Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1870–1885
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1870

On a Class of Nonlinear Elliptic Problems Involving the $\alpha(z)$-Biharmonic Operator with an $l(z)$-Hardy Term

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Abstract By applying the Mountain Pass Theorem, we establish the existence of a weak solution for a class of nonlinear elliptic problem involving an $\alpha(z)$-biharmonic operator and with an $l(z)$-hardy term in a bounded domain of $\mathbb{R}^N$. Provided that certain additional assumptions are made regarding the nonlinearities, the corresponding functional will satisfy the Palais-Smale condition.

Keywords $\alpha(z)$-biharmonic operator, variable exponents, $l(z)$-Hardy term, Hardy-Rellich inequality, Mountain Pass Theorem

MSC(2010) 58E05, 35J60, 47J05.

1. Introduction

In this paper, we investigate the existence of weak solution of the following elliptic problem involving an $\alpha(z)$-biharmonic operator and $l(z)$-hardy term

$$\begin{cases} \Delta^2_{\alpha(z)}\upsilon = \lambda_1 \dfrac{|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}} + \lambda_2 Q(z)|\upsilon|^{\beta(z)-2}\upsilon + \lambda_3 g(z,\upsilon) & in \quad \mathfrak{D}, \\ \\ \Delta\upsilon = \upsilon = 0 & on \quad \partial\mathfrak{D}, \end{cases} \tag{1.1}$$

where $\mathfrak{D}$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary. We indicate by $\gamma(z) := dist(z, \partial\mathfrak{D})$ the distance from the point $z \in \mathfrak{D}$ to the boundary $\partial\mathfrak{D}$, $\Delta^2_{\alpha(z)}\upsilon = \Delta\left(|\Delta\upsilon|^{\alpha(z)-2}\Delta\upsilon\right)$ is the $\alpha(z)$-biharmonic operator, the exponents α, β and l are continuous functions on $\overline{\mathfrak{D}}$, λ_1, λ_2, λ_3 are three positive parameters, $g : \mathfrak{D} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and Q is an indefinite weight function.

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Nonlinear singular elliptic problems have been a popular topic of study in recent years. They arise in some parts of science, such as boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, nonlinear electrorheological fluids and the flow in porous media. This has led to a great deal of excitement and interest from a number of authors in recent years, as the investigation of the existence and multiplicity of solutions for problems involving biharmonic, α-biharmonic and $\alpha(z)$-biharmonic operators, where α is a continuous function, has attracted significant interest (see [2, 4, 13–15, 18, 20, 23–25]).

The same problem, for $\lambda_2 = \lambda_3 = 0$ is studied by Laghzal and Touzani [18]. The authors determined that there is at least one non-decreasing sequence of non-negative eigenvalues for their problem.

In [1], Taarabti, El Allali and Hadddouch studied the existence of solutions to a nonhomogeneous eigenvalue problem with $\lambda_1 = \lambda_2 = 0$, by considering different situations with respect to the growth and they proved that a continuous family of eigenvalues exists.

In [16], the present author studied the existence of the following fourth-order, nonlinear elliptic problem

$$\begin{cases} \Delta^2_{\alpha(z)}\upsilon + a(z)|\upsilon|^{\alpha(z)-2}\upsilon = \lambda f(z,\upsilon) & \text{in } \mathfrak{D}, \\ \\ \upsilon = \Delta\upsilon = 0 & \text{on } \partial\mathfrak{D}, \end{cases}$$

for $\lambda > 0$, by using the Mountain Pass Theorem.

The remaining sections are organised as follows. In Section 2, we present fundamental results for the generalized Lebesgue-Sobolev $L^{\alpha(z)}(\mathfrak{D})$ and $W^{m,\alpha(z)}(\mathfrak{D})$. Moreover, the Mountain Pass Theorem is recalled (Theorem 2.2). Section 3, we prove that weak solutions exist for (1.1) by presenting several lemmas.

2. Preliminaries

For the reader's convenience, we recall in what follows some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces $L^{\alpha(z)}(\mathfrak{D})$ and $W^{m,\alpha(z)}(\mathfrak{D})$ where $\mathfrak{D}$ is an open subset of $\mathbb{R}^{\mathbb{N}}$(see for example [3, 6, 7, 9, 11, 12, 17, 19, 22]).

Let

$$C_+(\overline{\mathfrak{D}}) = \{\alpha \in C(\overline{\mathfrak{D}}) : \alpha(z) > 1, \text{ for every } z \in \overline{\mathfrak{D}}\}.$$

For every $\alpha \in C_+(\overline{\mathfrak{D}})$, we define

$$\alpha^+ = \max\{\alpha(z);\ z \in \overline{\mathfrak{D}}\} \textit{ and } \alpha^- = \min\{\alpha(z);\ z \in \overline{\mathfrak{D}}\}.$$

The generalized Lebesgue space $L^{\alpha(z)}(\mathfrak{D})$ is defined as

$$L^{\alpha(z)}(\mathfrak{D}) = \left\{\upsilon : \mathfrak{D} \to \mathbb{R},\ \textit{measurable and } \int_{\mathfrak{D}} |\upsilon(z)|^{\alpha(z)} dz < \infty\right\}.$$

We endow it with the Luxemburg norm

$$\|\upsilon\|_{\alpha(z)} = \inf\left\{\theta > 0 : \int_{\mathfrak{D}} \left|\frac{\upsilon(z)}{\theta}\right|^{\alpha(z)} dz \le 1\right\}.$$

Proposition 2.1. *[9, 10, 21] The space $\left(L^{\alpha(z)}(\mathfrak{D}), \|.\|_{\alpha(z)}\right)$ is separable, uniformly convex, reflexive and its conjugate space is $L^{\beta(z)}(\mathfrak{D})$ where $\beta(z)$ is the conjugate function of $\alpha(z)$, i.e*

$$\frac{1}{\alpha(z)} + \frac{1}{\beta(z)} = 1 \quad for\ every\ z \in \mathfrak{D}.$$

For $\upsilon \in L^{\alpha(z)}(\mathfrak{D})$ and $v \in L^{\beta(z)}(\mathfrak{D})$, the Hölder inequality

$$\Big| \int_{\mathfrak{D}} \upsilon(z) v(z) dz \Big| \leq \left(\frac{1}{\alpha^-} + \frac{1}{\beta^-}\right) \|\upsilon\|_{\alpha(z)} \|v\|_{\beta(z)} \leq 2\|\upsilon\|_{\alpha(z)} \|v\|_{\beta(z)},$$

holds true.

We denote by modular ρ the quantity

$$\rho(\upsilon) = \int_{\mathfrak{D}} |\upsilon|^{\alpha(z)} dz.$$

Proposition 2.2. *[9, 17] if $\upsilon \in L^{\alpha(z)}(\mathfrak{D})$ and $\alpha^+ < \infty$, then*

1. *$\|\upsilon\|_{\alpha(z)} < 1 (= 1; > 1)$ equivalent $\rho(\upsilon) < 1 (= 1; > 1)$;*
2. *if $\|\upsilon\|_{\alpha(z)} > 1$, then $\|\upsilon\|_{\alpha(z)}^{\alpha^-} \leq \rho(\upsilon) \leq \|\upsilon\|_{\alpha(z)}^{\alpha^+}$;*
3. *if $\|\upsilon\|_{\alpha(z)} < 1$, then $\|\upsilon\|_{\alpha(z)}^{\alpha^+} \leq \rho(\upsilon) \leq \|\upsilon\|_{\alpha(z)}^{\alpha^-}$.*

For every positive integer m, the Sobolev space with the variable exponent $W^{m,\alpha(z)}$ is given by

$$W^{m,\alpha(z)}(\mathfrak{D}) = \left\{ \upsilon \in L^{\alpha(z)}(\mathfrak{D}) : D^\alpha \upsilon \in L^{\alpha(z)}(\mathfrak{D}), \quad |\alpha| \leq m \right\},$$

equipped with the norm

$$\|\upsilon\|_{m,\alpha(z)} = \sum_{|\kappa| \leq m} \Big| D^\kappa \upsilon \Big|_{\alpha(z)},$$

where $D^\kappa \upsilon = \frac{\partial^{|\kappa|}}{\partial z_1^{\kappa_1} \partial z_2^{\kappa_2} \ldots \partial z_N^{\kappa_N}} \upsilon$, with $\kappa = (\kappa_1, \ldots, \kappa_N)$ a multi-index and $|\kappa| = \sum_{i=1}^N \kappa_i$.

The space $W^{m,\alpha(z)}(\mathfrak{D})$ is also a separable and reflexive Banach space. We refer the reader to the papers to [8, 9, 17].

Theorem 2.1. *Let us consider the case where α and s are elements of $C(\overline{\mathfrak{D}})$, such that $s(z) \leq \alpha_m^*(z)$ for all $z \in \mathfrak{D}$, In this situation, a continuous embedding is available:*

$$W^{m,\alpha(z)}(\mathfrak{D}) \subset L^{s(z)}(\mathfrak{D}),$$

where

$$\alpha_m^*(z) = \begin{cases} \frac{N\alpha(z)}{N - m\alpha(z)} & if \quad m\alpha(z) < N, \\ +\infty \quad if & m\alpha(z) \geq N. \end{cases}$$

A change from the symbol $\leq$ to $<$ results in a compact embedding.

We'll call $W_0^{m,\alpha(z)}(\mathfrak{D})$ the closure of $C_0^\infty(\mathfrak{D})$ in $W^{m,\alpha(z)}(\mathfrak{D})$.

In this paper, we shall look in the following space for a weak solution to problem (1.1).

$$\mathcal{W} = W_0^{1,\alpha(z)}(\mathfrak{D}) \cap W^{2,\alpha(z)}(\mathfrak{D}),$$

equipped with the norm

$$\|\upsilon\|_{\mathcal{W}} = \|\upsilon\|_{1,\alpha(z)} + \|\upsilon\|_{2,\alpha(z)}.$$

As stated in [26], the norm $\|.\|_{\mathcal{W}}$ is equivalent to the norm $\|\Delta.\|_{\alpha(z)}$ in the space $\mathcal{W}$. Therefore, the norms $\|.\|_{2,\alpha(z)}$, $\|.\|_{\mathcal{W}}$ and $\|\Delta.\|_{\alpha(z)}$ are equivalent. We may consider the following norm to be an equivalent norm in the space $\mathcal{W}$:

$$\|\upsilon\| = \|\Delta\upsilon\|_{\alpha(z)},$$

i.e.

$$\|\upsilon\| = \inf\Big\{\theta > 0 : \int_{\mathfrak{D}} \Big|\frac{\Delta\upsilon(z)}{\theta}\Big|^{\alpha(z)} dz \le 1\Big\}.$$

From Proposition 2.2 we get the following modular-type inequalities.

Proposition 2.3. *[9, 17] If $\upsilon \in L^{\alpha(z)}(\mathfrak{D})$ and $\alpha^+ < \infty$, then*

1. $\|\upsilon\| < 1 (= 1; > 1)$ *equivalent* $\displaystyle\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)} dz < 1 (= 1; > 1)$;
2. *if* $\|\upsilon\| > 1$, *then* $\|\upsilon\|^{\alpha^-} \le \displaystyle\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)} dz \le \|\upsilon\|^{\alpha^+}$;
3. *if* $\|\upsilon\| < 1$, *then* $\|\upsilon\|^{\alpha^+} \le \displaystyle\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)} dz \le \|\upsilon\|^{\alpha^-}$.

In the third section we need the $l(\cdot)$-Hardy-Rellich inequality, which is given in the following lemma.

Lemma 2.1. *[18] Assume that $1 < l^- \le l^+ < \alpha^- \le \alpha^+ < \frac{N}{2}$ and $l^+ < \alpha_2^*(z)$, for any $z \in \overline{\mathfrak{D}}$. Then there exists a positive constant C such that the $l(\cdot)$-Hardy-Rellich inequality*

$$\int_{\mathfrak{D}} \frac{1}{\alpha(z)} |\Delta\upsilon|^{\alpha(z)} dz \le C \int_{\mathfrak{D}} \frac{1}{l(z)} \frac{|\upsilon|^{l(z)}}{\gamma(z)^{2l(z)}} dz,$$

holds in one of the following cases for all $\upsilon \in W_0^{2,\alpha(z)}(\mathfrak{D})$:

- $|\upsilon| \le \gamma(z)^2$ *and* $|\Delta\upsilon| \ge 1$.
- $|\upsilon| \ge \gamma(z)^2$ *and* $|\Delta\upsilon| \le 1$.

Now, we present the theorem underlying this work, the Mountain Pass Theorem:

Theorem 2.2. *Let $(X, \|.\|_X)$ be a Banach space. Assume that $\phi \in C^1(X, \mathbb{R})$, $\phi(0) = 0$ and satisfies the three conditions*

1. *There exists $\varrho, b > 0$ such that $\phi(\upsilon) \ge b$ for $\|\upsilon\|_X = \varrho$.*
2. *There exists $\upsilon_0 \in X$ with $\|\upsilon_0\|_X > \varrho$ and such that $\phi(\upsilon_0) \le 0$.*

3. *ϕ satisfies the condition of $(PS)_c$, that is, any sequence $(\upsilon_n)_n \subset X$ such that $\phi(\upsilon_n) \to c$ and $\phi'(\upsilon_n) \to 0$ in X^* as $n \to \infty$, has a convergent subsequence..*

Then c is a critical value of ϕ, with

$$c = \inf_{\iota \in \Delta} \max_{h \in [0,1]} \phi(\iota(h)),$$

where Δ is the set of all paths connecting the origin to υ_0 of X:

$$\Delta = \{\iota \in C([0,1], X),\ \iota(0) = 0,\ \iota(1) = \upsilon_0\}.$$

3. Basic assumptions and technical lemmas

In this section, we look at the existence of weak solutions to (1.1). To establish the existence result, we outline the assumptions pertinent to our problem. We assume that $\mathfrak{D} \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\mathfrak{D}$, $\alpha \in C_+(\overline{\mathfrak{D}})$ satisfies the log-Hölder continuity condition, $\beta, l \in C_+(\overline{\mathfrak{D}})$ and $\alpha(z) < \frac{N}{2}$ with $1 < l^- < l^+ < \alpha^- < \alpha^+ < \beta^- < \beta^+ \leq \alpha_2^*(z)$ and $g : \mathfrak{D} \times \mathbb{R} \to \mathbb{R}$ is a function such that:

($\mathfrak{H}_1$) g is a Carathéodory function, such that

$$|g(z,\upsilon)| \leq r(z)|\upsilon|^{\frac{\alpha(z)}{s_1(z)}}, \text{ for all } (z,\upsilon) \in \mathfrak{D} \times \mathbb{R},$$

where $r \in L^{s_1(z)}(\mathfrak{D})$ is non-negative function and $\frac{1}{s_1(z)} + \frac{1}{\alpha(z)} = 1$.

($\mathfrak{H}_2$) There exists $\alpha^+ < \Theta < \beta^-$, such that

$$0 \leq \Theta G(z,\upsilon) \leq \upsilon g(z,\upsilon), \quad \text{for all } z \in \mathfrak{D},$$

where $G(z,\upsilon) = \int_0^\upsilon g(z,t)dt$.

($\mathfrak{H}_3$) The potential $Q \in L^\infty(\mathfrak{D}) \cap L^{s_2(z)}(\mathfrak{D})$ is non-negative function, and $\frac{1}{s_2(z)} + \frac{1}{\beta(z)} = 1$.

Before presenting our main result, let us first recall the definition of weak solutions to equation (1.1).

Definition 3.1. $\upsilon \in \mathcal{W}$ is a weak solution of (1.1), if for all $\varphi \in \mathcal{W}$

$$\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)-2}\Delta\upsilon\Delta\varphi dz - \lambda_1 \int_{\mathfrak{D}} \frac{|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\varphi dz - \lambda_2 \int_{\mathfrak{D}} Q(z)|\upsilon|^{\beta(z)-2}\upsilon\varphi dz - \lambda_3 \int_{\mathfrak{D}} g(z,\upsilon)\varphi dz = 0.$$

Theorem 3.1. *If the hypotheses ($\mathfrak{H}_1$)-($\mathfrak{H}_3$) are satisfied, then problem (1.1) has a non-trivial weak solution for all $\lambda_1 \in (0,\lambda_1^*)$, $\lambda_2 \in (0,\lambda_2^*)$ and $\lambda_3 \in (0,\lambda_3^*)$.*

The energy functional corresponding to the problem (1.1) is defined by the following equation

$$\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon) = \int_{\mathfrak{D}} \frac{1}{\alpha(z)}|\Delta\upsilon|^{\alpha(z)}dz - \int_{\mathfrak{D}} \frac{\lambda_1}{l(z)}\frac{|\upsilon|^{l(z)}}{\gamma(z)^{2l(z)}}dz - \int_{\mathfrak{D}} \lambda_2\frac{Q(z)}{\beta(z)}|\upsilon|^{\beta(z)}dz - \int_{\mathfrak{D}} \lambda_3 G(z,\upsilon)dz.$$

Lemma 3.1. *The functional* $\phi_{\lambda_1,\lambda_2,\lambda 3}$ *is well defined and* $C^1(\mathbb{W},\mathcal{R})$. *Moreover*

$$\langle \phi'_{\lambda_1,\lambda_2,\lambda 3}(\upsilon),\varphi\rangle = \int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)-2}\Delta\upsilon\Delta\varphi dz - \lambda_1\int_{\mathfrak{D}} \frac{|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\varphi dz$$
$$- \lambda_2\int_{\mathfrak{D}} Q(z)|\upsilon|^{\beta(z)-2}\upsilon\varphi dz - \lambda_3\int_{\mathfrak{D}} g(z,\upsilon)\varphi dz.$$

By combining $(\mathfrak{H}_1)$ with $(\mathfrak{H}_3)$, it is easy to see that $\phi'_{\lambda_1,\lambda_2,\lambda_3}$ belongs to the topological dual of $\mathcal{W}$.

Lemma 3.2. *There exist* $\lambda_1^*,\lambda_2^*,\lambda_3^*>0$ *such that for any* $\lambda_1\in(0,\lambda_1^*)$, $\lambda_2\in(0,\lambda_2^*)$ *and* $\lambda_3\in(0,\lambda_3^*)$ *there exist* $\varrho,b>0$ *such that* $\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon)\geq b$ *on* $\|\upsilon\|=\varrho$.

Proof. By $l(z)$-Hardy-Rellich inequality, we have

$$\int_{\mathfrak{D}} \frac{1}{\alpha(z)}|\Delta\upsilon|^{\alpha(z)}dz \leq C\int_{\mathfrak{D}} \frac{1}{l(z)}\frac{|\upsilon|^{l(z)}}{\gamma(z)^{2l(z)}}dz, \tag{3.1}$$

then

$$\int_{\mathfrak{D}} \frac{1}{\alpha(z)}|\Delta\upsilon|^{\alpha(z)}dz - \lambda_1\int_{\mathfrak{D}} \frac{1}{l(z)}\frac{|\upsilon|^{l(z)}}{\gamma(z)^{2l(z)}}dz$$
$$\geq \frac{1}{\alpha^+}\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)}dz - \frac{\lambda_1}{C\alpha^+}\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)}dz$$
$$\geq \left(\frac{1}{\alpha^+}-\frac{\lambda_1}{C\alpha^+}\right)\int_{\mathfrak{D}} |\Delta\upsilon|^{\alpha(z)}dz.$$

On the other hand, by the use of the Hölder inequality, we get

$$\int_{\mathfrak{D}} |G(z,\upsilon)|dz \leq \int_{\mathfrak{D}} \left|\frac{r(z)}{\beta(z)}|\upsilon|^{\beta(z)}\right| dz$$
$$\leq \frac{2}{\beta^-}\|r\|_{s_1(z)}\|\upsilon|^{\beta(z)}\Big|_{\alpha(z)}$$
$$\leq \frac{2}{\beta^-}\|r\|_{s_1(z)}\|\upsilon\|^{\beta^i}_{\alpha(z)},$$

with $\alpha(z)\leq\alpha_2^*(z)$. Theorem 2.1 which gives us the embedding $\mathcal{W}\hookrightarrow L^{\alpha(z)}(\mathfrak{D})$ is continuous, and we can find a constant $c_1>0$ such that:

$$\|\upsilon\|_{\alpha(z)}\leq c_1\|\upsilon\|, \forall \upsilon\in\mathcal{W}.$$

Then

$$\int_{\mathfrak{D}} |G(z,\upsilon)|dz \leq \frac{2c_1}{\beta^-}\|r\|_{s_1(z)}\|\upsilon\|^{\beta i},$$

where

$$i=\pm \quad \text{if} \quad \|\upsilon\|\gtrless 1.$$

And we obtain

$$\int_{\mathfrak{D}} \Big|\frac{Q(z)}{\beta(z)}|\upsilon|^{\beta(z)}\Big|dz \leq \frac{2}{\beta^-}\|Q\|_{s_2(z)}\|\upsilon|^{\beta(z)}\Big|_{s_2'(z)}$$
$$\leq \frac{2}{\beta^-}\|Q\|_{s_2(z)}\|\upsilon\|^{\beta^i}_{\beta(z)s_2'(z)}.$$

Since the embedding $\mathcal{W} \hookrightarrow L^{s'(z)\beta(z)}(\mathfrak{D})$ is continuous, we can find a constant $c_2 > 0$ such that:

$$\|\upsilon\|_{s_2'(z)\beta(z)} \leq c_2\|\upsilon\|, \forall \upsilon \in \mathcal{W}.$$

Then

$$\int_{\mathfrak{D}} \Big|\frac{Q(z)}{\beta(z)}|\upsilon|^{\beta(z)}\Big| dz \leq \frac{2c_2}{\beta^-}\|Q\|_{s_2(z)}\|\upsilon\|^{\beta^i},$$

with

$$i = \pm \quad \text{if} \quad \|\upsilon\| \gtrless 1.$$

The above gives us

$$\begin{aligned}
\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon) &= \int_{\mathfrak{D}} \frac{1}{\alpha(z)}|\Delta \upsilon|^{\alpha(z)} dz - \lambda_1 \int_{\mathfrak{D}} \frac{1}{l(z)} \frac{|\upsilon|^{l(z)}}{\gamma(z)^{2l(z)}} dz - \int_{\mathfrak{D}} \lambda_2 \frac{Q(z)}{\beta(z)}|\upsilon|^{\beta(z)} dz \\
&\quad - \int_{\mathfrak{D}} \lambda_3 G(z,\upsilon) dz \\
&\geq \left(\frac{1}{\alpha^+} - \frac{\lambda_1}{C\alpha^+}\right) \int_{\mathfrak{D}} |\Delta \upsilon|^{\alpha(z)} dz - \frac{2\lambda_2 c_2}{\beta^-}\|Q\|_{s_2(z)}\|\upsilon\|^{\beta^i} \\
&\quad - \frac{2\lambda_3 c_1}{\beta^-}\|r\|_{s_1(z)}\|\upsilon\|^{\beta^i} \\
&\geq \left(\frac{1}{\alpha^+} - \frac{\lambda_1}{C\alpha^+}\right)\|\upsilon\|^{\alpha^i} - \left(\frac{2\lambda_2 c_2}{\beta^-}\|Q\|_{s_2(z)} + \frac{2\lambda_3 c_1}{\beta^-}\|r\|_{s_1(z)}\right)\|\upsilon\|^{\beta^i},
\end{aligned}$$

and c_1, c_2 are positives constants, for any $\upsilon \in \mathcal{W}$, with $\|\upsilon\| = \varrho$, we have

$$\begin{aligned}
\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon) &\geq \left(\frac{1}{\alpha^+} - \frac{\lambda_1}{C\alpha^+}\right)\|\upsilon\|^{\alpha^i} - \left(\frac{2\lambda_2 c_2}{\beta^-}\|Q\|_{s_2(z)} + \frac{2\lambda_3 c_1}{\beta^-}\|r\|_{s_1(z)}\right)\|\upsilon\|^{\beta^i} \\
&= \left(\frac{1}{\alpha^+} - \frac{\lambda_1}{C\alpha^+}\right)\varrho^{\alpha^i} - \left(\frac{2\lambda_2 c_2}{\beta^-}\|Q\|_{s_2(z)} + \frac{2\lambda_3 c_1}{\beta^-}\|r\|_{s_1(z)}\right)\varrho^{\beta^i} \\
&= \varrho^{\beta^i}\left(\left(\frac{1}{\alpha^+} - \frac{\lambda_1}{C\alpha^+}\right)\varrho^{\alpha^i-\beta^i} - \frac{2\lambda_2 c_2}{\beta^-}\|Q\|_{s_2(z)} - \frac{2\lambda_3 c_1}{\beta^-}\|r\|_{s_1(z)}\right),
\end{aligned}$$

where

$$i = \pm \quad \text{if} \quad \|\upsilon\| \gtrless 1.$$

Putting

$$\lambda_1^* = \frac{C\big(\beta^- - 6\big)}{\beta^-} < C, \quad \lambda_2^* = \frac{\varrho^{\alpha^i-\beta^i}}{\alpha^+ c_2\|Q\|_{s_2(z)}}, \quad \lambda_3^* = \frac{\varrho^{\alpha^i-\beta^i}}{\alpha^+ c_1\|r\|_{s_1(z)}},$$

then for any $\lambda_1 \in (0,\lambda_1^*)$, $\lambda_2 \in (0,\lambda_2^*)$, $\lambda_3 \in (0,\lambda_3^*)$ and $\upsilon \in \mathcal{G}$, with $\|\upsilon\| = \varrho$ sufficiently small, there exists $b = \frac{2\varrho^{\alpha^i-\beta^i}}{\alpha^+\beta^-}$ such that

$$\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon) \geq b > 0.$$

□

Lemma 3.3. *There exists $\upsilon_0 \in \mathcal{W}$ with $\|\upsilon_0\| > \varrho$ such that $\phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_0) < 0$.*

Proof. Now, we demonstrate that $\upsilon_0 \neq 0$ i.e. υ_0 is a weak nontrivial solution of problem (1.1). Let $z_0 \in \mathfrak{D}_0$. Since $\alpha, \beta \in C_+(\bar{\mathfrak{D}})$, we can choose $a > 0$ small enough such that $B_a(z_0) \subset \mathfrak{D}_0$ and $\alpha_0^- := \min_{z \in B_a(z_0)} \alpha(z) < \beta_0^+ := \max_{z \in B_a(z_0)} \beta(z)$. Now, let us choose $\psi \in C_0^\infty(\mathfrak{D})$ with $|\psi| \leq 1$, $\|\psi\|_{W^{2,\alpha(z)}(B_a(z_0)) \cap W_0^{1,\alpha(z)}(B_a(z_0))} \leq c(a)$ and $|\psi|_{L^{s(z)}(B_a(z_0))} > 0$. Thus, for any $0 < t < \delta$ we deduce from (3.1) that

$$
\begin{aligned}
&\phi_{\lambda_1,\lambda_2,\lambda 3}(t\psi) \\
=&\int_{\mathfrak{D}} \frac{1}{\alpha(z)} |\Delta t\psi|^{\alpha(z)} dz - \int_{\mathfrak{D}} \frac{\lambda_1}{l(z)} \frac{|t\psi|^{l(z)}}{\gamma(z)^{2\beta(z)}} dz - \int_{\mathfrak{D}} \lambda_2 \frac{Q(z)}{\beta(z)} |t\psi|^{\beta(z)} dz - \int_{\mathfrak{D}} \lambda_3 G(z, t\psi) dz \\
\leq& \frac{t^{\alpha_0^-}}{\alpha^-} \int_{\mathfrak{D}} |\Delta\psi|^{\alpha(z)} dz - \frac{\lambda_1 t^{\alpha_0^-}}{C} \int_{\mathfrak{D}} |\Delta\psi|^{\alpha(z)} dz - \frac{2\lambda_2 c t^{\beta^-}}{\beta^+} \int_{\mathfrak{D}} Q(z) |\psi|^{\beta(z)} dz \\
\leq& \frac{t^{\alpha_0^-}}{\alpha^-} \max\left\{c(a)^{\alpha_0^-}, c(a)^{p_0^+}\right\} - \frac{\lambda_1 t^{\alpha_0^-}}{\alpha^-} \min\left\{c(a)^{\alpha_0^-}, c(a)^{\alpha_0^+}\right\} - \frac{2\lambda_2 c t^{\beta^-}}{\beta^+} \int_{\mathfrak{D}} Q(z) |\psi|^{\beta(z)} dz.
\end{aligned}
$$

Since $\alpha_0^- < \beta_0^+$, we get $\phi_{\lambda_1,\lambda_2,\lambda_3}(t_1\psi) < 0$ by taking $0 < t_1 < \delta$ small enough. Hence, $\phi_{\lambda_1,\lambda_2,\lambda 3}(\upsilon_0) \leq \phi_{\lambda_1,\lambda_2,\lambda 3}(t_1\phi) < 0$. □

Lemma 3.4. *The functional $\phi_{\lambda_1,\lambda_2,\lambda 3}$ satisfies the Palais-Smale condition $(PS)_c$, for any $c \in \mathbb{R}$.*

Proof. Let (υ_n) be a $(PS)_c$ sequence for the functional $\phi_{\lambda_1,\lambda_2,\lambda 3}$ in $\mathcal{W}$ i.e. $\phi(\upsilon_n)$ is bounded and $\phi'_{\lambda_1,\lambda_2,\lambda 3}(\upsilon_n) \to 0$. Then the sequence υ_n is bounded in $\mathcal{W}$.

In fact, since $\phi_{\lambda_1,\lambda_2,\lambda 3}(\upsilon_n)$ is bounded, we have

$$
\begin{aligned}
C' \geq& \phi_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n) \\
=& \int_{\mathfrak{D}} \left(\frac{1}{\alpha(z)} |\Delta\upsilon_n|^{\alpha(z)} - \frac{\lambda_1}{l(z)} \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} - \lambda_2 \frac{Q(z)}{\beta(z)} |\upsilon_n|^{\beta(z)} \right) dz - \int_{\mathfrak{D}} \lambda_3 G(z, \upsilon_n) dz \\
\geq& \int_{\mathfrak{D}} \left(\frac{1}{\alpha(z)} |\Delta\upsilon_n|^{\alpha(z)} dz - \frac{\lambda_1}{l(z)} \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} dz - \lambda_2 \frac{Q(z)}{\beta(z)} |\upsilon_n|^{\beta(z)} \right) dz \\
&- \int_{\mathfrak{D}} \frac{\lambda_3 \upsilon_n}{\Theta} g(z, \upsilon_n) dz.
\end{aligned}
$$

Since

$$
\begin{aligned}
\left\langle \phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n), \upsilon_n \right\rangle =& \int_{\mathfrak{D}} |\Delta\upsilon_n|^{\alpha(z)} dz - \lambda_1 \int_{\mathfrak{D}} \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} dz - \lambda_2 \int_{\mathfrak{D}} Q(z) |\upsilon_n|^{\beta(z)} dz \\
&- \lambda_3 \int_{\mathfrak{D}} g(z, \upsilon_n) \upsilon_n dz,
\end{aligned}
$$

then

$$
\begin{aligned}
C' \geq& \frac{1}{\alpha^+} \int_{\mathfrak{D}} |\Delta\upsilon_n|^{\alpha(z)} dz - \frac{1}{l^-} \int_{\mathfrak{D}} \lambda_1 \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} dz - \frac{1}{\beta^-} \int_{\mathfrak{D}} \lambda_2 Q(z) |\upsilon_n|^{\beta(z)} dz \\
&+ \frac{1}{\Theta} \left\langle \phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n), \upsilon_n \right\rangle - \frac{1}{\Theta} \int_{\mathfrak{D}} |\Delta\upsilon_n|^{\beta(z)} dz + \frac{1}{\Theta} \int_{\mathfrak{D}} \lambda_1 \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} dz \\
&+ \frac{1}{\Theta} \int_{\mathfrak{D}} \lambda_2 Q(z) |\upsilon_n|^{\beta(z)} dz \\
\geq& \left(\frac{1}{\alpha^+} - \frac{1}{\Theta} \right) \int_{\mathfrak{D}} |\Delta\upsilon_n|^{\alpha(z)} dz + \left(\frac{1}{\Theta} - \frac{1}{l^-} \right) \int_{\mathfrak{D}} \lambda_1 \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}} dz \\
&+ \left(\frac{1}{\Theta} - \frac{1}{\beta^-} \right) \int_{\mathfrak{D}} \lambda_2 Q(z) |\upsilon_n|^{\beta(z)} dz + \frac{1}{\Theta} \left\langle \phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n), \upsilon_n \right\rangle.
\end{aligned}
$$

By contradiction, we assume that (υ_n) is unbounded in $\mathcal{W}$. In particular, for n large enough, we can choose $\|\upsilon_n\| \geq 1$. Therefore, there exists $C^{''} > 0$ in such a way that

$$-C^{''}\|\upsilon_n\| \leq \left\langle \phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n), \upsilon_n \right\rangle \leq C^{''}\|\upsilon_n\|,$$

because $\phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n) \to 0$. To that end,

$$\begin{aligned} C^{'} \geq & \left(\frac{1}{\alpha^+} - \frac{1}{\Theta}\right)\|\upsilon_n\|^{\alpha^-} + \left(\frac{1}{\Theta} - \frac{1}{l^-}\right)\int_{\mathfrak{D}} \lambda_1 \frac{|\upsilon_n|^{l(z)}}{\gamma(z)^{2l(z)}}dz \\ & + \left(\frac{1}{\Theta} - \frac{1}{\beta^-}\right)\int_{\mathfrak{D}} \lambda_2 Q(z)|\upsilon_n|^{\beta(z)}dz - \frac{1}{\Theta}C^{''}\|\upsilon_n\| \\ \geq & \left(\frac{1}{\alpha^+} - \frac{1}{\Theta}\right)\|\upsilon_n\|^{\alpha^-} - \frac{1}{\Theta}C^{''}\|\upsilon_n\|. \end{aligned}$$

If we divide by $\|\upsilon_n\|^{\alpha^-}$ in the last inequality and let $n \to \infty$, we get a contradiction. The consequence is that the sequence $\{\upsilon_n\}$ is bounded in $\mathcal{W}$. Without loss of generality, we assume that $\{\upsilon_n\}$ is weakly convergent to υ in $\mathcal{W}$. Then for all $s(z) < \alpha_2^*(z)$, $\{\upsilon_n\}$ converges strongly to υ in $L^{s(z)}(\mathfrak{D})$.

Since $\phi'_{\lambda_1,\lambda_2,\lambda 3} \longrightarrow 0$ in $\mathcal{W}^*$, we conclude that $\left\langle \phi'_{\lambda_1,\lambda_2,\lambda 3}(\upsilon_n), \upsilon_n - \upsilon \right\rangle \longrightarrow 0$. We also have $\left\langle \phi'_{\lambda_1,\lambda_2,\lambda 3}(\upsilon), \upsilon_n - \upsilon \right\rangle \longrightarrow 0$ as $n \longrightarrow \infty$ because υ_n converges weakly to υ in $\mathcal{W}$.

Thus,

$$\left\langle \phi'_{\lambda_1,\lambda_2,\lambda 3}(\upsilon_n) - \phi_{\lambda_1,\lambda_2,\lambda 3}(\upsilon), \upsilon_n - \upsilon \right\rangle \longrightarrow 0.$$

Using $\mathfrak{H}_1$, we get

$$\begin{aligned} & \left|\lambda_3 \int_{\mathfrak{D}} \left(g(z,\upsilon_n) - g(z,\upsilon)\right)(\upsilon_n - \upsilon)\, dz\right| \\ \leq & \lambda_3 \int_{\mathfrak{D}} |g(z,\upsilon_n) - g(z,\upsilon)||\upsilon_n - \upsilon| dz \\ \leq & \lambda_3 \int_{\mathfrak{D}} r(z)\left(|\upsilon_n|^{\alpha(z)} + |\upsilon|^{\alpha(z)-2}\upsilon_n\upsilon + |\upsilon|^{\alpha(z)} + |\upsilon_n|^{\alpha(z)-2}\upsilon_n\upsilon\right) dz. \end{aligned}$$

By the Hölder inequality, we have

$$\int_{\mathfrak{D}} r(z)|\upsilon_n|^{\alpha(z)}dz \leq 2\|r\|_{s_1(z)}\||\upsilon_n|^{\alpha(z)}|_{\alpha(z)} \leq \frac{\varepsilon}{4}.$$

Using Young's inequality, we get

$$\begin{aligned} \int_{\mathfrak{D}} r(z)|\upsilon_n|^{\alpha(z)-1}|\upsilon|dz \leq & \int_{\mathfrak{D}} r(z)\left(|\upsilon_n|^{\alpha(z)} + |\upsilon|^{\alpha(z)}\right) dz \\ \leq & 2\|r\|_{s_1(z)}\left(\|\upsilon_n\|^{\alpha(z)}\|_{\alpha(z)} + \||\upsilon|^{\alpha(z)}|_{\alpha(z)}\right) \\ \leq & \frac{\varepsilon}{4}. \end{aligned}$$

Similarly, we show that the last two terms are less than $\frac{\varepsilon}{4}$. Then

$$\left|\lambda_3 \int_{\mathfrak{D}} \left(g(z,\upsilon_n) - g(z,\upsilon)\right)(\upsilon_n - \upsilon)\, dz\right| \longrightarrow 0, \ when\ n \longrightarrow \infty. \tag{3.2}$$

On the other hand, we show that

$$\left|\lambda_2 \int_{\mathfrak{D}} Q(z)\big(|\upsilon_n|^{\beta(z)-2}\upsilon_n - |\upsilon|^{\beta(z)-2}\big)(\upsilon_n - \upsilon)dz\right| \longrightarrow 0,\ when\ n \longrightarrow \infty.$$

Indeed,

$$\int_{\mathfrak{D}} Q(z)|\upsilon_n|^{\beta(z)}dz \leq 2\|Q\|_{s_2(z)}\||\upsilon_n|^{\beta(z)}|_{\beta(z)} \leq \frac{\varepsilon}{4}.$$

Using Young's inequality, we get

$$\begin{aligned}
\int_{\mathfrak{D}} Q(z)|\upsilon_n|^{\beta(z)-1}|\upsilon|dz &\leq \int_{\mathfrak{D}} Q(z)\left(|\upsilon_n|^{\beta(z)} + |\upsilon|^{\beta(z)}\right)dz \\
&\leq 2\|Q\|_{s_2(z)}\left(|\upsilon_n|^{\beta(z)}|_{\beta(z)} + \||\upsilon|^{\beta(z)}|_{\beta(z)}\right) \\
&\leq \frac{\varepsilon}{4}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\lambda_2 \int_{\mathfrak{D}} Q(z)\big(|\upsilon_n|^{\beta(z)-2}\upsilon_n - |\upsilon|^{\beta(z)-2}\big)(\upsilon_n - \upsilon)dz\right| \\
\leq&\lambda_2 \int_{\mathfrak{D}} Q(z)\left(|\upsilon_n|^{\beta(z)} + |\upsilon|^{\beta(z)-2}\upsilon_n\upsilon + |\upsilon|^{\beta(z)} + |\upsilon_n|^{\beta(z)-2}\upsilon_n\upsilon\right)dz \\
\leq&\varepsilon.
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n - |\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n - \upsilon)dz\right| \\
\leq&\int_{\{z\in\mathfrak{D}:\gamma(z)>1\}}\left|\frac{|\upsilon_n|^{l(z)-2}\upsilon_n - |\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right||\upsilon_n - \upsilon|dz \\
&+\int_{\{z\in\mathfrak{D}:\gamma(z)\leq1\}}\left|\frac{|\upsilon_n|^{l(z)-2}\upsilon_n - |\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right||\upsilon_n - \upsilon|dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n - |\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n - \upsilon)dz\right| \\
\leq&\int_{\{z\in\mathfrak{D}:\gamma(z)>1\}}\left(|\upsilon_n|^{l(z)} + |\upsilon|^{l(z)-1}\upsilon_n + |\upsilon_n|^{l(z)-1}\upsilon + |\upsilon|^{l(z)}\right)dz \\
&+\int_{\{z\in\mathfrak{D}:\gamma(z)\leq1\}}\frac{1}{\gamma(z)^2}\frac{\big(|\upsilon_n|^{l(z)} + |\upsilon|^{l(z)-2}\upsilon_n\upsilon + |\upsilon_n|^{l(z)-2}\upsilon_n\upsilon + |\upsilon|^{l(z)}\big)}{\gamma(z)^{2(l(z)-1)}}dz.
\end{aligned}$$

By applying Hölder's inequality, we obtain

$$\begin{aligned}&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n-|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n-\upsilon)dz\right|\\ \leq& c_7\Bigg(\|\upsilon_n\|_{\beta(z)}^{l(z)}+\|\upsilon\|_{s_2(z)}^{l(z)-1}\|\upsilon_n\|_{\beta(z)}+\|\upsilon_n\|_{s_2(z)}^{l(z)-1}\|\upsilon\|_{\beta(z)}\\ &+\|\upsilon_n\|_{\alpha(z)}^{l(z)}+\|\frac{\upsilon_n}{\gamma(z)^2}\|_{\beta(z)}^{l(z)}+\|\frac{\upsilon}{\gamma(z)^2}\|_{s_2(z)}^{l(z)-1}\|\frac{\upsilon_n}{\gamma(z)^2}\|_{\beta(z)}\\ &+\|\frac{\upsilon_n}{\gamma(z)^2}\|_{s_2(z)}^{l(z)-1}\|\frac{\upsilon}{\gamma(z)^2}\|_{\beta(z)}+\|\frac{\upsilon}{\gamma(z)^2}\|_{\beta(z)}^{l(z)}\Bigg).\end{aligned}$$

By (3.1), we have

$$\begin{aligned}&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n-|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n-\upsilon)dz\right|\\ \leq& c_7\Bigg(\|\upsilon_n\|_{\beta(z)}^{l(z)}+\|\upsilon\|_{s_2(z)}^{l(z)-1}\|\upsilon_n\|_{\beta(z)}+\|\upsilon_n\|_{s_2(z)}^{l(z)-1}\|\upsilon\|_{\beta(z)}+\|\upsilon_n\|_{\alpha(z)}^{l(z)}\\ &+\frac{1}{C}\Bigg(\|\Delta\upsilon_n\|_{\beta(z)}^{l(z)}+\frac{1}{C}\|\Delta\upsilon\|_{s_2(z)}^{l(z)-1}\|\Delta\upsilon_n\|_{\beta(z)}\\ &+\frac{1}{C}\|\Delta\upsilon_n\|_{s_2(z)}^{l(z)-1}\|\Delta\upsilon\|_{\beta(z)}^{l(z)}+\|\Delta\upsilon\|_{\beta(z)}\Bigg)\Bigg).\end{aligned}$$

Then

$$\begin{aligned}&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n-|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n-\upsilon)dz\right|\\ \leq& c_7\Bigg(k_1\big(\|\upsilon_n\|^{l(z)}+\|\upsilon\|^{l(z)-1}\|\upsilon_n\|+\|\upsilon_n\|^{l(z)-1}\|\upsilon\|+\|\upsilon_n\|^{\beta(z)}\big)\\ &+\frac{1}{C}\big(\|\upsilon_n\|^{l(z)}+\frac{1}{C}\|\upsilon\|^{l(z)-1}\|\upsilon_n\|+\frac{1}{C}\|\upsilon_n\|^{l(z)-1}\|\upsilon\|+\|\upsilon\|^{l(z)}\big)\Bigg),\end{aligned}$$

where k_1 is a constant given by the embedding of $W_0^{2,\alpha(\cdot)}(\mathfrak{D})$ in $L^{\beta(\cdot)}(\mathfrak{D})$. Hence

$$\begin{aligned}&\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n-|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n-\upsilon)dz\right|\\ \leq& c_7\left(k_1+\frac{1}{C}\right)\left(\|\upsilon_n\|^{l(z)}+\|\upsilon\|^{l(z)}\right)\\ &+c_7\left(k_1+\frac{1}{C^2}\right)\left(\|\upsilon\|^{l(z)-1}\|\upsilon_n\|+\|\upsilon_n\|^{l(z)-1}\|\upsilon\|\right)\\ \leq&\varepsilon.\end{aligned}$$

Since

$$\left|\int_{\mathfrak{D}}\left(\frac{|\upsilon_n|^{l(z)-2}\upsilon_n-|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}}\right)(\upsilon_n-\upsilon)dz\right|\longrightarrow 0,\ when\ n\longrightarrow\infty. \tag{3.4}$$

The last step consists of using the following elementary inequalities (see [5]):

$$\left(|\nu|^{j-2}\nu - |\varsigma|^{j-2}\varsigma\right)(\nu-\varsigma) \geq \frac{1}{2^j}|\nu-\varsigma|^j, \quad j \geq 2, \tag{3.5}$$

$$\left(|\nu|^{j-2}\nu - |\varsigma|^{j-2}\varsigma\right)(\nu-\varsigma)(|\nu|+|\varsigma|)^{2-j} \geq (j-1)|\nu-\varsigma|^2, \quad 1<j<2, \tag{3.6}$$

for all $\varsigma, \nu \in \mathbb{R}^N$. Put

$$\mathfrak{U}_{\alpha(z)} := \{z \in \mathfrak{D} : \alpha(z) \geq 2\}, \quad \mathfrak{V}_{\alpha(z)} := \{z \in \mathfrak{D} : 1 < \alpha(z) < 2\},$$

Then, from (3.5) and (3.6) it follows that

$$\int_{\mathfrak{U}_{\alpha(z)}} |\Delta \upsilon_n - \Delta \upsilon|^{\alpha(z)} dz \leq c_8 \int_{\mathfrak{D}} \Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) dz, \tag{3.7}$$

$$\int_{\upsilon_{\alpha(z)}} |\nabla \upsilon_n - \nabla \upsilon|^{\alpha(z)} dz \leq c_8 \int_{\mathfrak{D}} \Lambda^{(N)}\left(\nabla \upsilon_n, \nabla \upsilon\right) dz, \tag{3.8}$$

$$\int_{\mathfrak{V}_{\alpha(z)}} |\Delta \upsilon_n - \Delta \upsilon|^{\alpha(z)} dz \leq c_9 \int_{\mathfrak{D}} \left(\Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right)\right)^{\frac{\alpha(z)}{2}} \left(\Upsilon^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right)\right)^{(2-\alpha(z))\frac{\alpha(z)}{2}} dz, \tag{3.9}$$

$$\int_{\mathfrak{V}_{\alpha(z)}} |\nabla \upsilon_n - \nabla \upsilon|^{\alpha(z)} dz \leq c_9 \int_{\mathfrak{D}} \left(\Lambda^{(N)}\left(\nabla \upsilon_n, \nabla \upsilon\right)\right)^{\frac{\alpha(z)}{2}} \left(\upsilon^{(N)}\left(\nabla \upsilon_n, \nabla \upsilon\right)\right)^{(2-\alpha(z)\frac{\alpha(z)}{2}} dz, \tag{3.10}$$

where $\Lambda^{(k)}, \Upsilon^{(k)} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}, k = 1, N$, are defined by the following expressions

$$\Lambda^{(k)}(\nu,\varsigma) := \left(|\nu|^{\alpha(z)-2}\nu - |\varsigma|^{\alpha(z)-2}\varsigma\right)(\nu-\varsigma), \quad \Upsilon^{(k)}(\nu,\varsigma) := |\nu| + |\varsigma|,$$

for all $\varsigma, \nu \in \mathbb{R}^k, k = 1, N$.

Now, according to the definition of the function $\phi_{\lambda_1,\lambda_2,\lambda_3}$ and relations (3.7), (3.2), (3.3) and (3.4) we have

$$\begin{aligned}
0 &\leq \int_{\mathfrak{D}} \left(|\Delta \upsilon_n|^{\alpha(z)-2}\Delta \upsilon_n - |\Delta \upsilon|^{\alpha(z)-2}\Delta \upsilon\right)(\Delta \upsilon_n - \Delta \upsilon)\, dz \\
&= \left\langle \phi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon_n) - \varphi'_{\lambda_1,\lambda_2,\lambda_3}(\upsilon), \upsilon_n - \upsilon \right\rangle + \lambda_1 \int_{\mathfrak{D}} \frac{\left(|\upsilon_n|^{l(z)-2}\upsilon_n - |\upsilon|^{l(z)-2}\upsilon\right)}{\gamma(z)^{2l(z)}}(\upsilon_n - \upsilon) dz \\
&\quad + \lambda_2 \int_{\mathfrak{D}} Q(z)\left(|\upsilon_n|^{\beta(z)-2}\upsilon_n - |\upsilon|^{\beta(z)-2}\right)(\upsilon_n - \upsilon) dz + \lambda_3 \int_{\mathfrak{D}} \left(g(z,\upsilon_n) - g(z,\upsilon)\right)(\upsilon_n - \upsilon)\, dz, \\
&\to 0
\end{aligned}$$

when $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int_{\mathfrak{D}} \Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) dz = \lim_{n\to\infty} \int_{\mathfrak{D}} \Lambda^{(N)}\left(\nabla \upsilon_n, \nabla \upsilon\right) dz = 0. \tag{3.11}$$

We can therefore assume that $0 \leq \int_{\mathfrak{D}} \Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) dz < 1$.

Then, if $\int_{\mathfrak{D}} \Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) dz = 0$, then $\Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) = 0$ since $\Lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) \geq 0$ in $\mathfrak{D}$.

If $0 < \int_{\mathfrak{D}} \lambda^{(1)}\left(\Delta \upsilon_n, \Delta \upsilon\right) dz < 1$, then, due to the Young inequality

$$AB \leq \frac{A^d}{d} + \frac{B^{d'}}{d'}, \quad \forall A, B > 0, \quad \frac{1}{d} + \frac{1}{d'} = 1, \quad d,\ d' \in (1, +\infty),$$

with

$$A = \left(\Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\frac{\alpha(z)}{2}} \left(\int_{\mathfrak{V}_{\alpha(z)}} \Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) dz\right)^{\frac{-\alpha(z)}{2}},$$

$$B = \left(\Upsilon^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{(2-\alpha(z))\frac{\alpha(z)}{2}},$$

$$d = \frac{2}{\alpha(z)} \text{ and } d' = \frac{2}{2-\alpha(z)},$$

we conclude that

$$\begin{aligned}
&\left(\int_{\mathfrak{V}_{\alpha(z)}} \Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) dz\right)^{-\frac{1}{2}} \int_{V_{\alpha(z)}} \left(\Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\frac{\alpha(z)}{2}} \\
&\times \left(\Upsilon^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{(2-\alpha(z))\frac{\alpha(z)}{2}} dz \\
\le& \int_{\mathfrak{V}_{\alpha(z)}} \left(\Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\frac{\alpha(z)}{2}} \left(\int_{\mathfrak{V}_{\alpha(z)}} \Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) dz\right)^{-\frac{\alpha(z)}{2}} \\
&\times \left(C^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{(2-\alpha(z))\frac{\alpha(z)}{2}} dz \\
\le& \int_{\mathfrak{V}_{\alpha(z)}} \left(\Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) \left(\int_{\mathfrak{V}_{\alpha(z)}} \Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) dz\right)^{-\frac{1}{2}} + \left(\Upsilon^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\alpha(z)}\right) dz \\
\le& 1 + \int_{\mathfrak{D}} \left(\Upsilon^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\alpha(z)} dz.
\end{aligned}$$

Hence, by relation (3.9),

$$\begin{aligned}
&\frac{1}{c_9} \int_{\mathfrak{V}_{\alpha(z)}} |\Delta\upsilon_n - \Delta\upsilon|^{\alpha(z)} dz \\
&\le \left(\int_{\mathfrak{V}_{\alpha(z)}} \Lambda^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right) dz\right)^{\frac{1}{2}} \left(1 + \int_{\mathfrak{D}} \left(\Upsilon^{(1)}\left(\Delta\upsilon_n, \Delta\upsilon\right)\right)^{\alpha(z)} dz\right).
\end{aligned}$$

We also have

$$\begin{aligned}
&\frac{1}{c_9} \int_{\mathfrak{U}_{\alpha(z)}} |\nabla\upsilon_n - \nabla\upsilon|^{\alpha(z)} dz \\
&\le \left(\int_{\mathfrak{U}_{\alpha(z)}} \Lambda^{(N)}\left(\nabla\upsilon_n, \nabla\upsilon\right) dz\right)^{\frac{1}{2}} \left(1 + \int_{\mathfrak{D}} \left(\Upsilon^{(N)}\left(\nabla\upsilon_n, \nabla\upsilon\right)\right)^{\alpha(z)} dz\right).
\end{aligned} \tag{3.12}$$

By (3.7), (3.9), (3.11) and (3.12), we have

$$\int_{\mathfrak{D}} |\Delta\upsilon_n - \Delta\upsilon|^{\alpha(z)} dz = \int_{\mathfrak{U}_{\alpha(z)}} |\Delta\upsilon_n - \Delta\upsilon|^{\alpha(z)} dz + \int_{\mathfrak{V}_{\alpha(z)}} |\Delta\upsilon_n - \Delta\upsilon|^{\alpha(z)} dz \to 0,$$

if $n \to \infty$. In a similar way, from (3.8), (3.10), (3.11) and (3.12) we get

$$\int_{\mathfrak{D}} |\nabla\upsilon_n - \nabla\upsilon|^{\alpha(z)} dz = \int_{\mathfrak{U}_{\alpha(z)}} |\nabla\upsilon_n - \nabla\upsilon|^{\alpha(z)} dz + \int_{\mathfrak{V}_{\alpha(z)}} |\nabla\upsilon_n - \nabla\upsilon|^{\alpha(z)} dz \to 0.$$

Therefore,

$$|\upsilon_n - \upsilon|^{\alpha^+} \leq \int_{\mathfrak{D}} \left(|\Delta \upsilon_n - \Delta \upsilon|^{\alpha(z)} + |\nabla \upsilon_n - \nabla \upsilon|^{\alpha(z)} \right) d \to 0,$$

when $n \to \infty$. So, the sequence $\{\upsilon_n\}$ strongly converges to $\upsilon \in \mathcal{W}$ and the functional $\phi_{\lambda_1,\lambda_2,\lambda 3}$ satisfies the $(PS)_c$ condition in $\mathcal{W}$.

Proof of Theorem 3.1: Set

$$\Delta = \{\iota \in C([0,1], \mathcal{W}), \quad \iota(0) = 0, \quad \iota(1) = \upsilon_0\},$$
$$c = \inf_{\iota \in \Delta} \max_{h \in [0,1]} \phi(\iota(h)).$$

The energy functional $\phi_{\lambda_1,\lambda_2,\lambda 3}$ satisfies the geometric conditions of the Mountain Pass Theorem according to Lemmas 3.2, 3.3 and 3.4. Hence c is a critical value of $\phi_{\lambda_1,\lambda_2,\lambda 3}$ associated with a critical point $\upsilon \in \mathcal{W}$, which is exactly a solution of (1.1). □

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

This work does not have any conflicts of interest.

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