^+}_{0,s,p(x,y)}$.
4. $\|u_k\|_{0,s,p(x,y)} \longrightarrow 0$ *(resp* ∞*)* $\Leftrightarrow \rho^0_{p(.,.)}(u_k) \longrightarrow 0$ *(resp* ∞*).*

Theorem 2.1. *[2] Let Ω be a Lipschitz bounded domain in $\mathbb{R}^n$ and let $s \in (0,1)$. Let $p(x,.)$ satisfy* (1.1) *with $sp^+ < N$. If $r : \Omega \to (1, +\infty)$ is a continuous variable exponent such that*

$$1 < r^- < r(x) < p^*_s(x) \ for\ all\ x \in \bar{\Omega},$$

then, there exists a constant $C = C(N, s, p, r, \Omega) > 0$ such that for any $u \in W^{s,p(x,y)}(Q)$,

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{W^{s,p(x,y)}(Q)}.$$

That is, the space $W^{s,p(x,y)}(Q)$ is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

Remark 2.1. Theorem 2.1 remains true if we replace $W^{s,p(x,y)}(Q)$ by $X_0^{s,p(x,y)}$.

We define $E = X_0^{s,p(x,y)} \times X_0^{s,q(x,y)}$ as the solution space corresponding to our system $(\mathcal{P}_s)$, equipped with the norm $\|(u,v)\| = \|u\|_{0,s,p(x,y)} + \|v\|_{0,s,q(x,y)}$.

Under the conditions on $p(x,.)$ and $q(x,.)$, the product space $(E, \|(u,v)\|)$ is reflexive, separable, and a Banach space because these properties are inherited from the individual spaces $X_0^{s,p(x,y)}$ and $X_0^{s,q(x,y)}$. Reflexivity is guaranteed, as it ensures the boundedness and weak compactness properties necessary for reflexivity. Separability follows from the density of smooth functions in each Sobolev space with variable exponents, which implies that the product of the two separable spaces remains separable. Finally, the Banach property holds because both $X_0^{s,p(x,y)}$ and $X_0^{s,q(x,y)}$ are complete, and the norm on the product space $\|(u,v)\|$ is derived naturally, preserving completeness.

Now we recall the Mountain Pass and the Fountain Theorems which we use to prove our main results.

Theorem 2.2. *[1] (Mountain Pass Theorem). Let $\mathcal{J}$ be a functional of class C^1 on a Banach space X. Suppose that $\mathcal{J}$ satisfies the Palais-Smale condition and such that:*

- $\mathcal{J}(0) = 0$ *;* $\exists \rho, \alpha > 0 : \|u\|_x = \rho \Longrightarrow \mathcal{J}(u) \geq \alpha$*;*
- $\exists u_1 \in X : ||u_1||_X \geq \rho$ *and* $\mathcal{J}(u_1) < \alpha$.

Let $P = \{p \in C^0([0,1], X), p(0) = 0, p(1) = u_1\}$ be the set of paths from 0 to u_1 . Then $\beta = \inf\limits_{p \in P} \sup\limits_{u \in p} \mathcal{J}(u) \geq \alpha$ is a critical value of $\mathcal{J}$.

Theorem 2.3. *[32] (Fountain Theorem). Let X be a Banach space with the norm $\|.\|_X$ and let X_j be a sequence of subspaces of X with $dim X_j < \infty$ for each $j \in \mathbb{N}$. Further, $X = \overline{\bigoplus\limits_{j \in \mathbb{N}} X_j}$, the closure of the direct sum of all X_j. Set*

$\mathbb{Y}_k = \oplus_{j=1}^{k} X_j$, $\mathbb{Z}_k = \overline{\oplus_{j=k}^{\infty} X_j}$. *Assume that* $\Psi \in C^1(X, \mathbb{R})$ *satisfies the (PS) condition and* $\Psi(-u) = \Psi(u)$. *For every* $k \in \mathbb{N}$, *suppose that there exist* $R_k > r_k > 0$ *such that*

(A1) $\inf\limits_{\substack{u\in\mathbb{Z}_k \\ \|u\|_X = r_k}} \Psi(u) \longrightarrow +\infty$ *as* $k \to \infty$.

(A2) $\max\limits_{\substack{u\in\mathbb{Y}_k \\ \|u\|_X = R_k}} \Psi(u) \leqslant 0$.

Then Ψ *has an unbounded sequence of critical values.*

Remark 2.2. Since X is a separable and reflexive space, there exist $\{e_i\}_{i=1}^{\infty} \subset X$ and $\{f_i\}_{i=1}^{\infty} \subset X^*$ such that

$$f_i(e_j) = \delta_{i,j} = \begin{cases} 1 \; if \; i = j, \\ 0 \; if \; i \neq j, \end{cases}$$

and $X = \overline{span}\{e_i, \; i = 1, 2, ...\}$ and $X^* = \overline{span}\{f_i, \; i = 1, 2, ...\}$. For $k = 1, 2, ...$, we define

$$X_i = span\{e_i\}, \; \mathbb{Y}_k = \bigoplus_{i=0}^{k} X_i, \text{ and } \mathbb{Z}_k = \bigoplus_{i=k}^{\infty} X_i.$$

Lemma 2.2. *(see, [31]) Let* $r \in C_+(\mathbb{R}^N)$ *such that* $1 < r^- \leqslant r(x) \leqslant r^+ < \min\{p_s^*(x), q_s^*(x)\} \;\; \forall x \in \mathbb{R}^N$. *For* $k = 1, 2, ...,$ *set*

$$\eta_k = \sup_{\substack{u\in\mathbb{Z}_k \\ \|u\|_X \leqslant 1}} \int_{\mathbb{R}^N} |u|^{r(x)} \, dx.$$

Then $\eta_k \longrightarrow 0$ *as* $k \to +\infty$.

3. Existence of eigenvalues for system $(\mathcal{P}_s)$

Definition 3.1. We say that $(u, v) \in E$ is a weak solution of $(\mathcal{P}_s)$ if

$$\begin{aligned} &\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega |u(x)|^{\bar{p}(x)-2} u(x)\varphi(x) dx \\ &+ \int_Q \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega |v(x)|^{\bar{q}(x)-2} v(x)\psi(x) dx \\ &-\lambda \left(\int_\Omega \alpha(x)|u(x)|^{\alpha(x)-2}u(x)|v(x)|^{\beta(x)}\varphi(x)dx \right. \\ &\left. + \int_\Omega \beta(x)|u(x)|^{\alpha(x)}|v(x)|^{\beta(x)-2}v(x)\psi(x)dx \right) \\ &= 0, \end{aligned} \tag{3.1}$$

for all $(\varphi, \psi) \in E$.

It is clear that problem $(\mathcal{P}_s)$ has a variational structure. The energy functional corresponding to problem $(\mathcal{P}_s)$ is defined as $J_\lambda : E \longrightarrow \mathbb{R}$

$$\begin{aligned} J_\lambda(u,v) = & \int_Q \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega \frac{1}{\bar{p}(x)} |u(x)|^{\bar{p}(x)} dx \\ & + \int_Q \frac{1}{q(x,y)} \frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega \frac{1}{\bar{q}(x)} |v(x)|^{\bar{q}(x)} dx \\ & -\lambda \int_\Omega |u(x)|^{\alpha(x)} |v(x)|^{\beta(x)} dx. \end{aligned}$$

By a direct calculation we have that $J_\lambda \in C^1(E, \mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} < J'_\lambda(u,v), (\varphi,\psi) >= & \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}} dxdy \\ & + \int_Q \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}} dxdy \\ & + \int_\Omega |u(x)|^{\bar{p}(x)-2} u(x)\varphi(x) dx + \int_\Omega |v(x)|^{\bar{q}(x)-2} v(x)\psi(x) dx \\ & -\lambda \int_\Omega \alpha(x) |u(x)|^{\alpha(x)-2} u(x) |v(x)|^{\beta(x)} \varphi(x) dx \\ & -\lambda \int_\Omega \beta(x) |u(x)|^{\alpha(x)} |v(x)|^{\beta(x)-2} v(x)\psi(x) dx \end{aligned}$$

for any $(\varphi, \psi) \in E$.

3.1. Some important lemmas

In the following lemma we prove that the energy functional J_λ satisfies the first geometrical condition of the Mountain Pass Theorem.

Lemma 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p, q : \bar{Q} \longrightarrow (1, +\infty)$ be two continuous variable exponents with $sp(x,y) < N$ and $sq(x,y) < N$ for all $(x,y) \in \bar{Q}$ satisfying (1.1). Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $R, k > 0$ such that $J_\lambda(u,v) \geq k > 0$ for any $(u,v) \in E$ with $\|(u,v)\| = R$.*

Proof. Let $\overline{\alpha}, \overline{\beta} \in L^\infty(\Omega)$ be continuous such that

$$\frac{\alpha(x)+\overline{\alpha}(x)}{p_s^*(x)} + \frac{\beta(x)+\overline{\beta}(x)}{q_s^*(x)} = 1. \tag{3.2}$$

Put $p_1(x) = \dfrac{p_s^*(x)}{\alpha(x)+\overline{\alpha}(x)}$ and $q_1(x) = \dfrac{q_s^*(x)}{\beta(x)+\overline{\beta}(x)}$. Since $\alpha(x)p_1(x) < p_s^*(x)$ and $\beta(x)q_1(x) < q_s^*(x)$ for all $x \in \overline{\Omega}$, by Theorem 2.1, there exist $c_1, c_2 > 0$ such that

$$\|u\|_{\alpha(x)p_1(x)} \leqslant c_1 \|u\|_{0,s,p(x,.)} \text{ and } \|v\|_{\beta(x)q_1(x)} \leqslant c_2 \|v\|_{0,s,q(x,.)}. \tag{3.3}$$

Fix $R_1,\ R_2 \in (0, \frac{1}{2})$ such that $R_1 < \dfrac{1}{2c_1}$ and $R_2 < \dfrac{1}{2c_2}$. Then, by relation (3.3), we have

$$\|u\|_{\alpha(x)p_1(x)} < \frac{1}{2} \text{ for all } u \in X_0^{s,p(x,y)} \text{ with } \|u\|_{0,s,p(x,.)} = R_1, \tag{3.4}$$

and

$$\|v\|_{\beta(x)q_1(x)} < \frac{1}{2} \text{ for all } v \in X_0^{s,q(x,y)} \text{ with } \|v\|_{0,s,q(x,.)} = R_2. \tag{3.5}$$

Using Proposition 2.1 and combining relations (3.3)-(3.5), we get

$$\int_\Omega |u(x)|^{\alpha(x)p_1(x)} \leqslant c_1^{\alpha^- p_1^-} \|u\|_{0,s,p(x,.)}^{\alpha^- p_1^-} \leqslant c_1^{\alpha^-} \|u\|_{0,s,p(x,.)}^{\alpha^-} \tag{3.6}$$

for all $u \in X_0^{s,p(x,y)}$ with $\|u\|_{0,s,p(x,.)} = R_1$, and

$$\int_\Omega |v(x)|^{\beta(x)q_1(x)} \leqslant c_2^{\beta^- q_1^-} \|v\|_{0,s,q(x,.)}^{\beta^- q_1^-} \leqslant c_2^{\beta^-} \|v\|_{0,s,q(x,.)}^{\beta^-} \tag{3.7}$$

for all $v \in X_0^{s,q(x,y)}$ with $\|v\|_{0,s,q(x,.)} = R_2$.

Using Young's inequality and Proposition 2.1, for all $(u,v) \in E$ with $\|(u,v)\| = R = R_1 + R_2 < 1$ there exists $c = \max\{c_1, c_2\}$ such that

$$\begin{aligned}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx &\leqslant \int_\Omega \frac{1}{p_1(x)}|u|^{\alpha(x)p_1(x)}dx + \int_\Omega \frac{1}{q_1(x)}|v|^{\beta(x)q_1(x)}dx \\ &\leqslant c\big(\|u\|_{0,s,p(x,y)}^{\alpha^-} + \|v\|_{0,s,q(x,y)}^{\beta^-}\big).\end{aligned}$$

Then, by Proposition 2.4, for any $(u,v) \in E$ with $\|(u,v)\| = R$, we get

$$\begin{aligned}J_\lambda(u,v) &\geqslant \frac{1}{p^+}\|u\|_{0,s,p(x,y)}^{p^+} + \frac{1}{q^+}\|v\|_{0,s,q(x,y)}^{q^+} - c\lambda\left(\|u\|_{0,s,p(x,y)}^{\alpha^-} + \|v\|_{0,s,q(x,y)}^{\beta^-}\right) \\ &\geqslant \min\left(\frac{1}{p^+}, \frac{1}{q^+}\right)\|(u,v)\|^{\max(p^+,q^+)} - c\lambda\|(u,v)\|^{\min(\alpha^-,\beta^-)} \\ &\geqslant R^{\min(\alpha^-,\beta^-)}\left[\min\left(\frac{1}{p^+}, \frac{1}{q^+}\right)R^{\max(p^+,q^+)-\min(\alpha^-,\beta^-)} - c\lambda\right].\end{aligned}$$

By the above inequality, we can choose λ^* in order to

$$MR^{\max(p^+,q^+)-\min(\alpha^-,\beta^-)} - c\lambda > 0,$$

where $M = \min\left(\frac{1}{p^+}, \frac{1}{q^+}\right)$. Therefore, if

$$\lambda^* = \frac{M}{2c}R^{\max(p^+,q^+)-\min(\alpha^-,\beta^-)}, \tag{3.8}$$

then, for any $\lambda \in (0, \lambda^*)$ and any $(u,v) \in E$ with $\|(u,v)\| = R$, there exists $k > 0$ such that $J(u,v) \geq k > 0$. □

The following lemma shows that the functional J_λ satisfies the second geometrical condition of the Mountain Pass Theorem.

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $s \in (0,1)$. Let $p, q : \bar{Q} \longrightarrow (1, +\infty)$ be two continuous variable exponents with $sp(x,y) < N$ and $sq(x,y) < N$ for all $(x,y) \in \bar{Q}$ satisfying* (1.1). *Then there exists $(u,v) \in E \setminus \{0,0\}$ such that for $\|(u,v)\| > R$ we have $J_\lambda(u,v) \leqslant 0$.*

Proof. Let $(\varphi, \phi) \in E \setminus (\{0,0\})$ and $t > 1$,

$$\begin{aligned} J_\lambda(t\varphi, t\phi) &= \int_Q \frac{t^{p(x,y)}}{p(x,y)} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega \frac{t^{\bar{p}(x)}}{\bar{p}(x)} |\varphi(x)|^{\bar{p}(x)} dx \\ &\quad + \int_Q \frac{t^{q(x,y)}}{q(x,y)} \frac{|\phi(x)-\phi(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega \frac{t^{\bar{q}(x)}}{\bar{q}(x)} |\phi(x)|^{\bar{q}(x)} dx \\ &\quad - \lambda \int_\Omega t^{\alpha(x)+\beta(x)} |\varphi(x)|^{\alpha(x)} |\phi(x)|^{\beta(x)} dx \\ &\leqslant \frac{t^{p^+}}{p^-} \left[\int_Q \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega |\varphi(x)|^{\bar{p}(x)} dx \right] \\ &\quad + \frac{t^{q^+}}{q^-} \left[\int_Q \frac{|\phi(x)-\phi(y)|^{p(x,y)}}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega |\phi(x)|^{\bar{q}(x)} dx \right] \\ &\quad - \lambda t^{\alpha^-+\beta^-} \int_\Omega |\varphi(x)|^{\alpha(x)} |\phi(x)|^{\beta(x)} dx \\ &\leqslant \frac{t^{p^+}}{p^-} \rho^0_{p(x,y)}(\varphi) + \frac{t^{q^+}}{q^-} \rho^0_{q(x,y)}(\phi) \\ &\quad - \lambda t^{\alpha^-+\beta^-} \int_\Omega |\varphi(x)|^{\alpha(x)} |\phi(x)|^{\beta(x)} dx. \end{aligned}$$

Since $\alpha^- + \beta^- \geqslant p^+$ and $\alpha^- + \beta^- \geqslant q^+$, we get

$$J_\lambda(tu, tv) \longrightarrow -\infty \qquad as \qquad t \longrightarrow +\infty.$$

Hence, there exist $t_0 > 0$ and $(u,v) = (t_0\varphi, t_0\phi) \in E$ such that $\|(u,v)\| > R$ and $J_\lambda((u,v)) < 0$. □

Lemma 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $s \in (0,1)$. Let $p, q : \bar{Q} \longrightarrow (1, +\infty)$ be two continuous variable exponents with $sp(x,y) < N$ and $sq(x,y) < N$ for all $(x,y) \in \bar{Q}$ satisfying* (1.1)*. Then $J_\lambda^{'}$ is of (S_+) type, i.e. if $(u_n, v_n) \rightharpoonup (u,v)$ in E and $\limsup_{n\to+\infty} \left\langle J_\lambda^{'}(u_n,v_n) - J_\lambda^{'}(u,v), (u_n - u, v_n - v) \right\rangle \leqslant 0$, this implies that $(u_n, v_n) \longrightarrow (u,v)$ in E.*

Proof. The proof is similar to [4, Lemma 3.5]. □

3.2. Proof of Theorem 1.1

In Lemma 3.1 and Lemma 3.2 we have shown that the functional energy satisfies the first and the second geometric condition of the Mountain Pass Theorem. In order to complete the proof of the theorem, we prove that J_λ satisfies the Palais-Smale condition on E. Indeed, since by Lemma 3.3, J'_λ is of type (S^+), to show that J_λ satisfies the Palais-Smale condition on E, it is enough to verify that any Palais-Smale sequence $\{(u_n, v_n)\}$ is bounded.

Let $\{(u_n, v_n)\}$ be a Palais-Smale sequence for the functional J_λ. Then, there exists a constant $C > 0$ such that $J_\lambda(u_n, v_n) \leqslant C$ and $\lim_{n\to+\infty} ||J'_\lambda(u_n, v_n)||_* \longrightarrow 0$.

Arguing by contradiction, we suppose that $\{(u_n, v_n)\}$ is unbounded in E. Without loss of generality, we can assume that $\|u_n\|_{s,p(x,y)} > \|v_n\|_{s,q(x,y)}$ for all $n \geq 1$.

By the definition of J_λ, we have

$$\begin{aligned}
c \geqslant J_\lambda(u_n, v_n) &= \int_Q \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega \frac{1}{\bar{p}(x)}|u_n|^{\bar{p}(x)}dx \\
&\quad + \int_Q \frac{1}{q(x,y)} \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega \frac{1}{\bar{q}(x)}|v_n|^{\bar{q}(x)}dx \\
&\quad - \lambda \int_\Omega |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx \\
&\geq \int_Q \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega \frac{1}{\bar{p}(x)}|u_n|^{\bar{p}(x)}dx \\
&\quad + \int_Q \frac{1}{q(x,y)} \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega \frac{1}{\bar{q}(x)}|v_n|^{\bar{q}(x)}dx \\
&\quad - \lambda \int_\Omega \frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)}|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx \\
&\geq \frac{1}{p^+}\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{p^+}\int_\Omega |u_n|^{\bar{p}(x)}dx \\
&\quad + \frac{1}{q^+}\int_Q \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \frac{1}{q^+}\int_\Omega |v_n|^{\bar{q}(x)}dx \\
&\quad - \frac{\lambda}{\alpha^- + \beta^-}\int_\Omega (\alpha(x)+\beta(x))\, |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-+\beta^-}\right)\left[\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n|^{\bar{p}(x)}dx\right] \\
&\quad + \left(\frac{1}{q^+} - \frac{1}{\alpha^-+\beta^-}\right)\left[\int_Q \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_n|^{\bar{q}(x)}dx\right] \\
&\quad + \frac{1}{\alpha^-+\beta^-}\left[D_1J_\lambda(u_n,v_n)(u_n) + D_2J_\lambda(u_n,v_n)(v_n)\right] \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^-+\beta^-}\right)\left[\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u_n|^{\bar{p}(x)}dx\right] \\
&\quad + \left(\frac{1}{q^+} - \frac{1}{\alpha^-+\beta^-}\right)\left[\int_Q \frac{|v_n(x)-v_n(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy + \int_\Omega |v_n|^{\bar{q}(x)}dx\right] \\
&\quad - \frac{1}{\alpha^-+\beta^-}\|D_1J_\lambda(u_n,v_n)\|_{\left(X_0^{s,p(x,y)}\right)^*}\|u_n\|_{s,p(x,y)} \\
&\quad - \frac{1}{\alpha^-+\beta^-}\|D_2J_\lambda(u_n,v_n)\|_{\left(X_0^{s,q(x,y)}\right)^*}\|v_n\|_{s,q(x,y)},
\end{aligned}$$

where

$$\begin{aligned}
D_1J_\lambda(u_n,v_n)(\varphi) &= \int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \int_\Omega |u_n(x)|^{\bar{p}(x)-2}u_n(x)\varphi(x)dx \\
&\quad - \lambda\int_\Omega \alpha(x)|u_n(x)|^{\alpha(x)-2}u_n(x)|v_n(x)|^{\beta(x)}\varphi(x)dx,
\end{aligned}$$

and

$$D_2J_\lambda(u_n,v_n)(\psi) = \int_Q \frac{|v_n(x)-v_n(y)|^{q(x,y)-2}(v_n(x)-v_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy$$

$$+ \int_\Omega |v_n(x)|^{\bar{q}(x)-2} v_n(x)\psi(x)dx$$
$$- \lambda \int_\Omega \beta(x)|u_n(x)|^{\alpha(x)}|v_n(x)|^{\beta(x)-2} v_n(x)\psi(x)dx.$$

Therefore,

$$c \geq \left(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\right)\|u_n\|^{p-}_{s,p(x,y)} - \frac{1}{\alpha^- + \beta^-}\left(\|D_1 J_\lambda(u_n, v_n)\|_{\left(X_0^{s,p(x,y)}\right)^*}\right.$$
$$\left. + \|D_2 J_\lambda(u_n, v_n)\|_{\left(X_0^{s,q(x,y)}\right)^*}\right)\|u_n\|_{s,p(x,y)}.$$

In view of $p^- > 1$, $\alpha^- + \beta^- > p^+$ and the boundedness of $||J'_\lambda(u_n, v_n)||_*$, the above inequality cannot hold if $n \to +\infty$. Therefore, $\{(u_n, v_n)\}$ is bounded in E and J_λ satisfies the Palais-Smale condition.

Therefore, by the Mountain Pass Theorem, we conclude that, for any $\lambda \in (0, \lambda^*)$, the functional J_λ has at least one nontrivial critical point (u, v) in E. Hence, any $\lambda \in (0, \lambda^*)$ is an eigenvalue of system $(\mathcal{P}_s)$.

4. Existence of infinitely many eigenvalues of system $(\mathcal{P}_s)$

In this section, we prove the existence of infinitely many eigenvalues of system $(\mathcal{P}_s)$. Our strategy consists in applying the Fountain Theorem of Bartdch.

4.1. Proof of Theorem 1.2

Since E is a reflexive and separable Banach space, we can define two spaces Y_k and Z_k as in Remark 2.2. In the above section, we have proved that J_λ satisfies the Palais-Smale condition. Obviously, $J_\lambda(-(u, v)) = J_\lambda(u, v)$. Now, we will study the geometrical conditions of the functional J_λ.

- **Claim 1:** First geometric condition $(A1)$ of the Fountain Theorem.
  Let $(u, v) \in Z_k$ with $\|(u, v)\| > 1$. Using Young's inequality and Proposition 2.4, we have

  $$J_\lambda(u, v) \geqslant \frac{1}{p^+}\|u\|^{p^-}_{0,s,p(x,y)} + \frac{1}{q^+}\|v\|^{q^-}_{0,s,q(x,y)} - c_3\lambda\left(\|u\|^{2\alpha^+}_{2\alpha(x)} + \|v\|^{2\beta^+}_{2\beta(x)} + |\Omega|\right).$$

  Set $a = \max\{2\alpha^+, 2\beta^+\}$, $b = \min\{2\alpha^+, 2\beta^+\}$ and define

  $$\eta_k = \sup\left\{\int_\Omega |u|^{2\alpha(x)}dx, (u, v) \in \mathbb{Z}_k, \|(u, v)\| \leq 1\right\}$$

  and

  $$\zeta_k = \sup\left\{\int_\Omega |v|^{2\beta(x)}dx, (u, v) \in \mathbb{Z}_k, \|(u, v)\| \leq 1\right\}.$$

Then, we get

$$\begin{aligned} J_\lambda(u,v) \geqslant & \frac{1}{\max(p^+,q^+)}\|(u,v)\|^{\min(p^-,q^-)} - c_4\lambda\left(\eta_k\|(u,v)\|\right)^{2\alpha^+} \\ & -c_4\lambda\left(\zeta_k\|(u,v)\|\right)^{2\beta^+} - c_4\lambda\,|\Omega| \\ \geqslant & \frac{1}{\max(p^+,q^+)}\|(u,v)\|^{\min(p^-,q^-)} - c_5\lambda\kappa_k^b\|(u,v)\|^a - c_5\lambda\,|\Omega|\,, \end{aligned}$$

where $\kappa_k = \eta_k + \zeta_k \longrightarrow 0$ with the same argument in Lemma 2.2. At this stage, we fix

$$r_k = \left(\frac{1}{2\max(p^+,q^+)c_5\lambda\kappa_k^b}\right)^{\frac{1}{a-\min(p^-,q^-)}}.$$

It is easy to see that $r_k \longrightarrow +\infty$ as $k \longrightarrow \infty$. Thanks to Lemma 2.2 and the fact that $a > \min(p^-,q^-)$, we get that for any $(u,v) \in Z_k$ with $\|(u,v)\| = r_k$,

$$J_\lambda(u,v) \geqslant \frac{1}{2\max(p^+,q^+)}r_k^{\min(p^-,q^-)} - c_5\lambda\,|\Omega| \longrightarrow +\infty \, as\, k \longrightarrow +\infty.$$

This ends Claim 1.

- **Claim 2:** Second geometric condition $(A2)$ of the Fountain Theorem.
  For every $x \in \bar{\Omega}, s,t \in \mathbb{R}$ the inequality:

$$|s|^{\alpha(x)}\,|t|^{\beta(x)} \geq c_6\left(|s|^{\alpha^-+\beta^-} + |t|^{\alpha^-+\beta^-} - 1\right)$$

holds true. Indeed, consider the compact subset K of $\mathbb{R}^2$ defined by

$$K = \left\{(s,t) \in \mathbb{R}^2 : \frac{|s|^{\alpha^-} + |t|^{\beta^-}}{2} = 1\right\}.$$

For every $(s,t) \in K$, we introduce the function

$$H(x,\tau) = \left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)}$$

defined on $\bar{\Omega} \times \mathbb{R}$. Then

$$\begin{aligned} \tau\frac{\partial H}{\partial \tau} =& \frac{s}{\alpha^-+\beta^-}\tau^{\frac{1}{\alpha^-+\beta^-}}\alpha(x)\left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)-2}\left(\tau^{\frac{1}{\alpha^-+\beta^-}}s\right)\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)} \\ & + \frac{t}{\alpha^-+\beta^-}\tau^{\frac{1}{\alpha^-+\beta^-}}\beta(x)\left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)-2}\left(\tau^{\frac{1}{\alpha^-+\beta^-}}t\right) \\ =& \frac{\alpha(x)}{\alpha^-+\beta^-}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)} \\ & + \frac{\beta(x)}{\alpha^-+\beta^-}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)} \\ =& \frac{\alpha(x)+\beta(x)}{\alpha^-+\beta^-}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}s\right|^{\alpha(x)}\left|\tau^{\frac{1}{\alpha^-+\beta^-}}t\right|^{\beta(x)} \\ \geq& H(x,\tau) > 0. \end{aligned}$$

Fix $M > 0$ such that $|s|^{\alpha^-+\beta^-} + |t|^{\alpha^-+\beta^-} \geq 2M$. Then, we have

$$H(x,\tau) \geq \frac{H(x,M)}{M}|\tau| \quad \text{for any} \quad |\tau| \geq M.$$

Set

$$\tilde{c} = \min_{x\in\bar{\Omega}} \min_{(s,t)\in K} \frac{H(x,M)}{M}.$$

Then

$$\left|\tau^{\frac{1}{\alpha^-+\beta^-}} s\right|^{\alpha(x)} \left|\tau^{\frac{1}{\alpha^-+\beta^-}} t\right|^{\beta(x)} \geq \tilde{c}|\tau| - c' \geq c_6(|\tau|-1), \quad \forall \tau \in \mathbb{R},$$

where $c_7 = \min(\tilde{c}, c')$. Moreover, every $(s', t') \in \mathbb{R}^2$ can be rewritten as

$$(s',t') = \left((|s'|^{\alpha^-+\beta^-} + |t'|^{\alpha^-+\beta^-})^{\frac{1}{\alpha^-+\beta^-}} s, (|s'|^{\alpha^-+\beta^-} + |t'|^{\alpha^-+\beta^-})^{\frac{1}{\alpha^-+\beta^-}} t \right),$$

where $(s,t) \in K$. Therefore, there is $c_7 > 0$ such that

$$|s'|^{\alpha(x)} |t'|^{\beta(x)} \geq c_7 \left(|s'|^{\alpha^-+\beta^-} + |t'|^{\alpha^-+\beta^-} - 1 \right) \quad \forall (s',t') \in \mathbb{R}^2.$$

For any $(u,v) \in Y_k$ with $||(u,v)|| = 1$ and $t > 1$, we have

$$\begin{aligned} J_\lambda(tu,tv) &= \int_Q \frac{t^{p(x,y)}}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega \frac{t^{\bar{p}(x)}}{\bar{p}(x)} |u(x)|^{\bar{p}(x)} dx \\ &+ \int_Q \frac{t^{q(x,y)}}{q(x,y)} \frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega \frac{t^{\bar{q}(x)}}{\bar{q}(x)} |v(x)|^{\bar{q}(x)} dx \\ &- \lambda \int_\Omega |tu(x)|^{\alpha(x)} |tv(x)|^{\beta(x)} dx \\ &\leqslant \frac{t^{p^+}}{p^-} \left[\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy + \int_\Omega |u(x)|^{\bar{p}(x)} dx \right] \\ &+ \frac{t^{q^+}}{q^-} \left[\int_Q \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sq(x,y)}} dxdy + \int_\Omega |v(x)|^{\bar{q}(x)} dx \right] \\ &- c_7 \lambda t^{\alpha^-+\beta^-} \int_\Omega |u(x)|^{\alpha^-+\beta^-} - c_7 \lambda t^{\alpha^-+\beta^-} \int_\Omega |v(x)|^{\alpha^-+\beta^-} - c_8 \\ &\leqslant \frac{t^{p^+}}{p^-} \rho_{p(x,y)}(u) + \frac{t^{q^+}}{q^-} \rho_{q(x,y)}(v) \\ &- c_7 \lambda t^{\alpha^-+\beta^-} \int_\Omega |u(x)|^{\alpha^-+\beta^-} - c_7 \lambda t^{\alpha^-+\beta^-} \int_\Omega |v(x)|^{\alpha^-+\beta^-} - c_8. \end{aligned}$$

Since $\alpha^- + \beta^- > max(p^+, q^+)$, we conclude that

$$J_\lambda(tu,tv) \longrightarrow -\infty \quad as \quad t \longrightarrow +\infty.$$

This ends Claim 2.

Therefore, by Fountain Theorem, we conclude that, the functional J_λ has infinitely many nontrivial critical points (u,v) in E. Hence, system $(\mathcal{P}_s)$ has infinitely many eigenvalues.

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgments

The authors would like to thank the anonymous referees for their suggestions and helpful comments which have improved the presentation of the original manuscript.

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