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# Fractional Order Dynamical Behavior of Dengue Hemorrhagic Fever with Saturation Factor

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**Abstract** In this study, we propose a modified fractional order derivative in the scope of Caputo to investigate the dynamics of Dengue fever transmission. The existence, uniqueness and boundedness of the fractional model were established using a fixed-point approach. The stability analysis of the model was done with respect to the reproduction number which was found to be stable locally and globally at infection free and endemic state respectively. The fractional order (DHF) model was estimated using the fractional Adams-Bashforth predictor-corrector technique. Additionally, the numerical validation was done to ascertain the impact of various parameters on the dynamics as a whole, as well as the effect of vaccines on the model. The graphical solutions show that the fractional order ($\alpha$) and vaccinations affect the dynamics of the model when they are varied within the model. The findings indicate that saturation of infectious individuals in the system helps to flatten out the infection transmission.

**Keywords** Fractional order, stability, dengue fever, Adam's Bashforth, Banach and Schauder's theorem

**MSC(2010)** 34H99, 49K15, 92B05.

## 1. Introduction

The dengue virus is the primary cause of dengue fever, sometimes referred to as a vector-borne illness. It has flavivirus serotypes, ranging from (DENV 1-4). This disease poses a threat to the majority of countries worldwide. The Eastern Mediterranean, Americas, Africa, and particularly the Western Pacific and South-East Asian regions are the areas most severely impacted by dengue fever worldwide. With thousands of fatalities and over 390 million cases, dengue fever is regarded as the worst vector-borne disease after malaria [5] [17] [39]. Approximately 100 nations are at risk of contracting dengue fever globally, according to a 2012 estimate. The disease is carried by a variety of mosquito species, including Aedes and particularly Aegypti [25]. In contrast to other dengue fevers, which often produce only modest morbidity and death, classical dengue fever, also known as break bone fever, heals its victims in one to two weeks after the fever first arises [13].

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Dengue hemorrhagic fever (DHF) or the shock syndrome (SS) can strike some individuals [15]. The World Health Organization (WHO) reports a larger number of (DHF) each year worldwide [37]. The primary method of transmission of the dengue virus to humans is via the bites of female mosquitoes carrying the virus [9]. There is currently no known efficient remedy for the dengue virus, with the exception of fluid substitution therapy, which can be started early. Traditional therapeutic methods are also possible. [10]. In addition to the lack of therapy for dengue virus infections, there is currently no effective vaccine available to vaccinate vulnerable persons. Despite the lack of an efficient dengue virus vaccine on the market, the WHO recommended some advancement in the field of dengue vaccine development. Regarding the dengue fever vaccine, a study that was released in 2015 said that the first vaccine was created in Mexico. [38]. There are several mathematical models in the research [24], [22] [16], [36], that address the dengue dynamics. The aforementioned references present several viewpoints on the dengue infection dynamics, including dynamic analysis, vaccine, and optimum control analysis. [33] contains a few recent scientific publications that provided actual data on dengue infection. In [1], a mathematical framework of an infection with dengue was created, addressing the disease using actual data from Pakistan and providing some practical methods for eradicating mosquito-borne disease. Analysis of the dynamics of dengue with identical strains and their reinfection has been found in [2]. A basic discussion of dengue modeling in both deterministic and stochastic approaches is done in [8]. In [35] a hybrid system for dengue prediction is examined. In [21], the authors examined the dynamics of dengue disease and its mutual infection with the Zika virus, demonstrating the protective effects of immunization against dengue hemorrhagic fever (DHF).

Many academics have used fractional differential equations to demonstrate a variety of infectious and non-infectious diseases in the modern era [31], [30], [4], [40]. Omame et al. investigated the analytical solution of a fractional order mathematical framework for a tumor displaying mutations in cells. [37]. The study [28] investigates bifurcation and optimal control in co-infection scenarios of ZIKA and SARS-CoV-2. Andrew et al. [29] used a fractional approach to investigate the dynamics of complex systems. The paper in [28] provides basic rules for modeling complicated systems, which facilitates the creation of axiomatic methods in the field. Atede et al. [3] focused on Nigerian actual data as a base of study and investigated a fractional-order vaccination model for COVID-19 that took ecological propagation into consideration. Studies of TB in relation to internal reactivated and foreign re-infection have been conducted using Frégal-order models [7].

## 1.1. Objective

This work attempts to examine the dynamics and management of dengue hemorrhagic fever (DHF) transmission in the population using a fractional-order model and also attempts to show how the fractional order variable and memory indices influence the dynamics of the disease.

## 1.2. Preliminaries and definitions

We include some notations, definitions, and established results that are required for the sequel. This study uses Liouville-Caputo's fractional derivative.

**Definition 1.1.** Let $J$ be a real function in the range $C_\chi, where\ \chi \in R, t > 0$. There exists $n > \chi$, which is Real, then $J(t) = t^n k(t)$. [6], [14].

**Definition 1.2.** [32], [14] The fractional integrating operator of order $\alpha \geq 0$ of the Riemann Liouville for a function $J \in L^1(a, b)$ is defined as

$$I^\alpha J(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)} J(\tau) d\tau, \quad t > 0, \alpha > 0,$$

where $\Gamma$ is the Gamma function and $I^0 J(t) = f(t)$.

## 2. Formulation of model

Fractional order model are used in mathematics to represent the elaborate behaviors of disease structures, providing a more practical representation than integer models. This technique expands the capacity of integer-order models and has proven to be an indispensable tool for infectious disease modeling. Important components such as past disease status, memory of prior patterns of disease when the fractional order is less than 1 and variation in genetic profiles are all included in these models. Furthermore, as opposed to ordinary differential equations, which are bounded by smaller regions, fractional order derivatives have a global character. The efficiency of fractional calculus advances as the uniformity of range of the system gets better [18].

Moving forward, In order to illustrate the dynamics of dengue hemorrhagic fever (DHF) transmission between mosquito and human populations, we build a fractional mathematical model for the disease. The model makes some assumptions; three mosquito compartments and four human compartments make up our seven compartmental model, which is equipped with biological parameters. Three classes comprise the whole mosquito population which are ($S_m$ ) susceptible, Exposed ($E_m$ ) and Infected ($I_m$). The human population as a whole is split into four classes: Susceptible ($S_h$) ,Exposed ($E_h$), Infected ($I_h$) and Recovered ($R_h$). This is represented with a schematic flow below.

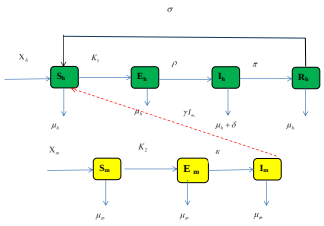


**Figure 1.** The Schematic flow of the proposed model.

The set of fractional differential equations is derived from figure 1:

$$\left.\begin{aligned}
{}_0^cD_t^\alpha S_h(t) &= X_h - (K_1 + \mu_h)S_h + \sigma R_h + \gamma I_m,\\
{}_0^cD_t^\alpha E_h(t) &= K_1 S_h - (\rho + \mu_h)E_h,\\
{}_0^cD_t^\alpha I_h(t) &= \rho E_h - (\mu_h + \delta + \pi)I_h,\\
{}_0^cD_t^\alpha R_h(t) &= \pi I_h - (\mu_h + \sigma)R_h,\\
{}_0^cD_t^\alpha S_m(t) &= X_m - (K_2 + \mu_m)S_m,\\
{}_0^cD_t^\alpha E_m(t) &= K_2 S_m - (\mu_m + \varepsilon)E_m,\\
{}_0^cD_t^\alpha I_m(t) &= \varepsilon E_m - (\mu_m + \gamma)I_m.
\end{aligned}\right\} \tag{2.1}$$

Here, ${}_0^cD_t^\alpha(.)$ with $0 < \alpha \leq 1$, denotes the Caputo Fractional derivative of order $\alpha$ where $K_1 = \frac{e_h I}{1+a_h I}$ and $K_2 = \frac{e_m I}{1+a_m I}$ ; $e_m$ and $e_h$ are the effective contact rate for mosquito and human respectively $a_h$, $a_m$ are the saturation factors for human and mosquito populations.

**Table 1.** Description of parameters

| Parameters | Bioliogical Meaning |
|---|---|
| $X_h$ | Human recruitment rate |
| $K_1$ | Incidence rate for human |
| $K_2$ | Incidence rate for mosquitoes |
| $\sigma$ | Relapse rate |
| $\rho$ | Progression rate to infectious class |
| $\pi$ | Rate of progression to recovered |
| $X_m$ | Mosquito recruitment rate |
| $\varepsilon$ | Progression rate to infected mosquito |
| $\delta$ | Rate of disease induced death |
| $\mu_m$ | Natural death rate for mosquitoes |
| $\mu_h$ | Natural death rate for human |
| $\gamma$ | Transmission rate from $I_m$ to $S_h$ |

# 3. Qualitative analysis

## 3.1. Uniqueness and existence of solution of the fractional order model

Initial conditions and integrals are embedded in the fractional model 2.1 to establish the uniqueness and existence

$$\left.\begin{aligned}
S_h(t) &= S_h(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[X_h - (K_1+\mu_h)S_h + \sigma R_h + \gamma I_m]d\tau,\\
E_h(t) &= E_h(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[K_1 S_h - (\rho+\mu_h)E_h]d\tau,\\
I_h(t) &= I_h(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[\rho E_h - (\mu_h+\delta+\pi)I_h]d\tau,,\\
R_h(t) &= R_h(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[\pi I_h - (\mu_h+\sigma)R_h]d\tau,\\
S_m(t) &= S_m(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[X_m - (K_2+\mu_m)S_m]d\tau,\\
E_m(t) &= E_m(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[K_2 S_m - (\mu_m+\varepsilon)E_m]d\tau,\\
I_m(t) &= I_m(0) + \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{(\alpha-1)}[\varepsilon E_m - (\mu_m+\gamma)I_m]d\tau.
\end{aligned}\right\} \quad (3.1)$$

The fractional model 2.1 can be alternatively given as

$$\left.\begin{aligned}
{}_0^cD_t^\alpha S_h(t) &= t^{\alpha-1}F_1(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha E_h(t) &= t^{\alpha-1}F_2(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha I_h(t) &= t^{\alpha-1}F_3(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha R_h(t) &= t^{\alpha-1}F_4(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha S_m(t) &= t^{\alpha-1}F_5(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha E_m(t) &= t^{\alpha-1}F_6(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),\\
{}_0^cD_t^\alpha I_m(t) &= t^{\alpha-1}F_7(S_m, E_m, I_m, S_h, E_h, I_h, R_h, t),
\end{aligned}\right\} \tag{3.2}$$

where

$$\begin{aligned}
F_1 &= X_h - (K_1 + \mu_h)S_h + \sigma R_h + \gamma I_m,\\
F_2 &= K_1 S_h - (\rho + \mu_h)E_h,\\
F_3 &= \rho E_h - (\mu_h + \delta + \pi)I_h,\\
F_4 &= \pi I_h - (\mu_h + \sigma)R_h,\\
F_5 &= X_m - (K_2 + \mu_m)S_m,\\
F_6 &= K_2 S_m - (\mu_m + \varepsilon)E_m,\\
F_7 &= \varepsilon E_m - (\mu_m + \gamma)I_m.
\end{aligned} \tag{3.3}$$

This can be generalized as; ${}_0^cD_t^\alpha F(t) = t^{\alpha-1}G(t, F(t))$, with $F(0) = F_0$.

Using the last expression we obtain:

$$F(t) = F_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{(\alpha-1)}G(\tau, F(\tau))d\tau,$$

where $F(t) = [S_m(t), E_m(t), I_m(t), S_h(t), E_h(t), I_h(t), R_h(t)]$. Lets us consider a Banach space of the form $C = B \times B \times B \times B$ and $L \in [0, T]$ with a norm operator $\|F\| = \max|S_m(t) + E_m(t) + I_m(t) + S_h(t) + E_h(t) + I_h(t) + R_h(t)|$ We observe whether the terms of non-linear fulfill the following conditions:
$(B_1)$ : Let $F_1 \in C$. There exist $L > 0$ and $R > 0$ such that

$$|G(t, F_1(t))| < L\,|F_1(t)| + R.$$

$(B_2)$ : Let $F_1, F_2 \in C$. There exists $Q > 0$ which is a constant such that

$$\|G(t, F_1(t) - G(t, F_2(t)\| < Q\,|F_1(t) - F_2(t)|\,. \tag{3.4}$$

Considering equation 3.4 we have been able to establish the uniqueness and existence of the fractional order model 2.1.

## 3.2. Positivity and boundedness of the fractional model

To establish the non-negativity of the solution, we assume that the parameters of the fractional model remain positive.

$$\left.\begin{aligned}
&{}_0^cD_t^\alpha S_h(t)\,\big|_{s_h(0)=0} = X_h \geq 0,\\
&{}_0^cD_t^\alpha E_h(t)\,\big|_{E_h(0)=0} = K_1 S_h \geq 0,\\
&{}_0^cD_t^\alpha I_h(t)\,\big|_{I_h(0)=0} = \rho E_h \geq 0,\\
&{}_0^cD_t^\alpha R_h(t)\,\big|_{R_h(0)=0} = \pi I_h \geq 0,\\
&{}_0^cD_t^\alpha S_m(t)\,\big|_{s_m(0)=0} = X_m \geq 0,\\
&{}_0^cD_t^\alpha E_m(t)\,\big|_{E_m(0)=0} = K_2 S_m \geq 0,\\
&{}_0^cD_t^\alpha I_m(t)\,\big|_{I_m(0)=0} = \varepsilon E_m \geq 0.
\end{aligned}\right\} \tag{3.5}$$

From equation 3.5 the solution sets of the model remain non-negative (positive).

**Theorem 3.1.** *For $t > 0$ the fractional model 2.1 remains bounded.*

**Proof.** Let $N_h(t) = [S_h(t) + E_h(t) + I_h(t) + R_h(t)]$ which can be expressed as

$${}_0^cD_t^\alpha N_h(t) = X_h - \mu_h((S_h(t) + E_h(t) + I_h(t) + R_h(t)).$$

Simplifying further we obtain

$${}_0^cD_t^\alpha N_h(t) = X_h - \mu_h N_h(t). \tag{3.6}$$

Using the approach of integrating factor on equation 3.6 we obtain

$$N_h(t) \leq \frac{X_h}{\mu_h} + Ae^{-\mu_h t}, \tag{3.7}$$

where $A$ denotes the constant of integration. As $t \to \infty$, equation 3.7 becomes

$$N_h(t) \leq \frac{X_h}{\mu_h}.$$

Therefore the viable region of the fractional model 2.1 for human class is

$$\Pi_h = [(S_h(t), E_h(t), I_h(t), R_h(t)) \in R_+^4 : N_h(t) \leq \frac{X_h}{\mu_h}].$$

In the same manner we compute for the vector (mosquito) population

$$\Pi_m = [(S_m(t), E_m(t), I_m(t) \in R_+^3 : 0 \leq N_m(t) \leq \frac{X_m}{\mu_m}].$$

Additionally, let $[(S_m(t) + E_m(t) + I_m(t) + S_h(t) + E_h(t) + I_h(t) + R_h(t)]$ be positive and remain within limit (bounded). Then there exist positive constants $L_1, L_2, L_3, L_4, L_5, L_6, L_7$ such that

$$\|S_h(t)\| \leq L_1, \|E_h(t)\| \leq L_2, \|I_h(t)\| \leq L_3 \, \|R_h(t)\| \leq L_4,$$

$$\|S_m(t)\| \leq L_5, \|E_m(t)\| \leq L_6, \|I_m(t)\| \leq L_7.$$

Hence the solution sets of fractional model 2.1 is bounded.

The fractional model 2.1 can be translated into a fixed point example (problem). Let $J : C \to C$ be given as

$$J(F(t)) = F_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} G(\tau, F(\tau)) d\tau, \tag{3.8}$$

provided that the assumptions given above are valid with the mapping $G : [0, B]xC \to R$ Then there exists a solution that satisfies the fractional model 2.1. □

**Proof.** Let $K$ be bounded in $C$ and let $a_1 > 0$. Then there exists $|G(\tau, F(\tau))| \leq a_1$, for all $F \in K$. We obtain $\|J(F)\| \leq F_0 + \frac{a_1}{\Gamma(\alpha)} \max \int_0^t (t-\tau)^{(\alpha-1)} d\tau$, $\|J(F)\| \leq F_0 + \frac{a_1}{\Gamma(\alpha)} \max \int_{t_0}^1 (1-\tau)^{(\alpha-1)} d\tau$ and $\|J(F)\| \leq F_0 + \frac{a_1}{\Gamma(\alpha)} t^{(\alpha-1)} \Phi(\alpha)$. From the last expression $J$ is bounded uniformly. Let's assume that $b_1 < b_2 \leq B$. With this we obtain $|G(F(b_2)) - G(F(b_1))| \leq \frac{1}{\Gamma(\alpha)} (K|F(t)| + L)\Phi(\alpha) b_2^{(\alpha-1)} - b_1^{(\alpha-1)})$. If $b_2 \to b_1$ then $|G(F(b_2)) - G(F(b_1))| \to 0$. Therefore, by the theorem of Arzela-Ascoli $J$ is continuous. Subsequently the fractional order model 2.1 has at least one solution by the approach of Schauder's theorem [14]. □

## 3.3. Points of equilibrium

Equilibrium point are the steady state solution of fractional model 2.1. Two types of equilibrium points will be discussed; the absence of infection points and endemic equilibrium. We solve this by equating the state variables to zero

$$\begin{aligned} {}_0^c D_t^\alpha S_m(t) = {}_0^c D_t^\alpha E_m(t) = {}_0^c D_t^\alpha I_m(t) = {}_0^c D_t^\alpha S_h(t) \\ = {}_0^c D_t^\alpha E_h(t) = {}_0^c D_t^\alpha I_h(t) = {}_0^c D_t^\alpha R_h(t) = 0, \end{aligned}$$

which can be alternatively given as:

$$\begin{aligned} &X_h - (K_1 + \mu_h) S_h + \sigma R_h + \gamma I_m = 0, \\ &K_1 S_h - (\rho + \mu_h) E_h = 0, \\ &\rho E_h - (\mu_h + \delta + \pi) I_h = 0, \\ &\pi I_h - (\mu_h + \sigma) R_h = 0, \\ &X_m - (K_2 + \mu_m) S_m = 0, \\ &K_2 S_m - (\mu_m + \varepsilon) E_m = 0, \\ &\varepsilon E_m - (\mu_m + \gamma) I_m = 0. \end{aligned} \tag{3.9}$$

At the disease free equilibrium point we obtain

$$P^0 = (S_m^0, E_m^0, I_m^0, S_h^0, E_h^0, I_h^0, R_h^0) = (\frac{X_m}{\mu_m}, 0, 0, \frac{X_h}{\mu_h}, 0, 0, 0). \tag{3.10}$$

At the endemic equilibrium point we obtain

$$\begin{aligned} P^* &= (S_m^*, E_m^*, I_m^*, S_h^*, E_h^*, I_h^*, R_h^*) \\ &= (\tfrac{X_m}{(K_2+\mu_m)}, \tfrac{K_2 S_m^*}{(\mu_m+\varepsilon)}, \tfrac{\varepsilon E_m^*}{(\mu_m+\gamma)}, \tfrac{\sigma R_h^* - X_h}{(K_1+\mu_h)}, \tfrac{K_1 S_h^*}{(\rho+\mu_h)}, \tfrac{\rho E_h^*}{(\mu_h+\delta+\pi)}, \tfrac{\pi I_h^*}{(\mu_h+\sigma)}). \end{aligned} \tag{3.11}$$

## 3.4. Fundamental reproduction number

The fundamental reproduction number of the fractional model 2.1 is computed using the next generation matrix technique [11], [23]. The human and mosquito compartments will be done differently. $R_{0h} = \Delta(M_h V_h^{-1})$, $R_{0m} = \Delta(M_m V_m^{-1})$ where $M_h$ and $M_v$ are the matrices of new infection for human and mosquito compartments respectively, while $V_h$ and $V_m$ are the matrices of secondary infections in the human mosquito compartments respectively and $\Delta$ denotes the maximum absolute value of the matrix.

$$M_h = \begin{bmatrix} 0 & e_h S \\ 0 & 0 \end{bmatrix},$$

$$V_h = \begin{bmatrix} -(\rho + \mu_h) & 0 \\ \rho & -(\delta + \mu_h + \pi) \end{bmatrix}.$$

The spectral radius $\Delta$ of $M_h V_h^{-1}$ yields

$$R_{0h} = \frac{e_h X_h \rho}{\mu_h(\rho + \mu_h)(\delta + \mu_h + \pi)}, \tag{3.12}$$

$$M_m = \begin{bmatrix} 0 & e_m S \\ 0 & 0 \end{bmatrix},$$

$$V_m = \begin{bmatrix} -(\varepsilon + \mu_m) & 0 \\ \varepsilon & -\mu_m \end{bmatrix}.$$

The spectral radius $\Delta$ of $M_m V_m^{-1}$

$$R_{0m} = \frac{e_m X_m \varepsilon}{\mu_m^2(\varepsilon + \mu_m)}. \tag{3.13}$$

## 3.5. Stability analysis

The stability Analysis will be established at infection-free and at endemic equilibra.

**Theorem 3.2.** *If $R_{0h} < 1$ and $R_{0m} < 1$, then the fractional order model 2.1 is asymptotically stable locally at infection free equilibrium.*

**Proof.** Let

$$\begin{aligned} H_1 &= X_h - (K_1 + \mu_h)S_h + \sigma R_h + \gamma I_m, \\ H_2 &= K_1 S_h - (\rho + \mu_h)E_h, \\ H_3 &= \rho E_h - (\mu_h + \delta + \pi)I_h, \\ H_4 &= \pi I_h - (\mu_h + \sigma)R_h, \\ H_5 &= X_m - (K_2 + \mu_m)S_m, \\ H_6 &= K_2 S_m - (\mu_m + \varepsilon)E_m, \\ H_7 &= \varepsilon E_m - (\mu_m + \gamma)I_m. \end{aligned} \tag{3.14}$$

Using the Jacobian function we have:

$$J_0(S_h, E_h, I_h, R_h, S_m, E_m, I_m),$$

$$J_0 = \begin{bmatrix} -\mu_h & 0 & e_h S & \sigma & 0 & 0 & \gamma \\ K_1 & -(\rho+\mu_h) & e_h S & 0 & 0 & 0 & 0 \\ 0 & \rho & -(\mu_h+\delta+\pi) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi & -(\mu_h+\sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_m & 0 & e_m S \\ 0 & 0 & 0 & 0 & 0 & -(\mu_m+\varepsilon) & e_m S \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\mu_m+\gamma) \end{bmatrix}. \tag{3.15}$$

Steadily solving equation 3.15 we obtain the following;

$$\lambda_1 = -\mu_h, \lambda_2 = -(\pi+\sigma), \lambda_3 = -(\mu_m), \lambda_4 = -(\rho+\mu_h),$$
$$\lambda_5 = -((\mu_h+\delta+\pi) + \tfrac{\rho X_h e_h}{\mu_h(\mu_h+\rho)}), \lambda_6 = -(\mu_h+\delta+\pi)$$

and $\lambda_7 = -((\mu_m) + \frac{\varepsilon X_m e_m}{\mu_m(\mu_m+\varepsilon)})$.

From $\lambda_5$ and $\lambda_7$ we can easily establish that $R_{0h} < 1$ and $R_{0m} < 1$. $\lambda_5 = -((\mu_h+\delta+\pi) + \frac{\rho X_h e_h}{\mu_h(\mu_h+\rho)})$ can be re-written as

$$\lambda_5 = -(\mu_h+\delta+\pi)(1 - \frac{\rho X_h e_h}{(\mu_h+\delta+\pi)(\mu_h+\rho)\mu_h}) \leq -(\mu_h+\delta+\pi)(1-R_{0h}). \tag{3.16}$$

Subsequently from equation 3.16 we can deduce that $R_{0h} < 1$. With the same approach, using $\lambda_7 = -((\mu_m) + \frac{\varepsilon X_m e_m}{\mu_m(\mu_m+\varepsilon)})$ we have

$$\lambda_7 = -(\mu_m)(1 - \frac{\varepsilon X_m e_m}{\mu_m^2(\mu_m+\varepsilon)}) \leq -(\mu_m)(1-R_{0m}). \tag{3.17}$$

Similarly from equation 3.17, $R_{0m} < 1$, which completes the proof. □

Next we establish the stability of the endemic equilibrium point (EE) globally.

**Theorem 3.3.** *If $R_{0h} > 1$ and $R_{0m} > 1$, then the fractional order 2.1 is globally asymptotically stable at (EE).*

**Proof.** According to Lasalle"s principle [19], we develop the Lyapunov function which is given below

$$\begin{aligned} &L(S_m, E_m, I_m, S_h, E_h, I_h, R_h) \\ =&[S_h - S_h^*(\ln\frac{S_h}{S_h^*})] + [E_h - E_h^*(\ln\frac{E_h}{E_h^*})] + [I_h - I_h^*(\ln\frac{I_h}{I_h^*})] + [R_h - R_h^*(\ln\frac{R_h}{R_h^*})] \\ &+ [S_m - S_m^*(\ln\frac{S_m}{S_m^*})] + [E_m - E_m^*(\ln\frac{E_m}{E_m^*})] + +[I_m - I_m^*(\ln\frac{I_m}{I_m^*})]. \end{aligned} \tag{3.18}$$

Differentiating equation 3.18 along the solution part of model 2.1 we obtain

$$\begin{aligned}\frac{dL}{dt} =&(1-\frac{S_h^*}{S_h})_0^c D_t^\alpha S_h(t) + (1-\frac{E_h^*}{E_h})_0^c D_t^\alpha E_h(t) + (1-\frac{I_h^*}{I_h})_0^c D_t^\alpha I_h(t)\\ &+(1-\frac{R_h^*}{R_h})_0^c D_t^\alpha R_h(t) + (1-\frac{S_m^*}{S_m})_0^c D_t^\alpha S_m(t)\\ &+(1-\frac{E_m^*}{E_m})_0^c D_t^\alpha E_m(t) + (1-\frac{I_m^*}{I_m})_0^c D_t^\alpha I_m(t).\end{aligned} \tag{3.19}$$

Replacing ${}_0^c D_t^\alpha S_h(t), {}_0^c D_t^\alpha E_h(t), {}_0^c D_t^\alpha I_h(t), {}_0^c D_t^\alpha R_h(t), {}_0^c D_t^\alpha S_m(t), {}_0^c D_t^\alpha E_m(t), {}_0^c D_t^\alpha I_m(t)\}$ in equation 3.19 we obtain

$$\begin{aligned}\frac{dL}{dt} =&(1-\frac{S_h^*}{S_h})[X_h-(K_1+\mu_h)S_h+\sigma R_h+\gamma I_m]\\ &+(1-\frac{E_h^*}{E_h})[K_1S_h-(\rho+\mu_h)E_h]+(1-\frac{I_h^*}{I_h})[\rho E_h-(\mu_h+\delta+\pi)I_h]\\ &+(1-\frac{R_h^*}{R_h})[\pi I_h-(\mu_h+\sigma)R_h]+(1-\frac{S_m^*}{S_m})[X_m-(K_2+\mu_m)S_m]\\ &+(1-\frac{E_m^*}{E_m})[K_2S_m-(\mu_m+\varepsilon)E_m]+(1-\frac{I_m^*}{I_m})[\varepsilon E_m-(\mu_m+\gamma)I_m].\end{aligned} \tag{3.20}$$

For $S_m = S_m^*, E_m = E_m^*, I_m = I_m^*, S_h = S_h^*, I_h = I_h^*, R_h = R_h^*$, we obtain $\frac{dL}{dt} = 0$. Hence the fractional order model 2.1 equipped with positive starting points, $S_h = E_h = I_h = R_h = S_m = E_m = I_m \geq 0$ are found in

$$\Pi = \{S_m, E_m, I_m, S_h, E_h, I_h, R_h \in R_+^7; 0 \leq N_m \leq \frac{X_m}{\mu_m}; 0 \leq N_h \leq \frac{X_h}{\mu_h}\}.$$

As the system approaches the endemic equilibrium point $P*$, it suffices that $P*$ is GAS. This completes the proof. □

## 3.6. Numerical solution for model 2.1 in the sense of Caputo operator

We apply the Caputo fractional operator to study the dynamics of the fractional-order model 2.1. The proposed nonlinear fractional-order system's numerical simulation is provided by the Adams-type estimator-corrector approach [10]. We consider the following Cauchy-type with respect to the order $\alpha$ Caputo operator.

$${}_0^c D_t^\alpha \chi(t) = \Theta(t, \chi(t)),$$

$$\chi^{(\kappa)}(0) = \chi_0^\kappa,$$

$0 < \alpha \leq 1, \quad 0 < t \leq \beta$, where $\kappa = 0, 1, 2...n-1$.

The above system can be transformed into Voltera equation of the form

$$\chi(t) = \sum_{\kappa=0}^{n-1} \chi_0^{(\kappa)} \frac{t^\kappa}{\kappa!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} \chi(\tau, \chi(\tau)) d\tau. \tag{3.21}$$

This estimated corrector method is proposed with respect to algorithm of Adams bashforth-Moulten for us to achieve numerical values, taking the step size to be $b = \frac{\beta}{N}$, $t_i = ih$ where $i = 0.1.2...N \in Z^+$, *and* $X_i \approx X(t_i)$, which can be discretized using the corrector formular [23].

$$
\begin{aligned}
S_{h(n+1)} &= \sum_{i=0}^{n-1} S_{0h}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[X_h - (K_1+\mu_h)S_{hi} + \sigma R_{hi} + \gamma I_{mi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[X_h - (K_1+\mu_h)S_{h(n+1)}^{PF} + \sigma R_{h(n+1)}^{PF} + \gamma I_{h(n+1)}^{PF}], \\
E_{h(n+1)} &= \sum_{i=0}^{n-1} E_{0h}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[K_1 S_{hi} - (\rho+\mu_h)E_{hi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[K_1 S_{h(n+1)}^{PF} - (\rho+\mu_h)E_{h(n+1)}^{PF}], \\
I_{h(n+1)} &= \sum_{i=0}^{n-1} I_{0h}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[\rho E_{hi} - (\mu_h+\delta+\pi)I_{hi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[\rho E_{h(n+1)}^{PF} - (\mu_h+\delta+\pi)I_{h(n+1)}^{PF}], \\
R_{h(n+1)} &= \sum_{i=0}^{n-1} R_{0h}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[\pi I_{hi} - (\mu_h+\sigma)R_{hi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[\pi I_{h(n+1)}^{PF} - (\mu_h+\sigma)R_{h(n+1)}^{PF}], \\
S_{m(n+1)} &= \sum_{i=0}^{n-1} S_{0m}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[X_m - (K_2+\mu_m)S_{mi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[X_m - (K_2+\mu_m)S_{m(n+1)}^{PF}], \\
E_{m(n+1)} &= \sum_{i=0}^{n-1} E_{0m}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[K_2 S_{mi} - (\mu_m+\varepsilon)E_{mi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[K_2 S_{m(n+1)}^{PF} - (\mu_m+\varepsilon)E_{m(n+1)}^{PF}], \\
I_{m(n+1)} &= \sum_{i=0}^{n-1} I_{0m}^{(i)} \tfrac{t_{n+1}^i}{i!} + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{i,n+1})[\varepsilon E_{mi} - (\mu_m+\gamma)I_{mi}] \\
&\quad + \tfrac{b^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} (M_{n+1,n+1})[\varepsilon E_{m(n+1)}^{PF} - (\mu_m+\gamma)I_{m(n+1)}^{PF}].
\end{aligned}
\tag{3.22}
$$

$$
M_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1}, \\ 1. \end{cases} \tag{3.23}
$$

The next stage is to create the concurrence predictor formula , the predictor formula

which can be given as $X^{PF}_{n+1}$.

$$\left.\begin{aligned}
S^{PF}_{h(n+1)} &= \sum_{i=0}^{n-1} S^{(i)}_{0h}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[X_h - (K_1+\mu_h)S_{hi} + \sigma R_{hi} + \gamma I_{mi}],\\
E^{PF}_{h(n+1)} &= \sum_{i=0}^{n-1} E^{(i)}_{0h}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[K_1 S_{hi} - (\rho+\mu_h)E_{hi}],\\
I^{PF}_{h(n+1)} &= \sum_{i=0}^{n-1} I^{(i)}_{0h}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[\rho E_{hi} - (\mu_h+\delta+\pi)I_{hi}],\\
R^{PF}_{h(n+1)} &= \sum_{i=0}^{n-1} R^{(i)}_{0h}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[\pi I_{hi} - (\mu_h+\sigma)R_{hi}],\\
S^{PF}_{m(n+1)} &= \sum_{i=0}^{n-1} S^{(i)}_{0m}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[X_m - (K_2+\mu_m)S_{mi}],\\
E^{PF}_{m(n+1)} &= \sum_{i=0}^{n-1} E^{(i)}_{0m}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[K_2 S_{mi} - (\mu_m+\varepsilon)E_{mi}],\\
I^{PF}_{m(n+1)} &= \sum_{i=0}^{n-1} I^{(i)}_{0m}\frac{t^i_{n+1}}{i!} + \frac{b^\alpha}{\Gamma(\alpha+1)}\sum_{i=0}^{n}(L_{i,n+1})[\varepsilon E_{mi} - (\mu_m+\gamma)I_{mi}],
\end{aligned}\right\} \tag{3.24}$$

where $L_{i,n+1} = (n+1-i)^\alpha - (n-i)^\alpha$.

## 3.7. Congenital trace and memory tracking

To capture the dynamics of dengue hemorrhagic fever (DHF) model 2.1, we utilize the defined Caputo operator in fractional order model 2.1 with $0 < \alpha \leq 1$. Let $X(t)$ represent the fractional order derivative [40]. Then

$$D^\alpha_t \chi(t) = \Psi(\chi(t), t). \tag{3.25}$$

Using the approach of $L1$ scheme, the numerical estimate of the functional derivative of $X(t)$ is given by:

$$D^\alpha_t \chi(t) \approx \frac{(dt)^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{\kappa=0}^{J-1}[\chi(t_{\kappa+1}) - \chi(t_\kappa)][(J-\kappa)^{1-\alpha} - (J-1-\kappa)^{1-\alpha}\right]. \tag{3.26}$$

$L1$ scheme is one of the most accurate approaches used for partitioning fractional derivative due to its memory term and rate of convergence. Using equations 3.25 and 3.26 we obtain

$$\begin{aligned}
\chi(t_J) \approx\ & D^\alpha_t \Gamma(2-\alpha)F(\chi(t),t) + \chi(t_J - 1)\\
& - \left[\sum_{\kappa=0}^{J-2}[\chi(t_{\kappa+1}) - \chi(t_\kappa)][(J-\kappa)^{1-\alpha} - (J-1-\kappa)^{1-\alpha}\right].
\end{aligned} \tag{3.27}$$

The fractional derivative solution can be termed the variance between the memory tracking and Markov term, where the Markov term is fitted by a gamma function. For $\alpha < 1$ , all past analyses of disease are taken into account for memory tracking, which is solely based on time $t$.

## 3.8. Numerical simulation

In this section numerical simulation are carried out using MATLAB software to check the effect of the fractional order $\alpha$ and vaccination therapy on the fractional order model 2.1, the Adams Bashforth Predictor-corrector technique has been inserted into the numerical scheme.

# 4. Discussion of results

The stability and mathematical well-posedness (accuracy) of the fractional order model 2.1 at $P^0$ and $P*$ are illustrated in 2(a) and 2(b) for fraction order $\alpha = 0.9$ with

$$X_h = 0.05, K_1 = 0.091193, \sigma = 0.086, \rho = 0.10, \pi = 0.005, X_m = 0.12,$$
$$\varepsilon = 0.0003465, \mu_m = 0.057, \delta = 0.001865, \mu_h = 0.001127, K_1 = 0.091196,$$

all taken as fitted values.

With the fitted values given, different values of fractional order are incorporated in each class to check the dynamics of the fractional model 2.1. Figures:2(c), 2(d), 2(e), 2(f) which is the human population density increase as the fractional order $\alpha$ increases over a period of 100 days.

Similarly Figures:2(g), 2(h), 2(i) and 2(j) also show an increment as the fractional order $\alpha$ increases, depicting the effect of the fractional order on the population density of mosquitoes.

In Figure:2(k), the effect of disease contact rate ($e_h$) is incorporated into the infectious human individuals for $\alpha = 0.9$ to check the behavioral solution, which also increases as the effective contact rate rises. Its attendant effect can be seen in Figure:2(l) which shows the disease induced death rate ($\delta_h$).If a proper control plan is not put in place it might tend to endemic point.

In Figure:2(m), the saturation factor ($a_h$) for the fractional order $\alpha = 0.9$ is incorporated into the infectious human to check the dynamics, which depicts a decrease in this set of individuals translating to flattening out the disease (saturation in this context is treatment).

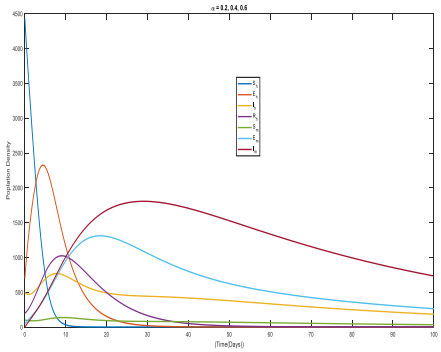
In Figure:2(n), over the course of 100 days the effective contact rate ($e_m$) is varied in the infectious mosquito population for the fractional order $\alpha = 0.9$ , and the increase is evident as the contact rate rises. Similarly, in Figure 2(j) the saturation factor ($a_h$) is inserted in the infectious mosquito where the fractional order is $\alpha = 0.9$ , demonstrating a sharp decrease in this class of mosquito. Saturation here represents measures such as insecticide use and fumigation.

The effect of the increasing rate of constant vaccination is depicted in Figure 2(o) and 2(p), where the exposed and infected compartments shows steady state solutions as the vaccination rate increases for fractional order $\alpha = 0.9$.

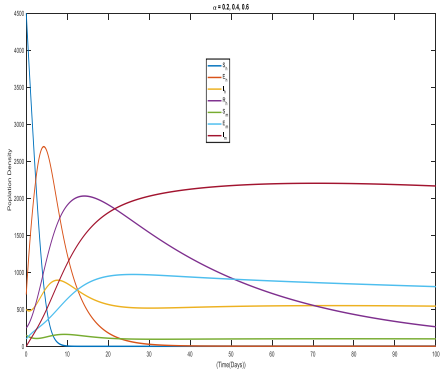
# 5. Conclusion

Using a modified fractional operator within the scope of Caputo, equipped with fractional order $\alpha$, we investigated a dengue hemorrhagic fever (DHF). Initially employing fixed-point theorem of Schauder and Banach types, we established the uniqueness and existence of the fractional model solutions. We also established the positivity and boundedness of the fractional order model within feasible regions. Furthermore, the equilibrium points for both disease-free and endemic status were verified, and the next-generation matrix technique was used to establish reproduction number ($R_0$), which was found to be less than 1, via the stability analysis using the Jacobian approach. The global stability analysis of the endemic state was verified using the Lyapunov technique for $R_0 > 1$. Through the numerical scheme,

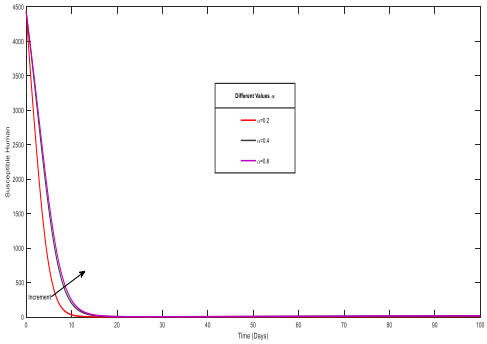
using the Adams-Bashforth predictor corrector approach we ascertained the stability of the model. The results explain the effect of different parameters on the model. Significantly, it is found that the present model's integration of memory impacts through Caputo FOD affects how quickly solution paths approach the steady state solution. These results add to the body of data supporting the idea that fractional operators give more insightful and dependable, understanding of the DHF fractional model.



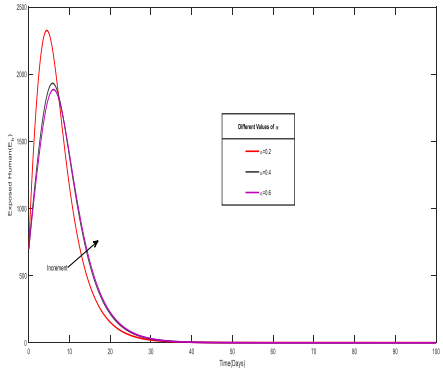
(a) At Infection-free.



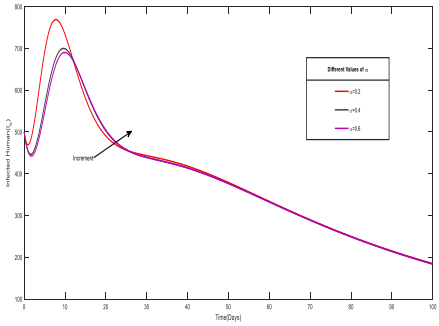
(b) At Endemic point.



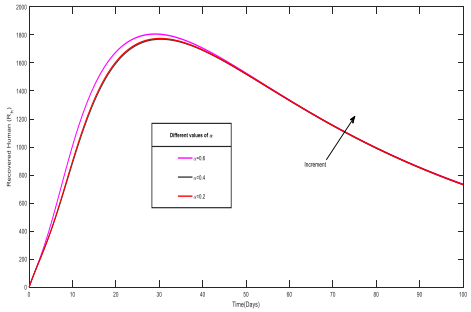
(c) Solution trajectories of different values of $\alpha$ on human susceptible.



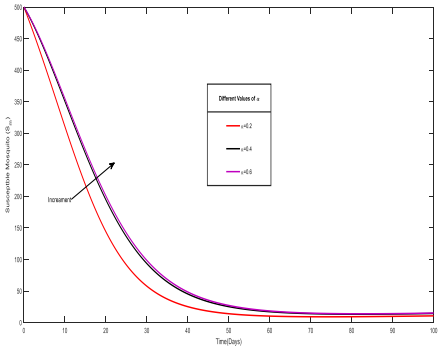
(d) Solution trajectories of different values of $\alpha$ on Exposed Human.



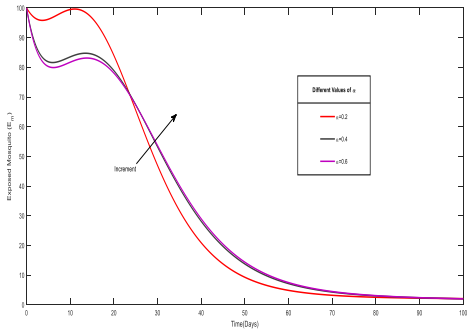

(e) Solution trajectories of different values of $\alpha$ on infected Human.



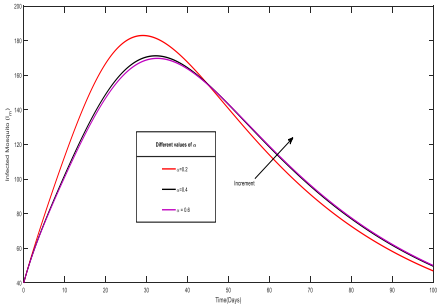

(f) Solution trajectories of different values of $\alpha$on recovered Human.



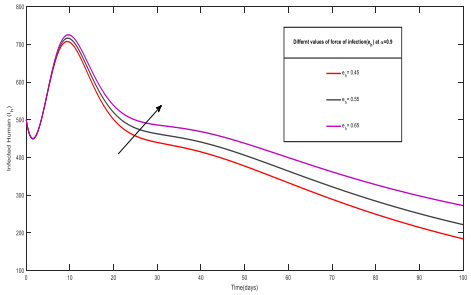

(g) Trajectory solutions of susceptible mosquito with increase in the values of fractionalorder $\alpha$.



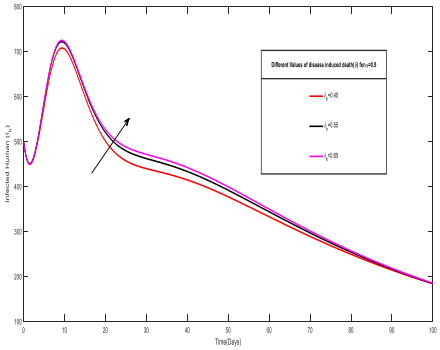

(h) Solution trajectories of Exposed mosquito with different values of $\alpha$, depicting increment, .



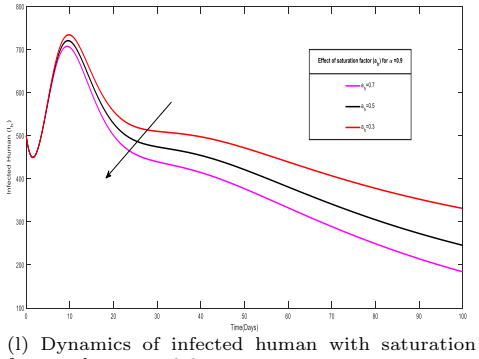

(i) Solution trajectories of infected mosquito with different values of $\alpha$.



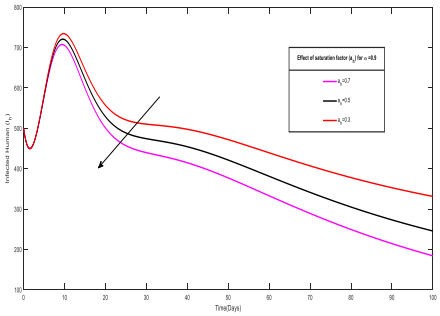

(j) Dynamics of the infected human with different values force of infection (eh) rate for the fractional order $\alpha = 0.9$.



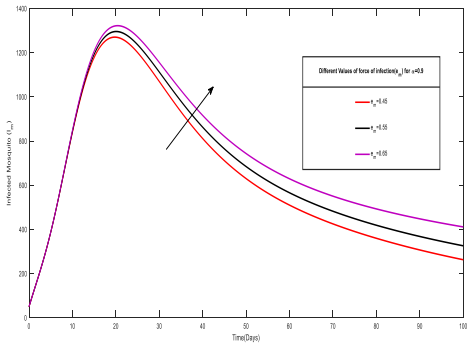
(k) Dynamics of the infected human with different values of disease induced death rate.



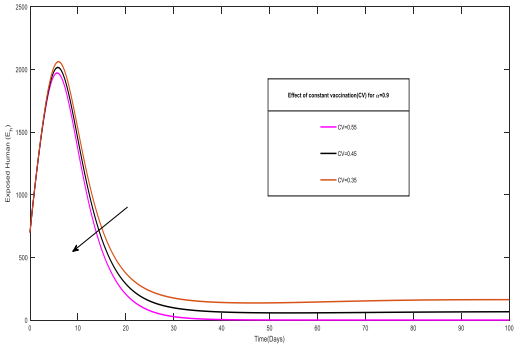
(l) Dynamics of infected human with saturation factor where $\alpha = 0.9$



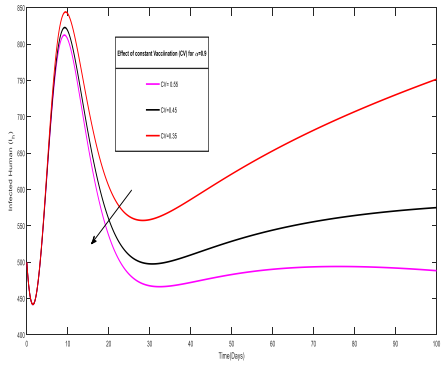
(m) Dynamics of infected mosquito with different values for force of infection.



(n) dynamics of infected mosquito with saturation factor for the fractional order $\alpha = 0.9$.



(o) The dynamical effect of constant vaccination on the exposed human individuals for $\alpha = 0.9$.



(p) dynamical effect of constant vaccination on the infected human for the fractional order $\alpha = 0.9$.

**Figure 2.** Mathematical well-posedness at infection free and endemic point

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