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Existence, Asymptotics and Computation of Solutions of Nonlinear Sturm-Liouville Problems

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Abstract This paper deals with the existence, asymptotics and computa-tion of solutions of nonlinear Sturm-Liouville problems with general separated boundary conditions. The approach centers first on converting these prob-lems into Hammerstein integral equations with modified argument, and then applying the Banach and Rothe fixed point theorems to solve them. This ap-proach not only enabled us to prove existence theorems for these problems, but also to derive general and accurate asymptotic formulae for their solutions. Finally, an illustrative numerical example is presented using the Picard's iter-ation method.

Keywords Approximate methods, asymptotic expansion, Jacobi elliptic functions, ordinary differential equation, nonlinear eigenvalue problem

MSC(2010) 34B15, 34B24, 34L16, 34L20, 47H30.

1. Introduction

Consider the nonlinear Sturm-Liouville equation

$$-u''(x) = \lambda u(x) - f(x, u(x)),\ x \in I := [a, b], \tag{1.1}$$

defined on the compact interval I and subject to the separated boundary conditions

$$a_1u(a) + a_2u'(a) = 0,\ |a_1| + |a_2| \neq 0,\ a_1, a_2 \in \mathbb{C}, \tag{1.2}$$
$$b_1u(b) + b_2u'(b) = 0,\ |b_1| + |b_2| \neq 0,\ b_1, b_2 \in \mathbb{C}. \tag{1.3}$$

Second-order nonlinear Sturm-Liouville equations are important due to their numerous real-world applications. The simple pendulum is a typical example, which is governed by the nonlinear equation

$$y'' + k^2 \sin(y) = 0, \tag{1.4}$$

where the constant $k \neq 0$ depends on the length of the pendulum and on gravity. Note that, by using the transformation $u := \sin\left(\frac{1}{2}y\right)$, one can reduce equation (1.4) to an equation of the form

$$-u'' = \lambda u - Au^3, \tag{1.5}$$

which is a special case of Sturm-Liouville equation (1.1) and is integrated by elliptic functions. For more details, see [12]. Equipped with boundary conditions of the

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form (1.3), a system of equations (1.1) and (1.3) is called a (nonlinear) Sturm-Liouville problem, which also occurs very frequently in applied mathematics. For details related to the Sturm-Liouville theory, the reader is referred to the books [18, 38]. These nonlinear problems have been studied for different purposes and under diverse forms by many authors. For a study of the existence of solutions or positive solutions see, for example, [2, 5, 8, 10, 16, 23, 24, 26, 30]. For a study of the asymptotic behavior of the solutions see, for example, [4, 6, 15, 22, 31, 32]. However, limited numerical studies have been carried out on them. The reader may see, for example, [1, 29, 34, 37, 39]. Note that some problems in the previously mentioned articles are special cases of problem (1.1)–(1.3).

In a recent article [17], we studied the existence and asymptotics of solutions of problem (1.1)–(1.3) when the nonlinear term f has the form Qu, with $Q \in L^1(I)$, using the homotopy perturbation method. We also have presented several numerical examples illustrating our theoretical results. In this paper, the objective is the same but the approach is different: we investigate the solvability of problem (1.1)–(1.3), as well as the asymptotic behavior of the pair (λ, u) as $|\lambda| \to \infty$, when the function f is nonlinear. We also present an iterative scheme that computes the solutions of problem (1.1)–(1.3). The approach adopted consists first in converting problem (1.1)–(1.3) into a nonlinear integral equation of Hammerstein type with modified argument, and then applying the Banach and Rothe fixed point theorems to solve it. This conversion is of interest for two primary reasons: it is better suited for proving the existence of solutions, and it enables the analysis of their qualitative properties. Moreover, Hammerstein integral equations have been intensively investigated in the literature, using many different approaches and methods, from both theoretical and numerical viewpoints, see, for example, [3, 7, 9, 11, 13, 21, 27]; see also [14, 25, 35] for integral equations with modified argument. For more details on the theory of integral equations, we refer the reader to the books [19, 20].

The outline of the paper is as follows: In Section 2, we state some preliminaries and notations. In Section 3, we begin by solving the integral equation in question using Rothe and Banach fixed point theorems. Then, we state and prove our existence theorems. As a direct consequence of these results, we show that if the boundary condition constants a_i and b_i, $i = 1, 2$, are real and f is continuous on $I \times V$, where V is some compact neighborhood of 0, then problem (1.1)–(1.3) has infinitely many solutions, which are twice continuously differentiable on I. Finally, we derive general and accurate asymptotic formulae for the solutions of problem (1.1)–(1.3) for sufficiently large $|\lambda|$. In the last section, we consider equation (1.5) with Dirichlet boundary conditions in order to illustrate our main results.

2. Preliminaries and notations

Some basic definitions and results are provided below (see, for example, [28, 33, 36]), which will be used in the next section.

Definition 2.1. Let $(\mathcal{X}, d)$ be a compact metric space. We say that $\mathcal{A} \subset C(\mathcal{X}, \mathbb{C})$ is uniformly bounded if there exists $M > 0$ such that $|g(x)| \leq M$, $\forall g \in \mathcal{A}$, $\forall x \in \mathcal{X}$.

Definition 2.2. Let $(\mathcal{X}, d)$ be a compact metric space. We say that $\mathcal{A} \subset C(\mathcal{X}, \mathbb{C})$ is equicontinuous if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) < \delta_\varepsilon$ it follows that $|g(x) - g(y)| < \varepsilon$, $\forall g \in \mathcal{A}$.

Theorem 2.1 (Arzela-Ascoli Theorem)**.** *Let $(\mathcal{X}, d)$ be a compact metric space and $\mathcal{A} \subset C(\mathcal{X}, \mathbb{C})$. Then, the following assertions are equivalent:*

1. *$\mathcal{A}$ is relatively compact, i.e., $\overline{\mathcal{A}}$ is compact,*
2. *$\mathcal{A}$ is uniformly bounded and equicontinuous.*

Theorem 2.2 (Rothe Fixed Point Theorem)**.** *Let E be a Banach space and $\mathcal{B} \subset E$ be a closed convex subset such that the zero of E is contained in the interior of $\mathcal{B}$, and let $T : \mathcal{B} \to E$ be a continuous map with $T(\mathcal{B})$ relatively compact in E and $T(\partial\mathcal{B}) \subset \mathcal{B}$, where $\partial\mathcal{B}$ denotes the boundary of $\mathcal{B}$. Then, there is a point $x^* \in \mathcal{B}$ such that $Tx^* = x^*$.*

Definition 2.3. Let $(\mathcal{X}, d)$ be a metric space. A mapping $T : \mathcal{X} \to \mathcal{X}$ is called a q-contraction if there exists a positive constant $q < 1$ such that

$$d(T(x), T(y)) < qd(x, y) \text{ for all } x, y \in \mathcal{X}. \tag{2.1}$$

Theorem 2.3 (Banach Fixed Point Theorem)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a q-contraction. Then, the following hold:*

1. *there is a unique point $x^* \in \mathcal{X}$ such that $Tx^* = x^*$;*
2. *the iterative sequence*

$$x_{n+1} = Tx_n, \;\; n = 0, 1, \cdots ; \tag{2.2}$$

   with arbitrary $x_0 \in \mathcal{X}$, converges to the unique fixed point x^* of T;

3. *the following error estimate holds for each $n \in \mathbb{N}$:*

$$d(x_n, x^*) \leq \frac{q^n}{1-q} d(x_1, x_0). \tag{2.3}$$

Lemma 2.1 ([17])**.** *The eigenparameter $\mu \in \mathbb{C}^*$ satisfies the problem*

$$v''(x) + \mu^2 v(x) = 0 \tag{2.4}$$

with the boundary conditions (1.3) *if and only if it is a zero of the characteristic function*

$$\Delta(\mu) = \left(\frac{1}{\mu} a_1 b_1 + a_2 b_2 \mu\right) \sin(\mu(b-a)) + (a_1 b_2 - a_2 b_1) \cos(\mu(b-a)). \tag{2.5}$$

The corresponding non-trivial solution is given by

$$v(x) = c\left(b_1 \frac{\sin(\mu(b-x))}{\mu} + b_2 \cos(\mu(b-x))\right), \tag{2.6}$$

where c is an arbitrary nonzero constant.

Throughout this paper, the following notations are adopted:

1. the maximum norm in the space $C(I, \mathbb{C})$ will be denoted by $\|.\|$ and the usual norm in the space $L^1(I, \mathbb{C})$ will be denoted by $\|.\|_1$, i.e.,

$$\|u\| := \max_{x \in I} |u(x)| \text{ and } \|u\|_1 := \int_a^b |u(x)| dx; \tag{2.7}$$

2. the closed ball in $C(I,\mathbb{C})$ of center 0 and radius $\rho > 0$ will be denoted by $\mathcal{B}_\rho$ and the closed disk in the complex plane of center 0 and radius $\rho > 0$ will be denoted by $\mathcal{D}_\rho$, i.e.,

$$\mathcal{B}_\rho := \{u \in C(I,\mathbb{C}) \mid \|u\| \leq \rho\} \ \ \textit{and}\ \mathcal{D}_\rho := \{u \in \mathbb{C} \mid |u| \leq \rho\}\,; \tag{2.8}$$

3. the set of zeros of the characteristic function Δ satisfying the condition

$$\int_a^b v^2(x,\mu)\,dx := \int_a^b v^2(x)\,dx \neq 0 \tag{2.9}$$

will be denoted by Ω_Δ, i.e.,

$$\Omega_\Delta := \left\{ \mu \in \mathbb{C}^* \mid \Delta(\mu) = 0 \ \textit{and} \ \int_a^b v^2(x,\mu)\,dx \neq 0 \right\}. \tag{2.10}$$

3. Main results

We begin by studying the following integral equation:

$$u(x) = \alpha v(x) + \frac{1}{\alpha\mu}\int_a^b\int_a^b K(x,t,s)F(t,s,u(t),u(s))\,ds\,dt =: Tu(x),\ \ x \in I, \tag{3.1}$$

where the pair $(\mu, v) \in \mathbb{C}^* \times C(I,\mathbb{C})$ is given, $\alpha \in \mathbb{C}^*$ is arbitrary and for $x, t, s \in I$,

$$K(x,t,s) = \frac{v(s)}{\beta^2} \times \begin{cases} v(x)\displaystyle\int_t^b \sin(\mu(s-t))v(s)\,ds - \beta\sin(\mu(x-t)), & t \leq x, \\ v(x)\displaystyle\int_t^b \sin(\mu(s-t))v(s)\,ds, & x \leq t, \end{cases} \tag{3.2}$$

where $\beta = \displaystyle\int_a^b v^2(x)\,dx \neq 0$ and

$$F(t,s,u(t),u(s)) = u(t)f(s,u(s)) - u(s)f(t,u(t)). \tag{3.3}$$

Lemma 3.1. *Let $v \in C(I,\mathbb{C})$ be such that $\int_a^b v^2(x)\,dx \neq 0$ and assume that the following conditions hold for α from* (3.1) *and for some $\rho > 0$:*

$(\mathcal{H}_1)$

$$f \in C(I \times \mathcal{D}_\rho, \mathbb{C}),$$

$(\mathcal{H}_2)$

$$\gamma := \frac{2(b-a)M_K\|M_\rho\|_1}{|\alpha\mu|} < 1 \ \textit{and}\ \frac{|\alpha|\|v\|}{1-\gamma} \leq \rho,$$

where $M_K > 0$ and $M_\rho \in L^1(I,\mathbb{R}^+)$ are such that

$$|K(x,t,s)| \leq M_K \ \textit{for all}\ x,t,s \in I, \tag{3.4}$$

$$|f(t,u)| \leq M_\rho(t) \ \textit{for a.e}\ t \in I \ \textit{and all}\ u \in \mathcal{D}_\rho. \tag{3.5}$$

Then, the integral equation (3.1) *has a solution $u^* \in \mathcal{B}_\rho$.*

Proof. We apply the Rothe Fixed Point Theorem, where the space $C(I,\mathbb{C})$ is equipped with the maximum norm $\|.\|$. Note that $(C(I,\mathbb{C}),\|.\|)$ is a Banach space. Let $\rho>0$ satisfy assumption $(\mathcal{H}_1)$ and let $u\in\mathcal{B}_\rho$. Clearly, Tu belongs to $C(I,\mathbb{C})$ for all $u\in\mathcal{B}_\rho$. We now show that $T(\mathcal{B}_\rho)$ is relatively compact in $C(I,\mathbb{C})$. First, we show that $T(\mathcal{B}_\rho)$ is equicontinuous. We have

$$\begin{aligned}|Tu(x)-Tu(y)| \leq &|\alpha||v(x)-v(y)|\\ &+\frac{1}{|\alpha\mu|}\int_a^b\int_a^b|K(x,t,s)-K(y,t,s)||F(t,s,u(t),u(s))|\,ds\,dt.\end{aligned}$$

Since f is continuous on $I\times\mathcal{D}_\rho$, there exists a positive constant M_f such that

$$|f(t,u)|\leq M_f \textit{ for all } t\in I \textit{ and all } u\in\mathcal{D}_\rho.$$

Using this,

$$\begin{aligned}|F(t,s,u(t),u(s))| =&|u(t)f(s,u(s))-u(s)f(t,u(t))|,\\ \leq&\, 2\rho M_f\end{aligned}$$

and

$$|Tu(x)-Tu(y)|\leq|\alpha||v(x)-v(y)|+\frac{2\rho M_f}{|\alpha\mu|}\int_a^b\int_a^b|K(x,t,s)-K(y,t,s)|\,ds\,dt.$$

By the uniform continuity of v and K, it follows that, for all $x,y\in I$ with $|x-y|\leq\delta$,

$$|Tu(x)-Tu(y)|\leq|\alpha|\varepsilon_1+\frac{2\rho M_f}{|\alpha\mu|}(b-a)^2\varepsilon_2,$$

which means that $T(\mathcal{B}_\rho)$ is equicontinuous. Next, we have

$$\begin{aligned}|Tu(x)|&\leq|\alpha|\|v\|+\frac{M_K}{|\alpha\mu|}\int_a^b\int_a^b|u(t)f(s,u(s))-u(s)f(t,u(t))|\,ds\,dt,\\ &\leq|\alpha|\|v\|+\frac{2\rho M_K M_f}{|\alpha\mu|}(b-a)^2.\end{aligned}$$

Hence, $T(\mathcal{B}_\rho)$ is uniformly bounded. By Arzela-Ascoli's theorem, $T(\mathcal{B}_\rho)$ is relatively compact in $C(I,\mathbb{C})$. On the other hand, we have

$$\begin{aligned}|Tu_1(x)-Tu_2(x)|\leq&\frac{M_K}{|\alpha\mu|}\int_a^b\int_a^b|u_1(t)f(s,u_1(s))-u_1(s)f(t,u_1(t))\\ &+u_2(s)f(t,u_2(t))-u_2(t)f(s,u_2(s))|\,ds\,dt.\end{aligned}$$

The term $u_1(t)f(s,u_1(s))-u_2(t)f(s,u_2(s))$ can be rewritten as

$$(u_1(t)-u_2(t))f(s,u_1(s))+u_2(t)(f(s,u_1(s))-f(s,u_2(s))).$$

Now, let us fix u_1 and $u_2\in\mathcal{B}_\rho$ with $\|u_1-u_2\|\leq\delta$. Then, by the uniform continuity of f on $I\times\mathcal{D}_\rho$, we have

$$|Tu_1(x)-Tu_2(x)|\leq\frac{2M_K}{|\alpha\mu|}(b-a)^2(M_f\delta+\rho\varepsilon),$$

which means that T is continuous. Next, let us show that $T(\partial \mathcal{B}_\rho) \subset \mathcal{B}_\rho$. Let $\|u\| = \rho$. Then, we have

$$|Tu(x)| \leq |\alpha| \|v\| + \frac{2(b-a)\rho M_K \|M_\rho\|_1}{|\alpha\mu|}.$$

Thus, by assumption $(\mathcal{H}_2)$, $T(\partial \mathcal{B}_\rho) \subset \mathcal{B}_\rho$. The Rothe Fixed Point Theorem then guarantees the existence of a solution $u^* \in \mathcal{B}_\rho$ of the fixed point equation $u^* = Tu^*$, which is equivalent to the integral equation (3.1). This completes the proof. □

Lemma 3.2. *Let* $v \in C(I, \mathbb{C})$ *be such that* $\int_a^b v^2(x)\,dx \neq 0$ *and assume that the following conditions hold for* α *from* (3.1) *and for some* $\rho > 0$*:*
$(\mathcal{H}_3)$ *there exist two positive functions* $L_\rho, M_\rho \in L^1(I, \mathbb{R}^+)$ *such that*

$$|f(t,u_1) - f(t,u_2)| \leq L_\rho(t)|u_1 - u_2| \text{ for a.e } t \in I \text{ and all } u_1, u_2 \in \mathcal{D}_\rho,$$
$$|f(t,u)| \leq M_\rho(t) \text{ for a.e } t \in I \text{ and all } u \in \mathcal{D}_\rho,$$

$(\mathcal{H}_4)$

$$q := \frac{2(b-a)M_K\left(\|M_\rho\|_1 + \rho\|L_\rho\|_1\right)}{|\alpha\mu|} < 1 \text{ and } \frac{|\alpha|\|v\|}{1-\gamma} \leq \rho.$$

Then, there is exactly one solution $u^* \in \mathcal{B}_\rho$ *of the integral equation* (3.1)*. Moreover, the sequence of successive approximations*

$$u_{n+1} = Tu_n, \; n = 0, 1, \cdots \tag{3.6}$$

converges to u^* *for any initial guess* $u_0 \in \mathcal{B}_\rho$*. The following error estimate holds:*

$$\|u_n - u^*\| \leq \frac{q^n}{1-q}\|u_1 - u_0\|, \; n = 0, 1, \cdots. \tag{3.7}$$

Proof. Here, we apply the Banach Fixed Point Theorem. Similarly, we equip the space $C(I, \mathbb{C})$ with the maximum norm. Since $\mathcal{B}_\rho$ is a closed subset of $C(I, \mathbb{C})$, we only need to show that $\mathcal{B}_\rho$ is an invariant set for the operator T, i.e., $T(\mathcal{B}_\rho) \subseteq \mathcal{B}_\rho$, and that T is a q-contraction. Let $\rho > 0$ satisfy assumption $(\mathcal{H}_3)$ and let $u \in \mathcal{B}_\rho$. Then, we have

$$\begin{aligned} |Tu(x)| &\leq |\alpha|\|v\| + \frac{M_K}{|\alpha\mu|}\int_a^b \int_a^b |u(t)f(s,u(s)) - u(s)f(t,u(t))|\,ds\,dt, \\ &\leq |\alpha|\|v\| + \frac{2(b-a)\rho M_K\|M_\rho\|_1}{|\alpha\mu|}. \end{aligned}$$

Thus, by assumptions $(\mathcal{H}_4)$, $Tu \in \mathcal{B}_\rho$. Next, we have, after a simple calculation,

$$\|Tu_1 - Tu_2\| \leq \frac{2(b-a)M_K\left(\|M_\rho\|_1 + \rho\|L_\rho\|_1\right)}{|\alpha\mu|}\|u_1 - u_2\| = q\|u_1 - u_2\|.$$

Therefore, by the first condition of assumption $(\mathcal{H}_4)$, it results that the operator T is a q-contraction. Applying the Banach Fixed Point Theorem completes the proof. □

Lemma 3.3. *Assumptions* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_4)$ *are equivalent to*

$$(\mathcal{H}_2')\ N_\rho \leq \rho \ \textit{and}\ |\alpha| \in J_\rho := \left[R_\rho^-, R_\rho^+\right] \tag{3.8}$$

and

$$(\mathcal{H}_4')\ N_\rho \leq \rho < R_\rho \left(\frac{R_\rho^+}{R_\rho^-}\right)^2 \ \textit{and}\ |\alpha| \in J_\rho' := \begin{cases} J_\rho, & \rho < R_\rho, \\ \left(N_\rho', R_\rho^+\right], & R_\rho \leq \rho, \end{cases} \tag{3.9}$$

respectively, where

$$N_\rho = \frac{8(b-a)M_K\|v\|\|M_\rho\|_1}{|\mu|},\ N_\rho' = \frac{N_\rho}{4\|v\|}\left(1 + \rho\frac{\|L_\rho\|_1}{\|M_\rho\|_1}\right), \tag{3.10}$$

and

$$R_\rho^\pm = \frac{\rho \pm \sqrt{\rho\left(\rho - N_\rho\right)}}{2\|v\|},\ R_\rho = \frac{\|M_\rho\|_1}{\|L_\rho\|_1}\frac{R_\rho^-}{R_\rho^+}. \tag{3.11}$$

Proof. We only prove the first equivalence. For the second one, the proof can proceed similarly. We have, after a simple calculation,

$$\begin{aligned}(\mathcal{H}_2) &\Leftrightarrow |\alpha| > \gamma' := \frac{2(b-a)M_K\|M_\rho\|_1}{|\mu|} \ \textit{and}\ \|v\||\alpha|^2 - \rho|\alpha| + \rho\gamma' \leq 0, \\ &\Leftrightarrow |\alpha| > \gamma' \ \textit{and}\ \left(\rho^2 - 4\rho\gamma'\|v\| \geq 0 \ \textit{and}\ |\alpha| \in J_\rho\right), \\ &\Leftrightarrow \rho - 4\gamma'\|v\| \geq 0 \ \textit{and}\ |\alpha| \in J_\rho,\end{aligned}$$

since for any $|\alpha| \in J_\rho$ we have $|\alpha| > \gamma'$. We thus complete the proof. □

Lemma 3.4. *If* $u(x)$ *is a solution of the integral equation* (3.1) *on* I*, then it satisfies*

$$\int_a^b v(x)u(x)\,dx = \alpha\int_a^b v^2(x)\,dx. \tag{3.12}$$

Proof. For simplicity, we rewrite the integral equation (3.1) as

$$u(x) = \alpha v(x) + \frac{1}{\alpha\mu}w(x), \tag{3.13}$$

where

$$w(x) = \int_a^b\int_a^b K(x,t,s)F(t,s,u(t),u(s))\,ds\,dt.$$

Using equation (3.2), we get

$$w(x) = \frac{v(x)}{\beta^2}\int_a^b G(t)H(t)\,dt - \frac{1}{\beta}\int_a^x \sin(\mu(x-t))H(t)\,dt,$$

where

$$G(t) = \int_t^b \sin(\mu(s-t))v(s)\,ds \ \textit{and}\ H(t) = \int_a^b v(s)F(t,s,u(t),u(s))\,ds. \tag{3.14}$$

Next, it is a simple matter to verify that

$$\int_a^b G(t)H(t)\,dt = \int_a^b v(s)\int_a^s \sin(\mu(s-t))H(t)\,dt\,ds.$$

Hence,

$$w(x) = \frac{v(x)}{\beta^2}\int_a^b v(s)\int_a^s \sin(\mu(s-t))H(t)\,dt\,ds - \frac{1}{\beta}\int_a^x \sin(\mu(x-t))H(t)\,dt. \quad (3.15)$$

Now, multiplying both sides in equation (3.13) by v and integrating over I, we get

$$\int_a^b v(x)u(x)\,dx = \alpha\int_a^b v^2(x)\,dx + \frac{1}{\alpha\mu}\int_a^b v(x)w(x)\,dx,$$

and by equation (3.15), we have

$$\int_a^b v(x)w(x)\,dx = 0.$$

This completes the proof. □

Now, we are ready to state and prove our main results.

Theorem 3.1. *Let $\mu \in \Omega_\Delta$ and assume that assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2')$ hold for some $\rho > 0$. Then, problem (1.1)–(1.3) has a solution $(\lambda_{\alpha,\mu}, u_{\alpha,\mu}) := (\lambda, u) \in \mathbb{C}\times C^2(I,\mathbb{C})$ for all $|\alpha| \in J_\rho$ such that*

$$\lambda_{\alpha,\mu} = \mu^2 + \frac{1}{\alpha\beta}\int_a^b v(x)f(x,u_{\alpha,\mu}(x))\,dx, \quad (3.16)$$

and

$$|u_{\alpha,\mu}(x)| \le \rho \text{ for all } x\in I \text{ and all } |\alpha| \in J_\rho, \quad (3.17)$$

$$|\lambda_{\alpha,\mu}| \le |\mu|^2 + \frac{\|v\|\|M_\rho\|_1}{|\alpha\beta|} \text{ for all } |\alpha| \in J_\rho. \quad (3.18)$$

Proof. Let $\mu \in \Omega_\Delta$. We first show that if u satisfies the integral equation (3.1), then it satisfies the Sturm-Liouville equation (1.1), too. According to (3.15), the integral equation (3.1) can be rewritten as

$$u(x) = \sigma v(x) - \frac{1}{\alpha\beta\mu}\int_a^x \sin(\mu(x-t))H(t)\,dt,$$

where H is given in (3.14) and

$$\sigma = \alpha + \frac{1}{\alpha\beta^2\mu}\int_a^b v(s)\int_a^s \sin(\mu(s-t))H(t)\,dt\,ds.$$

The first and second derivatives of u are

$$u'(x) = \sigma v'(x) - \frac{1}{\alpha\beta}\int_a^x \cos(\mu(x-t))H(t)\,dt, \quad (3.19)$$

$$u''(x) = \sigma v''(x) - \frac{1}{\alpha\beta}H(x) + \frac{\mu}{\alpha\beta}\int_a^x \sin(\mu(x-t))H(t)\,dt, \quad (3.20)$$

respectively. More precisely,

$$u''(x) = \sigma\left(v''(x) + \mu^2 v(x)\right) - \frac{1}{\alpha\beta}H(x) - \mu^2 u(x),$$

$$= -\frac{1}{\alpha\beta} H(x) - \mu^2 u(x),$$

since $v''(x) + \mu^2 v(x) = 0$. Now, replacing H by its expression yields

$$\begin{aligned} u''(x) &= -\frac{1}{\alpha\beta} \int_a^b v(s) \left(u(x) f(s, u(s)) - u(s) f(x, u(x))\right) ds - \mu^2 u(x), \\ &= -\left(\mu^2 + \frac{1}{\alpha\beta} \int_a^b v(s) f(s, u(s))\, ds\right) u(x) + \left(\frac{1}{\alpha\beta} \int_a^b v(s) u(s)\, ds\right) f(x, u(x)). \end{aligned}$$

According to Lemma 3.4, we have

$$\frac{1}{\alpha\beta} \int_a^b v(s) u(s)\, ds = 1$$

and consequently u'' reduces to

$$u''(x) = -\left(\mu^2 + \frac{1}{\alpha\beta} \int_a^b v(s) f(s, u(s))\, ds\right) u(x) + f(x, u(x)).$$

Take

$$\lambda = \mu^2 + \frac{1}{\alpha\beta} \int_a^b v(s) f(s, u(s))\, ds.$$

Then, u satisfies the Sturm-Liouville equation (1.1). Let us next show that u also satisfies the boundary conditions (1.3). We have

$$a_1 u(a) + a_2 u'(a) = \sigma \left(a_1 v(a) + a_2 v'(a)\right) = 0$$

and

$$\begin{aligned} A :&= b_1 u(b) + b_2 u'(b), \\ &= \sigma \left(b_1 v(b) + b_2 v'(b)\right) - \frac{1}{\alpha\beta\mu} \int_a^b \big(b_1 \sin(\mu(b-t)) + b_2 \mu \cos(\mu(b-t))\big) H(t)\, dt, \end{aligned}$$

that is,

$$A = -\frac{1}{\alpha\beta\mu} \int_a^b \big(b_1 \sin(\mu(b-t)) + b_2 \mu \cos(\mu(b-t))\big) H(t)\, dt.$$

According to Lemma 2.1, we have

$$\begin{aligned} A &= -\frac{1}{\alpha\beta c} \int_a^b v(t) H(t)\, dt, \\ &= -\frac{1}{\alpha\beta c} \int_a^b v(t) \int_a^b v(s) \left(u(t) f(s, u(s)) - u(s) f(t, u(t))\right) ds\, dt, \\ &= 0. \end{aligned}$$

Hence, u satisfies the boundary conditions (1.3). Moreover, (3.19) implies $u' \in C(I, \mathbb{C})$, and under assumption $(\mathcal{H}_1)$ of Lemma 3.1, (3.20) implies $u'' \in C(I, \mathbb{C})$. The proof is then completed by applying Lemmas 3.1 and 3.3. □

Theorem 3.2. *Let $\mu \in \Omega_\Delta$ and assume that assumptions $(\mathcal{H}_3)$ and $(\mathcal{H}'_4)$ hold for some $\rho > 0$. Then, problem (1.1)–(1.3) has a solution $(\lambda_{\alpha,\mu}, u_{\alpha,\mu}) := (\lambda, u) \in \mathbb{C} \times C^1(I, \mathbb{C})$ for all $|\alpha| \in J'_\rho$, which satisfies the relationship (3.16) and the estimates (3.17) and (3.18) for all $|\alpha| \in J'_\rho$. Moreover, $u_{\alpha,\mu}$ can be obtained as the limit of the Picard Iterations given by (3.6) for any initial guess $u_0 \in \mathcal{B}_\rho$.*

Proof. The proof is similar to that of Theorem 3.1, where we apply Lemma 3.2 instead of Lemma 3.1. □

Theorem 3.3. *Assume that the following holds:*
$(\mathcal{H}_5)$ *the set Ω_Δ has infinitely many elements μ_k, $k = 1, 2, \cdots$, such that*

$$|\mu_k| \to \infty, \ \textit{as } k \to \infty.$$

Assume, in addition, that assumption $(\mathcal{H}_1)$ (resp. $(\mathcal{H}_3)$) holds for some $\rho > 0$. Then, problem (1.1)–(1.3) has a solution $(\lambda_{\alpha,\mu_k}, u_{\alpha,\mu_k})$ for all $|\mu_k| \geq c(\rho)$ with suitable k and $c(\rho)$, and all $|\alpha| \in J_\rho$ (resp. $|\alpha| \in J'_\rho$), which is as in Theorem 3.1 (resp. 3.2).

Proof. We have from (3.2)

$$|K(x,t,s)| \leq \frac{2\|v\|^3(b-a)}{\beta^2} e^{|\Im(\mu)|(b-a)} \ \textit{for all } x,t,s \in I \ \textit{and all } \mu \in \Omega_\Delta,$$

where $\Im(\mu)$ denotes the imaginary part of μ. On the other hand, one can easily verify, from (2.5), that for large $|\Im(\mu)|$, $\Delta(\mu)$ has to grow like $e^{|\Im(\mu)|(b-a)}$, which means that $\Delta(\mu)$ cannot vanish for sufficiently large values of $|\Im(\mu)|$. It follows that the kernel K is bounded for all $x, t, s \in I$ and all $\mu \in \Omega_\Delta$. Hence, if assumption $(\mathcal{H}_5)$ holds, then both assumptions $(\mathcal{H}'_2)$ and $(\mathcal{H}'_4)$ hold for all $|\mu_k| \geq c(\rho)$ with suitable k and $c(\rho)$. Applying Theorem 3.1 (resp. 3.2) completes the proof. □

As a corollary of Theorem 3.3, we get the following result:

Corollary 3.1. *Let the boundary condition constants a_i and b_i, $i = 1, 2$, be real, and assume, in addition, that assumption $(\mathcal{H}_1)$ (resp. $(\mathcal{H}_3)$) holds for some $\rho > 0$. Then, problem (1.1)–(1.3) has a solution $(\lambda_{\alpha,\mu_k}, u_{\alpha,\mu_k})$ for all $|\mu_k| \geq c(\rho)$ with suitable k and $c(\rho)$, and all $|\alpha| \in J_\rho$ (resp. $|\alpha| \in J'_\rho$), which is as in Theorem 3.1 (resp. 3.2).*

Proof. It is well known that if the boundary condition constants a_i and b_i, $i = 1, 2$, are real, then Δ has infinitely many zeros μ_k, $k = 1, 2, \cdots$, such that (see, e.g., [38])

$$\mu_k^2 \in \mathbb{R}^* \ \textit{and} \ -\infty < \mu_1^2 < \mu_2^2 < \mu_3^2 < \cdots; \ \mu_k^2 \to \infty, \ \textit{as } k \to \infty.$$

Since all μ_k^2 are real, each μ_k is either real or pure imaginary, but clearly, the condition $\int_a^b v^2(x, \mu_k)\,dx \neq 0$ is satisfied in both cases. Hence, $\mu_k \in \Omega_\Delta$ for all positive integer k. Now, the conclusions of the corollary follow from Theorem 3.3. □

In the following, we derive general and accurate asymptotic formulae for the solution $(\lambda_{\alpha,\mu}, u_{\alpha,\mu})$ for fixed $\alpha \neq 0$ and sufficiently large $|\mu|$:

Theorem 3.4. *Let assumption $(\mathcal{H}_5)$ hold and assume, in addition, that $f(x,u)$ is continuous with respect to $x \in I$ and differentiable with respect to $u \in \mathcal{D}_\rho$ for some $\rho > 0$. Then, problem (1.1)–(1.3) has a solution $(\lambda_{\alpha,\mu_k}, u_{\alpha,\mu_k})$ for any fixed*

$|\alpha| \in J_\rho$ and all sufficiently large integer k, which is as in Theorem 3.1 and satisfies the following asymptotic formulae for all nonnegative integer j:

$$u_{\alpha,\mu_k}(x) = v_j(x) + O\left(\frac{1}{|\mu_k|^{j+1}}\right), \tag{3.21}$$

$$\lambda_{\alpha,\mu_k} = \mu_k^2 + \frac{1}{\alpha\beta}\int_a^b v(x)f(x,v_j(x))\,dx + O\left(\frac{1}{|\mu_k|^{j+1}}\right), \tag{3.22}$$

as $k \to \infty$, where

$$v_j(x) = \begin{cases} \alpha v(x), & \text{for } j = 0,\\ \alpha v(x) + \dfrac{1}{\alpha\mu_k}\displaystyle\int_a^b\int_a^b K(x,t,s)F(t,s,v_{j-1}(t),v_{j-1}(s))\,ds\,dt, & \text{for } j \geq 1. \end{cases}$$

Proof. The first conclusion immediately follows from Theorem 3.3. Next, since f is bounded for all $x \in I$ and all $u \in \mathcal{D}_\rho$ and K is bounded for all $x,t,s \in I$ and all $\mu_k \in \Omega_\Delta$, the second term in the integral equation (3.1) can be estimated so that

$$\left|\frac{1}{\alpha\mu_k}\int_a^b\int_a^b K(x,t,s)\Big(u(t)f(s,u(s)) - u(s)f(t,u(t))\Big)\,ds\,dt =: h_k(x)\right|, \tag{3.23}$$

$$\leq \frac{2M_K\rho\|M_\rho\|_1}{|\alpha\mu_k|}(b-a). \tag{3.24}$$

Since ρ and α are fixed, we have, as $|\mu_k| \to \infty$ (or as $k \to \infty$),

$$h_k(x) = O\left(\frac{1}{|\mu_k|}\right).$$

As a consequence,

$$u_{\alpha,\mu_k}(x) = \alpha v(x) + O\left(\frac{1}{|\mu_k|}\right) = v_0(x) + O\left(\frac{1}{|\mu_k|}\right), \ \textit{as } k \to \infty.$$

On the other hand, one can easily verify, under assumption $(\mathcal{H}_2')$, that $|u_{\alpha,\mu_k}(x)| \leq \rho$ implies $|v_j(x)| \leq \rho$ for all $j \geq 0$. Using this, the formula above and a Taylor expansion of f, with respect to the second variable, for sufficiently large k yields

$$f(x,u_{\alpha,\mu_k}(x)) = f\left(x, v_0(x) + O\left(\tfrac{1}{|\mu_k|}\right)\right) = f(x,v_0(x)) + O\left(\frac{1}{|\mu_k|}\right).$$

Now, inserting the two formulae above into the integral equation (3.1), we get

$$u_{\alpha,\mu_k}(x) = v_1(x) + O\left(\frac{1}{|\mu_k|^2}\right).$$

Similarly, using this new formula and a Taylor expansion for sufficiently large k gives

$$f(x,u_{\alpha,\mu_k}(x)) = f\left(x, v_1(x) + O\left(\frac{1}{|\mu_k|^2}\right)\right) = f(x,v_1(x)) + O\left(\frac{1}{|\mu_k|^2}\right).$$

Repeating this process $j-1$ times yields

$$u_{\alpha,\mu_k}(x) = v_j(x) + O\left(\frac{1}{|\mu_k|^{j+1}}\right), \; as\ k \to \infty,$$

and

$$f(x, u_{\alpha,\mu_k}(x)) = f\left(x, v_j(x)\right) + O\left(\frac{1}{|\mu_k|^{j+1}}\right), \; as\ k \to \infty.$$

Inserting this latter formula into equation (3.16) completes the proof. □

4. Numerical example

In this section, we give an example illustrating Theorems 3.1 and 3.2. Consider the nonlinear Sturm-Liouville problem

$$-u''(x) = \lambda u(x) - 2u^3(x) \;\; on\ I = [0,1], \tag{4.1}$$
$$u(0) = 0 = u(1), \tag{4.2}$$

with the exact solutions

$$u_{k,m}(x) = m\nu_{k,m} sn(\nu_{k,m}x, m), \;\; k = 1, 2, \cdots, \tag{4.3}$$
$$\lambda_{k,m} = \nu_{k,m}^2 \left(1 + m^2\right), \;\; k = 1, 2, \cdots, \tag{4.4}$$

where the parameter m is such that $0 < m^2 < 1$, the function sn is the elliptic sine given by

$$sn(z, m) = \sin(am(z, m)), \tag{4.5}$$

where $\varphi := am(z, m)$ is called the Jacobi amplitude and satisfies

$$z =: Y(\varphi, m) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - m^2 \sin^2(\theta)}}, \tag{4.6}$$

and where

$$\nu_{k,m} = 2kY(\tfrac{\pi}{2}, m), \;\; k = 1, 2, \cdots. \tag{4.7}$$

For more details see [12]. For comparison, the parameter m is determined by the normalization condition

$$\int_0^1 \sin(k\pi x) u_{k,m}(x)\, dx = \eta, \;\; k = 1, 2, \cdots, \tag{4.8}$$

for some nonzero real number η. We now identify the values of $\rho > 0$ for which assumptions $(\mathcal{H}_2')$ and $(\mathcal{H}_4')$ hold. Under the boundary conditions (4.2), Lemma 2.1 implies

$$v_k(x) = c_k \sin(k\pi x), \;\; k = 1, 2, \cdots, \tag{4.9}$$
$$\mu_k = k\pi, \;\; k = 1, 2, \cdots, \tag{4.10}$$

where k is taken to be positive. Moreover, under the normalization condition (4.8), Lemma 3.4 yields

$$\alpha = 2\eta, \tag{4.11}$$

where we set $c_k = 1$ for any positive integer k. After a simple calculation, we also find that

$$\|v_k\| = 1,\ k = 1, 2, \cdots,\ M_K = 4 + \tfrac{2}{\pi},\ M_\rho(t) = 2\rho^3 \ \textit{and}\ L_\rho(t) = 6\rho^2,$$

and for any positive integer k

$$N_{\rho,k} = N'_{\rho,k} = \rho^3\omega_k^2,\ R^{\pm}_{\rho,k} = \frac{\rho}{2}\left(1 \pm \sqrt{1-\rho^2\omega_k^2}\right)\ \textit{and}\ R_{\rho,k} = \frac{\rho}{3}\left(\tfrac{1-\sqrt{1-\rho^2\omega_k^2}}{\rho\omega_k}\right)^2,$$

where

$$\omega_k = \frac{4}{\pi}\sqrt{\frac{2\,(2\pi+1)}{k}}.$$

The assumption $N_{\rho,k} \le \rho$ implies the inequality $\rho\omega_k \le 1$, which means that assumption $(\mathcal{H}'_2)$ holds for all $\rho > 0$ and all $\alpha \neq 0$ satisfying

$$\rho \le \frac{1}{\omega_k},\ k = 1, 2, \cdots, \tag{4.12}$$

$$|\alpha| \in \left[\frac{\rho}{2}\left(1-\sqrt{1-\rho^2\omega_k^2}\right), \frac{\rho}{2}\left(1+\sqrt{1-\rho^2\omega_k^2}\right)\right],\ k = 1, 2, \cdots, \tag{4.13}$$

respectively. Similarly, the assumptions $N_{\rho,k} \le \rho$ and $\rho < R_{\rho,k}\left(\frac{R^+_{\rho,k}}{R^-_{\rho,k}}\right)^2$ imply the inequalities $\rho\omega_k \le 1$ and $\rho\omega_k < \frac{\sqrt{3}}{2}$, respectively, which means that assumption $(\mathcal{H}'_4)$ holds for all $\rho > 0$ and all $\alpha \neq 0$ satisfying

$$\rho < \frac{\sqrt{3}}{2\omega_k},\ k = 1, 2, \cdots, \tag{4.14}$$

$$|\alpha| \in \left(\rho^3\omega_k^2, \frac{\rho}{2}\left(1+\sqrt{1-\rho^2\omega_k^2}\right)\right],\ k = 1, 2, \cdots, \tag{4.15}$$

respectively, since $R_{\rho,k} < \rho$ for any positive integer k. Thus, Theorem 3.1 (resp. 3.2) applies for all $\rho > 0$ and all $\alpha \neq 0$ satisfying (4.12)–(4.13) (resp. (4.14)–(4.15)). In the following, we use the Picard Iterations given, for $k = 1, 2, \cdots$, by

$$\begin{aligned} u_{\alpha,k,n}(x) =& \alpha\sin(k\pi x) + \frac{1}{\alpha k\pi} \\ &\times \int_0^1\int_0^1 K_k(x,t,s)F(t,s,u_{\alpha,k,n-1}(t),u_{\alpha,k,n-1}(s))\,ds\,dt,\ n = 1, 2, \cdots, \end{aligned} \tag{4.16}$$

to compute the approximate eigenfunctions of problem (4.1)–(4.2), where

$$F(t,s,u(t),u(s)) = 2u(s)u(t)\left(u(s)-u(t)\right)\left(u(s)+u(t)\right)$$

and where, for $k = 1, 2\cdots$ and $0 \le s \le 1$,

$$K_k(x,t,s) = 2\sin(k\pi s) \times \begin{cases} \tilde{K}_k(x,t) - \sin(k\pi(x-t)), & 0 \le t \le x, \\ \tilde{K}_k(x,t), & x \le t \le 1, \end{cases}$$

where

$$\tilde{K}_k(x,t) = \sin(k\pi x)\left((1-t)\cos(k\pi t) + \frac{1}{k\pi}\sin(k\pi t)\right).$$

In order to compute the corresponding eigenvalues we use the relationship given, for $k = 1, 2, \cdots$, by

$$\lambda_{\alpha,k,n} = k^2\pi^2 + \frac{4}{\alpha}\int_0^1 \sin(k\pi x)u^3_{\alpha,k,n}(x)\,dx,\ \ n = 0, 1, \cdots. \tag{4.17}$$

α, ρ and the initial guess $u_{\alpha,k,0}$ are taken, for any positive integer k, as follows:

$$u_{\alpha,k,0} = \rho = \frac{1}{6} \ \textit{and}\ \alpha \in \left(\tfrac{4(2\pi+1)}{27\pi^2} \approx 0.11, \tfrac{1}{12}\left(1 + \sqrt{1 - \tfrac{8(2\pi+1)}{9\pi^2}}\right) \approx 0.13\right].$$

Numerical computations were performed using Python's SciPy library. Integrals in (4.16)–(4.17) were evaluated via the Composite Simpson's rule (scipy.integrate.simps) with a step size of $dx = 0.01$. Equation (4.8) was solved for the parameter m using Newton's method (scipy.optimize.newton) with a tolerance of $tol = 10^{-14}$. The exact solutions to problem (4.1)–(4.2) were computed using Jacobian elliptic functions (scipy.special.ellipj), parameterized by m^2 instead of m. Table 1 presents the first ten roots of (4.8) for $\eta = \alpha/2 = 0.0625$ and various values of k, as computed by Newton's method. The first ten approximate eigenvalues for $\alpha = 0.125$ are given in Table 2 for different k and n, alongside their exact values. Corresponding absolute errors are provided in Table 3. All values in Tables 1 and 2 are rounded to 12 decimal places. Finally, the first five approximate eigenfunctions are plotted alongside their exact counterparts in Figures 1–5 for various values of k and n.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
m	0.039769061900	0.019891907630	0.013262182903	0.009946876379	0.007957589680
	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
m	0.006631364830	0.005684047721	0.004973553525	0.004420943639	0.003978853892

Table 1. The first ten roots of equation (4.8) obtained for $\eta = 0.0625$ and different k.

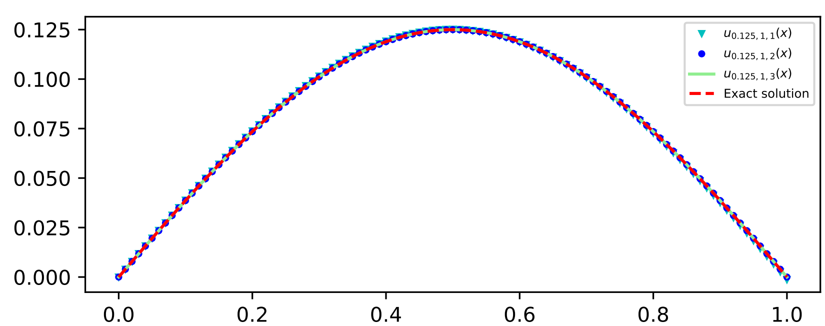

Figure 1. First approximate eigenfunction ($\alpha = 0.125$, $n = 1, 2, 3$) alongside the exact one.

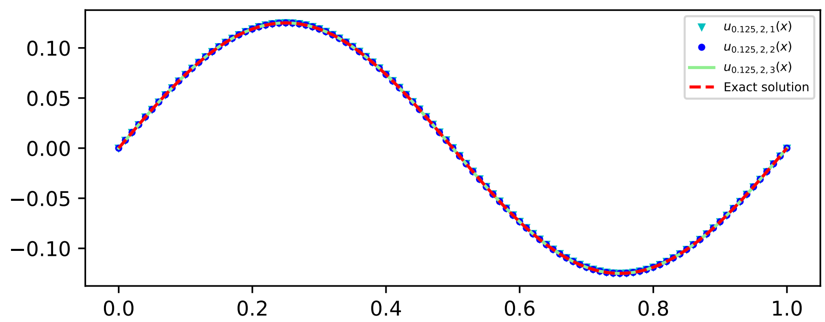

Figure 2. Second approximate eigenfunction ($\alpha = 0.125$, $n = 1, 2, 3$) alongside the exact one.

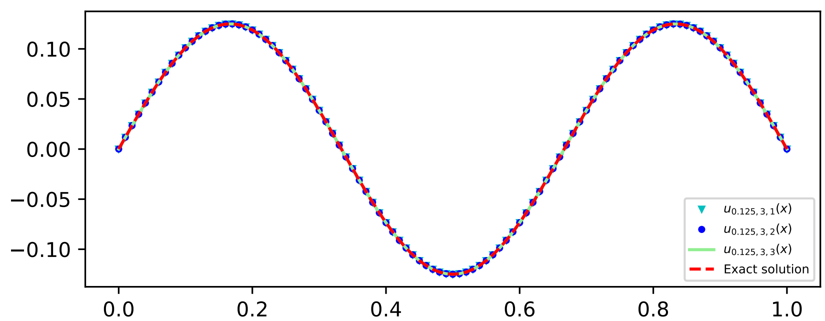

Figure 3. Third approximate eigenfunction ($\alpha = 0.125$, $n = 1, 2, 3$) alongside the exact one.

	Approximate eigenvalue			Exact eigenvalue
k	$n = 1$	$n = 2$	$n = 3$	$\eta = 0.0625$
1	9.893094765958	9.893039576552	9.893039583177	9.893039583178
2	39.501855516924	39.501854524610	39.501854524648	39.501854524664
3	88.849875923477	88.849876852115	88.849876852108	88.849876852144
4	157.937107943215	157.937107772424	157.937107772427	157.937107772492
5	246.763547137128	246.763547434372	246.763547434372	246.763547434474
6	355.329195944310	355.329195874655	355.329195874655	355.329195874797
7	483.634052989883	483.634053105859	483.634053105859	483.634053106051
8	631.678119171330	631.678119133238	631.678119133239	631.678119133483
9	799.461393905547	799.461393959306	799.461393959306	799.461393959606
10	986.983877609596	986.983877585384	986.983877585385	986.983877585738

Table 2. The first ten approximate eigenvalues obtained for $\alpha = 0.125$ and different k and n.

k	$n = 1$	$n = 2$	$n = 3$
1	$5.518278000060661 \times 10^{-5}$	$6.625999304787911 \times 10^{-9}$	$1.000088900582341 \times 10^{-12}$
2	$9.922600057166164 \times 10^{-7}$	$5.399414249041001 \times 10^{-11}$	$1.599431698195985 \times 10^{-11}$
3	$9.286670064057034 \times 10^{-7}$	$2.900435447372729 \times 10^{-11}$	$3.601030584832188 \times 10^{-11}$
4	$1.707230126157810 \times 10^{-7}$	$6.798472895752639 \times 10^{-11}$	$6.500044946733397 \times 10^{-11}$
5	$2.973459913846454 \times 10^{-7}$	$1.020055151457199 \times 10^{-10}$	$1.020055151457199 \times 10^{-10}$
6	$6.951302111701807 \times 10^{-8}$	$1.419948603142984 \times 10^{-10}$	$1.419948603142984 \times 10^{-10}$
7	$1.161679961114714 \times 10^{-7}$	$1.920170689118094 \times 10^{-10}$	$1.920170689118094 \times 10^{-10}$
8	$3.784691671171458 \times 10^{-8}$	$2.449951352900825 \times 10^{-10}$	$2.440856405883096 \times 10^{-10}$
9	$5.405900083133019 \times 10^{-8}$	$3.000195647473447 \times 10^{-10}$	$3.000195647473447 \times 10^{-10}$
10	$2.385797870374517 \times 10^{-8}$	$3.540208126651123 \times 10^{-10}$	$3.529976311256177 \times 10^{-10}$

Table 3. Absolute errors of the approximate eigenvalues.

k	$n = 1$	$n = 2$	$n = 3$
1	$1.30901854286883 \times 10^{-3}$	$3.31020211727373 \times 10^{-7}$	$1.54284230703682 \times 10^{-9}$
2	$4.69196060305883 \times 10^{-4}$	$4.64134699155369 \times 10^{-7}$	$3.05404940573184 \times 10^{-9}$
3	$9.57055224726985 \times 10^{-5}$	$3.10891285328421 \times 10^{-8}$	$4.56827733887749 \times 10^{-9}$
4	$1.17269442008003 \times 10^{-4}$	$2.88300581532952 \times 10^{-8}$	$6.03138130064739 \times 10^{-9}$
5	$5.07068105644256 \times 10^{-5}$	$1.36751118796907 \times 10^{-8}$	$7.46621653391343 \times 10^{-9}$
6	$5.21320529252213 \times 10^{-5}$	$8.90919773025022 \times 10^{-9}$	$8.91012959941539 \times 10^{-9}$
7	$2.93778813117671 \times 10^{-5}$	$1.17456294829176 \times 10^{-8}$	$1.03646652649391 \times 10^{-8}$
8	$2.92728711561232 \times 10^{-5}$	$1.14650860194576 \times 10^{-8}$	$1.17145832409515 \times 10^{-8}$
9	$1.89588493897559 \times 10^{-5}$	$1.34537208699991 \times 10^{-8}$	$1.30510574083975 \times 10^{-8}$
10	$1.87642096026538 \times 10^{-5}$	$1.38181665623737 \times 10^{-8}$	$1.38183241307765 \times 10^{-9}$

Table 4. Maximum absolute errors of the approximate eigenfunctions.

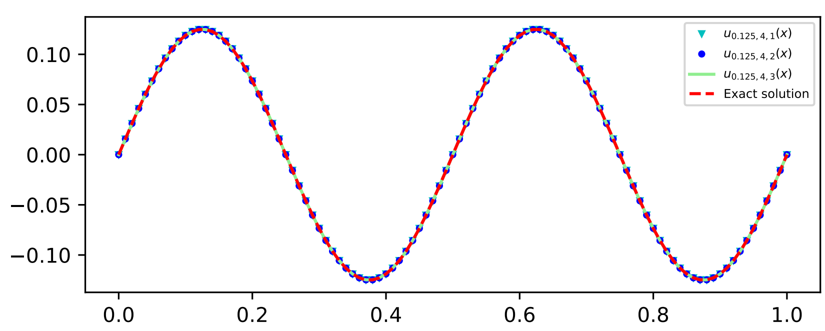

Figure 4. Fourth approximate eigenfunction ($\alpha = 0.125$, $n = 1, 2, 3$) alongside the exact one.

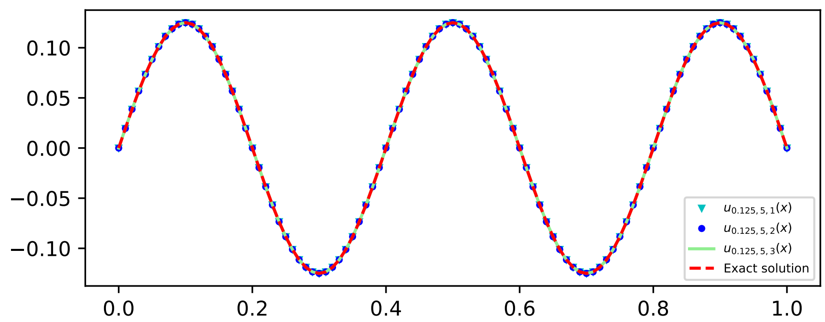

Figure 5. Fifth approximate eigenfunction ($\alpha = 0.125$, $n = 1, 2, 3$) alongside the exact one.

As shown by the absolute and maximum absolute errors in Tables 3 and 4, only three iterations yield a highly accurate approximation of the exact solution for problem (4.1)–(4.2).

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest

The author declares no conflicts of interest relating to this study.

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