Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1859–1869
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1859

On Local and Nonlocal Robin Boundary Value Problem with Critical Nonlinearity

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Abstract In this paper, using the Mountain Pass Theorem, we present results on compactness and the existence of solutions for a class of local and nonlocal p-Laplacian equations involving Robin boundary conditions, with critical nonlinearity and a small perturbation.

Keywords p-Laplacian, Robin problem, critical exponent

MSC(2010) 35J30, 35J60, 35J92.

1. Introduction

This paper deals with the following elliptic problem:

$$K\left(\int_{\Omega}|\nabla u|^p\,dx+\beta\int_{\partial\Omega}|u|^p d\sigma_x\right)\Delta_p u=\gamma a(x)|u|^{q-2}u+|u|^{p^*-2}u+g(x)\quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial u}+\beta|u|^{p-2}u=0\quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $p^*=Np/(N-p)$ is the critical Sobolev exponent, $1<p<N$, γ is a positive parameter and $a\in L^{p^*/(p^*-q)}(\Omega)$, $g\in L^{p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$ and $p^*=\frac{Np}{N-p}$.

Here the functional K verifies (K_1) $K:\ (0,+\infty)\to(0,+\infty)$ continuous and $k_0=\inf\limits_{s>0}K(s)>0$.

The problem (1.1) is called nonlocal because of the presence of the term $K(.)$, so it is no longer a pointwise identity. This leads us to some mathematical difficulties which make the study of such a class of problems particularly interesting.

In fact, equations such as (1.1) received more attention after Lions [11] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [21].

The critical exponent case poses a significant challenge due to the absence of compactness, rendering standard arguments ineffective. To our knowledge, only few results have studied the elliptic problems featuring critical exponents. Among these references, some of the most noteworthy include [3, 8, 9, 12–15, 18] and their associated literature. However, drawing inspiration from these seminal works from which we will draw certain insights, our aim is to generalize and partially extend

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corresponding results to accommodate cases where $p \neq 2$ and involve a perturbation g.

First, we deal with the case of a local problem: Suppose that the operator $K = Id$ and $1 < q < p < N$; then we can state the following compactness note.

Theorem 1.1. *There exists a constant $L > 0$ depending on p, q, and N such that ϕ_γ satisfies the Palais-Smale condition in the interval I^γ :*

$$I^\gamma = (-\infty, \frac{1}{2N} S^{\frac{N}{2p}} - L\gamma^{\frac{p^*}{p^*-q}}),$$

for every $\gamma > 0$ with g small enough with respect to the norm $\|.\|_*$.

Now, for the non-local case, we make the following assumption:

$$(K_2)\ \widehat{K}(t) \geq K(t)t \text{ for } t > 0 \text{ , with } \widehat{K}(t) = \textstyle\int_0^t K(s)ds.$$

Accordingly, we can report our main result.

Theorem 1.2. *Under the hypotheses $(K_1), (K_2)$ and $q \in (p, p^*)$, there exists $\gamma^* > 0$, such that problem (1.1) has at least a nontrivial solution for all $\gamma \geq \gamma^*$, provided g is small enough in the norm $\|.\|_*$ of $(W^{1,p}(\Omega))^*$.*

The existence of solutions for problem (1.1) remains largely uncharted territory within the realm of variational methods. As in our forthcoming paper, problem (1.1) can be construed as a Schrödinger equation entwined with a non-local term. The interplay between this nonlocal term and the critical nonlinearity prevents us from using the variational methods in a standard way. Establishing new estimates adjusted to Kirchhoff equations, which entail the utilization of Palais–Smale sequences, is imperative for our endeavor. Let us point out that although the idea was used before for other problems, adapting the procedure to our problem is not trivial at all, owing to the appearance of the non-local term and Robin boundary condition.

In [16], the authors presented a bifurcation-type theorem that describes the dependence of the set of positive solutions for a Robin problem with a concave-convex term.

The paper [10] addressed a nonlinear Robin problem driven by the (p, q)-Laplacian in addition to an indefinite potential term. It is shown that, under minimal conditions on the nonlinearity, the problem admits a nodal solution.

In [7], El Khalil investigates the existence of at least one nondecreasing sequence of positive eigenvalues by applying minimax arguments on a C^1-manifold.

Regarding the eigenvalues of the (p-q)-Laplacian with homogeneous Dirichlet boundary conditions, the author in [19] established the existence of two nontrivial (weak) solutions.

Throughout this paper, we consider the C^1-functional energy:

$$\begin{aligned}\Phi_\gamma(u) =& \frac{1}{p}\widehat{K}\left(\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p \, d\sigma_x\right) - \frac{\gamma}{q}\int_\Omega a(x)|u|^q dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx \\ & - \int_\Omega g(x)u dx.\end{aligned}$$

Note that

$$\Phi'_\gamma(u).v = K(\| u \|^p)\Big(\int_\Omega \|\nabla u|^{p-2}\nabla u.\nabla v dx + \int_{\partial\Omega} |u|^{p-2}uv \, d\sigma_x\Big)$$

$$- \gamma \int_\Omega a(x)|u|^{q-2}uvdx - \int_\Omega |u|^{p^*-2}uvdx - \int_\Omega g(x)vdx,$$

for all $v \in X$, where

$$X = W^{1,p}(\Omega) := \{u \in L^p(\Omega) \quad : \quad \int_\Omega |\nabla u|^p dx < \infty \ \}.$$

By a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [17,20], without (P.S) condition, there exists a sequence $(u_n)_n \subset W^{1,p}(\Omega)$ such that

$$\Phi_\gamma(u_n) \to c_\alpha \ \ \textit{and} \ \ \Phi'(u_n) \to 0,$$

where

$$c_\gamma = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) > 0$$

with

$$\Gamma = \{\alpha \in C([0,1], W^{1,p}(\Omega)) : \ \alpha(0) = 0, \Phi(\alpha(1)) < 0\}.$$

We recall that $u \in W^{1,p}(\Omega)$ is a weak solution of problem (1.1) if it verifies

$$K\left(\| u \|^p\right)\Big(\int_\Omega \nabla u|^{p-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta|u|^{p-2}u \, d\sigma_x\Big) - \int_\Omega \gamma a(x)|u|^{q-2}uvdx$$
$$- \int_\Omega |u|^{p^*-2}uvdx - \int_\Omega g(x)vdx = 0,$$

for all $v \in W^{1,p}(\Omega)$. So the critical points of Φ_γ are solutions of problem (1.1).

2. Auxiliary results and proofs

Let $L^s(\Omega)$ be the Lebesgue space equipped with the norm $|u|_s = \left(\int_\Omega |u|^s dx\right)^{\frac{1}{s}}$, $1 \le s < \infty$ and let $W^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\| u \| = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}.$$

From [1,6] the following norm,

$$\Big(\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} \beta|u|^p \, d\sigma_x\Big)^{\frac{1}{p}}, \quad \beta > 0$$

is equivalent with the usual $\|u\|$.

Now we can define the best Sobolev constant:

$$S = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \beta|u|^p \, d\sigma}{(\int_\Omega |u|^{p^*} dx)^{\frac{p}{p^*}}}.$$

2.1. Proof of Theorem 1.1:

Assume that (u_n) is a Palais-Smale sequence for Φ_γ. A standard argument leads to the boundedness of the sequence $(u_n)_n$. Then there exists a subsequence still denoted by $(u_n)_n$, and u in X verifying $u_n \rightharpoonup u$. Using the same arguments explored in [2], it follows that there exists a constant L depending only on p, q and N such that

$$\Phi_\gamma(u) \geq -L\gamma^{\frac{p^*}{p^*-q}}.$$

Putting $\omega_n = u_n - u$. Then by a lemma in Brezis and Lieb in [5], it follows that

$$\|\omega_n\|^p = \|u_n\|^p - \|u\|^p + o_n(1),$$

$$|\omega_n|_{p^*}^{p^*} = |u_n|_{p^*}^{p^*} - |u|_{p^*}^{p^*} + o_n(1).$$

Using the Lebesgue convergence theorem, we have,

$$\int_\Omega g u_n \, dx \to \int_\Omega g u dx,$$

$$\int_\Omega a(x)|u_n|^{q-2}u_n \, dx \to \int_\Omega a(x)|u|^{q-2}u \, dx$$

and

$$\int_{\partial\Omega} \beta|u_n|^{p-2}u_n \, d\sigma_x \to \int_{\partial\Omega} \beta|u|^{p-2}u \, d\sigma_x.$$

From the last three previous formulas, we obtain

$$\|\omega_n\|^p - |\omega_n|_{p^*}^{p^*} = o_n(1)$$

and

$$\frac{1}{p}\|\omega_n\|^p - \frac{1}{p^*}|\omega_n|_{p^*}^{p^*} = c - \Phi_\gamma(u) + o_n(1).$$

According to hypothesis that (u_n) is bounded in X. There exists $a \geq 0$ such that

$$\|\omega_n\|^p \to a.$$

Thus,

$$|\omega_n|_{p^*}^{p^*} \to a.$$

Let S denote the best Sobolev constant in the embedding $X \subset L^{p^*}(\Omega)$. Using the inequality (6) in Theorem 1 in [4],

$$(S-\varepsilon)|u|_{p^*}^p \leq \|u\|^p + B_\varepsilon|u|_p^p, \ \forall u \in W^{1,p}(\Omega),$$

which is a variant to Cherrier's inequality for the fourth order case, and passing to the limit, for arbitrary $\varepsilon > 0$, we get

$$(S-\varepsilon)a^{\frac{p}{p^*}} \leq a,$$

then,

$$Sa^{\frac{p}{p^*}} \leq a.$$

We claim that $a = 0$; if not, assume that $a > 0$. Then from the previous inequality

$$a \geq S^{\frac{N}{p}}.$$

On the other hand, we have

$$\frac{1}{N} a = c - \phi_\gamma(u),$$

thereby,

$$c \geq \frac{1}{N} S^{\frac{N}{p}} - L\gamma^{\frac{p^*}{p^*-\alpha}}.$$

However, this conclusion contradicts the hypothesis. Therefore,

$$a = 0$$

and

$$u_n \to u \ in \ X.$$

2.2. Proof of Theorem 1.2

In the sequel, we compare the minimax level c_γ with a suitable number which involves the constant S.

Lemma 2.1. *There exist $\sigma > 0, \rho > 0$ and $e \in W^{1,p}(\Omega)$ with $\| e \| > \rho$ such that*
(i) $\inf_{\|u\|=\rho} \Phi_\gamma(u) \geq \sigma > 0$;
(ii) $\Phi_\gamma(e) < 0$.

Proof.
(i) From the Hölder's inequality and the compact embedding theorem, we have

$$\begin{aligned}\Phi_\gamma(u) \geq & \frac{k_0}{p} \int_\Omega |\nabla u|^p dx + \frac{k_0 \beta}{p} \int_{\partial\Omega} |u|^p \, d\sigma_x - \frac{\gamma}{q} \int_\Omega a(x)|u|^q dx \\ & - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx - \int_\Omega g(x) u dx \\ \geq & C_0 \| u \|^p - \frac{C_1 \gamma}{q} |a|_\theta \| u \|^q - \frac{1}{p^* S^{\frac{p^*}{p}}} \| u \|^{p^*} - |g|_{p'} |u|_p \\ \geq & C_0 \| u \|^p - \frac{C_1 \gamma}{q} |a|_\theta \| u \|^q - C_2 \| u \|^{p^*} - C_3 \|g\|_* \| u \|, \end{aligned} \tag{2.1}$$

with $\theta = p^*/[p^* - q]$ and $C_0, C_1, C_2, C_3 > 0$.

Since $q \in (p, p^*)$, then for $\| u \| = \rho > 0$ small enough, we may find $\sigma > 0$ such that

$$\inf_{\|u\|=\rho} \Phi_\gamma(u) \geq \sigma > 0,$$

where $\|g\|_*$ is small.

(ii) Fix $v_0 \in C_0^\infty(\Omega) \backslash \{0\}$ with $v_0 \geq 0$ in Ω and $\| v_0 \| = 1$.

$$\Phi_\gamma(tv_0) \leq M|t|^p - \gamma |t|^\theta \int_\Omega a(x) v_0^\theta dx + C - \frac{|t|^{p^*}}{p^*} \int_\Omega v_0^{p^*} dx - |t| \int_\Omega g(x) v_0 dx,$$

with M and C two positive constants, then it follows that

$$\Phi_\gamma(tv_0) \to -\infty \text{ as } |\text{t}| \to \infty.$$

□

Lemma 2.2. $\lim_{\gamma\to+\infty} c_\gamma = 0$.

Proof. Let v_0 be the function defined in Lemma 2.1. Then there is $t_\gamma > 0$ such that $\Phi_\gamma(t_\gamma v_0) = \max_{t\geq 0} \Phi_\gamma(tv_0)$, thereafter,

$$K(\| t_\gamma v_0\|^p)t_\gamma^p \| v_0\|^p = \gamma t_\gamma^q \int_\Omega a(x)|v_0|^q dx + t_\gamma^{p^*} \int_\Omega |v_0|^{p^*} dx + t_\gamma^2 \int_\Omega g(x)v_0^2 dx. \quad (2.2)$$

From (K_2), there is $c > 0$, such that

$$\widehat{K}(s) \leq c|s| \quad for \quad all \quad s > s_0 > 0.$$

Hence

$$ct_\gamma^p \| v_0 \|^p \geq \gamma t_\gamma^q \int_\Omega a(x)|v_0|^q dx + t_\gamma^{p^*} \int_\Omega |v_0|^{p^*} dx + t_\gamma^2 \int_\Omega g(x)v_0^2 dx$$

and then t_γ is bounded, so there exists a sequence $\gamma_n \to +\infty$ and $t_* \geq 0$ with $t_{\gamma_n} \to t_*$ as $n \to +\infty$, and thus

$$K(\| t_{\gamma_n} v_0 \|^p)t_{\gamma_n}^p \| v_0 \|^p < C, \forall n \in \mathbb{N},$$

with C a positive constant, then we assert that

$$\gamma_n t_*^q \int_\Omega a(x)|v_0|^q dx + t_*^{p^*} \int_\Omega |v_0|^{p^*} dx \leq C, \forall n \in \mathbb{N}.$$

Hence, we claim that $t_* = 0$; if not, let $t_* > 0$ and then the last inequality becomes

$$\gamma_n t_*^q \int_\Omega a(x)|v_0|^q dx + t_*^{p^*} \int_\Omega |v_0|^{p^*} dx \to +\infty$$

as $n \to +\infty$, which is absurd, so $t_* = 0$.

Taking $\gamma_0(t) = te$, with $\gamma_0 \in \Gamma$, then we get

$$0 < c_\gamma \leq \max_{t\in[0,1]} \Phi_\gamma(\gamma_0(t)) \leq \frac{1}{p}\widehat{K}(t_\gamma^p).$$

As we have $\widehat{K}(t_\gamma^p) \to 0$, so $\lim_{\gamma\to\infty} c_\gamma = 0$.

□

As a consequence of Lemma 2.2, there exists $\gamma^* > 0$ such that for every $\gamma \geq \gamma^*$,

$$c_\gamma < (1 - \frac{p}{p^*})(k_0 S)^{\frac{N}{p}}.$$

Lemma 2.3. *Let* $(u_n)_n \subset W^{1,p}(\Omega)$, *with* $\Phi_\gamma(u_n) \to c_\gamma$, *and* $\Phi'(u_n) \to 0$. *Then* $(u_n)_n$ *is bounded in* $W^{1,p}(\Omega)$.

Proof. Assume that $\Phi_\gamma(u_n) \to c_\gamma$, and $\Phi'_\gamma(u_n) \to 0$. Then we have

$$\begin{aligned} pc_\gamma + o(1) + o(1) \| u_n \| &= p\Phi_\gamma(u_n) - (\Phi'_\gamma(u_n).u_n) \\ &\geq A_1\gamma(1 - \frac{p}{q})|a|_\theta \| u_n \|^q + A_2(1 - \frac{p}{p^*}) \| u_n \|^{p^*} \end{aligned}$$

$$+(p-1)\int_\Omega g(x)u_n dx,$$

where $A1$ and $A_2>0$. We conclude that $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. □

As mentioned previously, we apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n)_n\subset W^{1,p}(\Omega)$ such that $\Phi_\gamma(u_n)\to c_\gamma$ and $\Phi'_\gamma(u_n)\to 0$.

Because $(u_n)_n$ is a bounded sequence in $W^{1,p}(\Omega)$, passing to a subsequence, we may find $\gamma>0$ with

$$\| u_n \| \to \gamma,$$

and it follows from the continuity of K that

$$K(\| u_n \|^p)\to K(\gamma^p).$$

On the other hand, we know that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, then

$$u_n\to u \ in\ L^s(\Omega), \quad \text{for} \ \ 1<\text{s}<\text{p}^*,$$

and

$$u_n(x)\to u(x)\ a.e\ x\in\Omega.$$

By the Lebesgue Dominated Theorem,

$$\int_\Omega a(x) \mid u_n \mid^q dx \to \int_\Omega a(x)|u|^q dx.$$

Further,

$$|\nabla u_n |^p \rightharpoonup |\nabla u |^p +\mu \ \ \text{weak}^*-\text{sense of measure},$$
$$|u_n|^{p^*}\rightharpoonup |u|^{p^*}+\nu \ \ \text{weak}^*-\text{sense of measure}.$$

Afterwards, as a consequence of the concentration compactness principle due to Lion (cf. [11]), there is an at most countable index set I such that

$$\nu=\sum_{i\in I}\nu_i\delta_i, \mu\geq\sum_{i\in I}\mu_i\delta_i$$

and

$$S\nu_i^{p/p^*}\leq\mu_i,$$

for any $i\in I$ with $(\mu_i)_i,(\nu_i)_i\subset[0,\infty)$, δ_i is the Dirac mass and $(\mu_i)_i,(\nu_i)_i$ are positive measures. We claim that $I=\emptyset$, otherwise, we have $I\neq\emptyset$ and fix $i\in I$. Take $\psi\in C_0^\infty(\Omega,[0,1])$ such that $\psi\equiv 1$ if $|x|<1$ and $\psi\equiv 0$ when $|x|>2$ with $|\nabla\psi|_\infty\leq 2$. Set $\psi_\rho(x)=\psi((x-xi)/\rho)$ for $\rho>0$. Note that $(\psi_\rho u_n)$ is bounded thus $\Phi'_\gamma(u_n).(\psi_\rho u_n)\to 0$, that is

$$\begin{aligned}&K\left(\int_\Omega|\nabla u_n|^p\right)\Big(\int_\Omega|\nabla u_n |^{p-2}\ \nabla u_n.\nabla\psi_\rho u_n dx+\int_{\partial\Omega}\beta|u_n|^{p-2}u_n\psi_\rho u_n\, d\sigma_x\Big)\\
&=-\,K\left(\int_\Omega|\nabla u_n|^p\right)\times\Big(\int_\Omega|\nabla u_n |^p\ \psi_\rho\nabla u_n dx\\
&\quad+\int_{\partial\Omega}\beta|u_n|^p\psi_\rho u_n\, d\sigma_x\Big)+\int_\Omega|u_n|^{p^*-2}u_n.\psi_\rho u_n dx\end{aligned}$$

$$+\gamma\int_\Omega a(x)|u_n|^{q-2}u_n\psi_\rho u_n dx+\int_\Omega g(x)\psi_\rho u_n\,dx+O_n(1).$$

As it is known that $B_{2\rho}(x_i)$ is the support of the functional ψ_ρ and by applying Hölder inequality so, we obtain

$$\begin{aligned}|\int_\Omega|\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho u_n dx|&\le\int_{B_{2\rho}(x_i)}|\nabla u_n|^{p-1}\;|u_n\nabla\psi_\rho|dx\\&\le\Big(\int_{B_{2\rho}(x_i)}|\nabla u_n|^p\Big)^{\frac{1}{p'}}\Big(\int_{B_{2\rho}(x_i)}|u_n\nabla\psi_\rho|^p dx\Big)^{\frac{1}{p}}\\&\le C\Big(\int_{B_{2\rho}(x_i)}|u_n\nabla\psi_\rho|^p dx\Big)^{\frac{1}{p}}.\end{aligned}$$

By the Dominated Convergence Theorem, we entail that

$$\int_{B_{2\rho}(x_i)}|u_n\nabla\psi_\rho|^p dx\to 0$$

when $n\to\infty$ and $\rho\to 0$.

Thus,

$$\lim_{\rho\to 0}[\lim_n\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho]=0.$$

On the other hand, we recall that $K(\parallel u_n\parallel^p)$ converges to $K(\alpha^p)$, so we reach

$$\lim_{\rho\to 0}[\lim_n K(\parallel u_n\parallel^p)\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho]=0.$$

Similarly,

$$\lim_{\rho\to 0}\lim_n[K(\parallel u_n\parallel^p)\int_{\partial\Omega}\beta|u_n|^{p-2}u_n\psi_\rho u_n\,d\sigma_x]=0,$$

$$\lim_{\rho\to 0}\lim_n[\int_\Omega a(x)|u_n|^{q-2}u_n\psi_\rho u_n\,dx]=0,$$

$$\lim_{\rho\to 0}\lim_n[\int_\Omega g(x)\psi_\rho u_n\,dx]=0,$$

therefore

$$\int_\Omega K(\gamma^p)\psi_\rho d\mu+O_\rho(1)\le\int_\Omega\psi_\rho d\nu.$$

Tending ρ to zero we conclude that

$$\nu_i\ge K(\gamma^P)\mu_i\ge k_0\mu_i.$$

From the definition of ν and μ we have

$$\nu_i\ge(k_0S)^{N/p}.$$

It does not make sense. Indeed, let $i\in I$ be such that

$$\nu_i\ge(k_0S)^{N/p}.$$

Since $(u_n)_n$ is a $(PS)_{c_\gamma}$ for the functional Φ_γ, then

$$\begin{aligned} pc_\gamma &= p\Phi_\gamma(u_n) = p\Phi_\gamma(u_n) - \Phi'_\gamma(u_n).u_n + O_n(1) \\ &\geq (1-\frac{p}{p^*})\int_\Omega \psi_\rho |u_n|^{p^*} dx + O_n(1). \end{aligned} \tag{2.3}$$

Letting $n \to +\infty$, we obtain

$$pc_\gamma \geq (1-\frac{p}{p^*})\sum_{i\in I}\psi_\rho(x_i)\nu_i = (1-\frac{p}{p^*})\sum_{i\in I}\nu_i \geq (1-\frac{p}{p^*})\,(k_0 S)^{\frac{N}{p}},$$

which cannot occur (because $\lim_{\gamma\to\infty} c_\gamma = 0$), thereafter I is empty and thereby $u_n \to u$ in $L^{p^*}(\Omega)$.

On the other hand,

$$\begin{aligned} &K(\|u_n\|^p)\left(\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx \right.\\ &\left. + \int_{\partial\Omega} \beta\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u)\;d\sigma_x\right) \\ =&\Phi'_\gamma(u_n).\,(u_n-u) + \gamma\int_\Omega a(x)|u_n|^{q-2}u_n(u_n-u)dx + \int_\Omega g(x)(u_n-u)dx \\ &+ \int_\Omega |u_n|^{p^*-2}u_n(u_n-u)dx - K(\|u_n\|^p)\left(\int_\Omega |\nabla u|^{p-2}\nabla u\,(\nabla u_n - \nabla u)\,dx\right.\\ &\left.+\int_{\partial\Omega}\beta|u|^{p-2}u\,(u_n-u)\;d\sigma_x\right). \end{aligned}$$

In view of $u_n \rightharpoonup u$, a standard argument (similar to those found in [13]) shows that

$$\nabla u_n(x) \to \nabla u(x)\ a.e\ x\in\Omega,$$

and

$$u_n(x) \to u(x)\quad a.e\quad x\in\Omega,$$

then

$$\begin{aligned} &K(\|u_n\|^p)\Big(\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx \\ &+\int_{\partial\Omega}\beta\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u)\,d\sigma_x\Big) \to 0. \end{aligned}$$

Using the following inequality,

$\forall x, y \in \mathbb{R}^N$

$$|\,x-y\,|^\eta \leq 2^\eta(|\,x\,|^{\gamma-2}\,x - |\,y\,|^{\eta-2}\,y).(x-y)\quad if\quad \eta\geq 2,$$

$$|\,x-y\,|^2 \leq \frac{1}{\eta-1}(|\,x\,|+|\,y\,|)^{2-\eta}(|\,x\,|^{\gamma-2}\,x - |\,y\,|^{\eta-2}\,y).(x-y)\quad if\quad 1<\eta<2,$$

where $x.y$ is the inner product in $\mathbb{R}^N$, we get

$$c\;k_0\,\Big(\int_\Omega |\nabla u_n - \nabla u|^p\,dx + \int_{\partial\Omega}\beta|u_n-u|^p\,d\sigma_x\Big)$$

$$\leq K\left(\|u_n\|^p\right)\Big(\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx$$
$$+\int_{\partial\Omega}\beta\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u)\,d\sigma_x\Big).$$

Consequently,

$$\| u_n - u \| \to 0,$$

which will imply that

$$u_n \to u \ in \ W^{1,p}(\Omega).$$

Thus

$$\Phi_\gamma(u) = c_\gamma,\ \Phi'_\gamma(u) = 0$$

and we get the solution u_γ, which is a mountain pass type.

Acknowledgements

The anonymous referees deserve many thanks for their careful reading and comments.

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