Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1925–1939
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1925

An Analytical Technique on Numerical Solutions for EFKs of Fourth Order and Higher

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Abstract This study introduces an advanced analytical technique for solving the fourth-order Extended Fisher-Kolmogorov (EFK) equation, focusing on the application of Adomian decomposition methods (ADM) and modified Adomian decomposition methods (MADM) . The research outlines a systematic approach to deriving numerical solutions, facilitating the characterization and understanding of complex dynamics associated with the EFK equation. Additionally, the technique is generalized to higher-order extensions, enhancing its applicability in modeling various physical phenomena. The results illustrate the effectiveness of the proposed methods in achieving accurate solutions while addressing challenges inherent in higher-order differential equations.

Keywords Extended Fisher-Kolmogorov (EFK) equation, Adomian mechanism, initial conditions, higher order

MSC(2010) 34Nxx, 34N05.

1. Introduction

The Extended Fisher-Kolmogorov (EFK) equation represents a significant evolution of the original Fisher-Kolmogorov equation, which models the dynamics of biological populations and diffusion processes. The EFK equation is characterized by its nonlinear nature and has been crucial for understanding complex phenomena such as pattern formation in biological systems, bistability in reaction-diffusion models, and other ecological interactions [14]. Mathematically, it describes how population densities evolve over time, incorporating higher-order spatial derivatives that facilitate the modeling of more intricate spatial dynamics [20, 21]. Despite its importance, solving the EFK equation poses substantial challenges due to its nonlinear characteristics and the complex initial conditions often involved in practical applications. Traditional analytical methods may not suffice, necessitating the exploration of alternative approaches for obtaining approximate solutions because they frequently struggle with the nonlinear characteristics of the EFK equation. These methods often assume linearity or rely on perturbative approaches that fall short when confronted with strong nonlinearities.

This is where the Adomian Decomposition Method (ADM), a powerful semi-analytical technique for solving a broad class of nonlinear ordinary and partial

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differential equations, developed by George Adomian during the late 20th century [1, 3] becomes relevant.

In essence, ADM expresses the solution as a sum of functions, where each function corresponds to a term in the series. The nonlinearities in the equations are handled using Adomian polynomials, which facilitate the systematic computation of terms in the series expansion. This approach not only provides a straightforward means of addressing nonlinear equations but also enhances the flexibility in application to various fields, including physics, engineering, and applied mathematics.

Overall, the Adomian Decomposition Method stands out as an effective and accessible solution technique for tackling complex differential equations, making it a valuable tool in both theoretical and applied mathematics [4, 6, 12].

In another context, many modifications have been made to this method with the aim of improving it called Modified Adomian Decomposition Method (MADM). This technique provides a systematic framework for decomposing nonlinear problems into simpler components, enabling more manageable calculations and convergence to the true solution and it provides a structured way to address the nonlinear aspects of the EFK equation by decomposing the solution into an infinite series. [7, 16].

MADM's ability to manage various forms of nonlinear dynamics makes it suitable for a wide range of applications beyond the EFK equation, including scenarios that feature complex boundary conditions or initial value problems. This adaptability allows researchers to apply the same technique across different problems without significant reforms to the method and has been successfully applied to a range of differential equations, showcasing its adaptability and robustness in tackling nonlinear dynamics look in [5, 9, 13, 17]. The progression of differential equations from lower to higher orders, particularly the transition from fourth to sixth order, is rooted in the mathematical need to solve increasingly complex problems across various fields such as physics, engineering, and applied mathematics. High-order differential equations incorporate more derivatives and, consequently, provide richer models to capture the behavior of dynamic systems, wave propagation, thermal conduction, and other phenomena influenced by multiple variables. In the context of differential equations,"order" refers to the highest derivative present in the equation.

The extension from fourth to sixth order often involves sophisticated methods such as power series, Taylor expansions, and eigenfunction expansions. These techniques not only expand the class of functions that can be analyzed but also enhance the theoretical framework required for solving boundary value problems and initial value problems [15] . The need for higher-order derivatives arises in scenarios where the behavior of a system cannot be adequately described by simpler models. For instance, systems characterized by stiffness or complex interactions may necessitate these advanced formulations for accurate predictions and solutions. The exploration of higher-order differential equations reflects a broader trend in mathematics toward developing tools capable of addressing the multifaceted challenges posed by real-world applications [22].

In this paper, we aim to present a thorough investigation of the solution techniques for the EFK equation using the modified Adomian decomposition method. We first outline the theoretical foundations of the EFK equation, highlighting its relevance in contemporary research. Then, we detail the implementation of MADM, illustrating its effectiveness in deriving approximate solutions. Finally, we demonstrate the practical applicability of our findings through numerical examples, thereby providing a comprehensive analysis of the method's performance and its implications for future research in nonlinear dynamics.

2. Extended Fisher-Kolmogorov equation

The Extended Fisher-Kolmogorov (EFK) equation of fourth order is an extension of the classical Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) equation, which is a fundamental model in theoretical biology and mathematical physics for studying biological invasions, reaction-diffusion processes, and phase transitions [23]. Understanding the formation and origin of the EFK equation requires a deep dive into the background of these models. The original FKPP equation is expressed as:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru(1-u).$$

Here $u(x,t)$ represents the population density, D is the diffusion coefficient, and r is the intrinsic growth rate. The equation combines a linear diffusion term with nonlinear logistic growth. The significant leap from the basic FKPP equation to the extended versions—including the Ordinary Extended Fisher-Kolmogorov (OEFK) equation—occurs mainly due to the inclusion of higher-order spatial derivatives, which introduces greater complexity. The extended Fisher-Kolmogorov equation can generally be expressed as:

$$\frac{\partial u}{\partial t} = -\alpha\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \alpha > 0,$$

This equation was first put forward as an extension of the traditional Fisher-Kolmogorov (FK) equation in 1987 by Coullet, Elphick, and Repaux [10] and in 1988 by Dee and van Saarloos [11]. The occurrence of phase transitions, or solutions that spatially link two uniform states, is an issue of significant interest for these model equations. The following autonomous equation results from examining time-independent solutions:

$$u'''' - \zeta u'' + \delta u + u^3 = 0, x > 0, \tag{2.1}$$

whit $u(0) = 0$, $u'(0) = A$, $u''(0) = 0$, $u'''(0) = B$, where the constants A and B must be related by $B = \frac{1}{2\ A\ \mu}(A^2 - \frac{1}{2}),\ A \neq 0,\ B > 0,\ B > 0,\ A \in R/\left\{0\right\}.$

3. Polynomial of Adomian

The polynomial's Adomian method is an effective tool used for solving nonlinear problems, particularly in the context of differential equations. This method extends the classical Adomian decomposition technique by enabling the decomposition of non-linear operators into a series of multinomial terms.

The decomposition method decomposes the solution $u(x)$ and the nonlinearity $N(u)$ into series

$$u(x) = \sum_{n=0}^{\infty} u_n, \tag{3.1}$$

and

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials. The recursion relation in [2, 6] can be formulated as follows:

$$A_0 = N(u_0),$$
$$A_1 = N(u_0)u_1,$$
$$A_2 = N'(u_0)u_2 + \frac{1}{2!}N''(u_0)u_1^2,$$
$$\ldots$$

Hence

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}, \quad n \geq 0.$$

4. The solution's existence and uniqueness

In this part, Theorem 4.1 introduces the necessary condition that ensures the existence of a unique solution, Theorem 4.2 proves the convergence of the series solution (3.1) and Theorem 4.3 estimates the maximum absolute error of the truncated series (3.1).

Theorem 4.1. *There is a unique solution of problem (2.1) whenever* $0 < \alpha < 1$, *where*

$$\alpha = \left[\frac{1}{r^2} + \sum_{n=1}^{\infty}(-1)^n(\frac{2^n r^{(\frac{-1}{n+1})} + x^n}{2^{(n+1)} r^{(\frac{n+2}{n+1})}})\left((-1)^n e^{(-1)^n\sqrt{r}x} - e^{(-1)^{n+1}\sqrt{r}x}\right)\right].$$

Proof. Let $E = (C[J], \|.\|)$ be the Banach space of all continuous functions on J with the norm

$$\|u(x)\| = \max_{x\in J}|u(x)|.$$

Define a mapping $F : E \to E$ where $Fu(x) = \phi(x) - L^{-1}N(u)$. If u and $u^* \in E$, we arrive at

$$\begin{aligned}
\|Fu - Fu^*\| &= \max_{x\in J}|L^{-1}\left[N(u) - N(u^*)\right]| \\
&\leq \max_{x\in J} L^{-1}|N(u) - N(u^*)| \\
&\leq \varrho \max_{x\in J}|u - u^*| \int_0^x \int_0^x \ldots k - fold \ldots \int_0^x \int_0^x dxdx\ldots dxdx \\
&\leq \varrho\alpha \max_{x\in J}|u - u^*| \\
&\leq \varrho\alpha\|u - u^*\| \\
&\leq \ell\|u - u^*\|.
\end{aligned}$$

□

The Banach fixed-point theorem for contractions states that there is only one solution to issue (2.1) since the mapping F is a contraction under the constraint $0 < \alpha < 1$.

Theorem 4.2. *When $0 < \alpha < 1$ and $|u_1| < \infty$, the ADM series solution (3.1) of problem (2.1) converges.*

Proof. Let $n \geq m$ and let S_n and S_m be arbitrary partial sums. In Banach space E, we will demonstrate that S_n is a Cauchy sequence.

$$\begin{aligned}
\|S_n - S_m\| &= \max_{x\in J} |S_n - S_m| \\
&= \max_{x\in J} |\sum_{i=m+1}^{n} u_i(x)| \\
&= \max_{x\in J} |\sum_{i=m+1}^{n} L^{-1}A_i| \\
&= \max_{x\in J} |L^{-1} \sum_{i=m+1}^{n} A_i| \\
&= \max_{x\in J} |L^{-1}[N(S_n) - N(S_m)]| \\
&\leq \max_{x\in J} \varrho L^{-1}|N(S_n) - N(S_m)| \\
&\leq \varrho\alpha \|S_n - S_m\| \\
&\leq \ell \|S_n - S_m\|.
\end{aligned}$$

If $n = m + 1$, then,

$$\|S_{m+1} - S_m\| \leq \ell\|S_m - S_{m-1}\| \leq \ell^2\|S_{m-1} - S_{m-2}\| \leq ... \leq \ell^m\|S_1 - S_0\|.$$

The triangular inequality leads us to

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\leq (\ell^m + \ell^{m+1} + ... + \ell^{n-1})\|S_1 - S_0\| \\
&\leq \ell^m(1 + \ell + ... + \ell^{n-m-1})\|S_1 - S_0\| \\
&\leq \ell^m(\frac{1 - \ell^{n-m}}{1 - \ell})\|u_1(x)\|.
\end{aligned}$$

Whenever $0 < \ell < 1$, $(1 - \alpha^{n-m}) < 1$, then we get

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{x\in J} |u_1(x)|.$$

But $|u_1| < \infty$ as $m \to 0$, so $\|S_n - S_m\| \to 0$; therefore, $\{S_n\}$ is a Cauchy sequence in Banach space E. Hence the series $\sum_{n=0}^{\infty} u_n(x)$ converges and the proof is complete. □

Theorem 4.3. *It is calculated that the series solution (3.1) to problem (2.1) has a maximum absolute truncation error of:*

$$\max_{x\in J} \left| u(x) - \sum_{i=0}^{m} u_i(x) \right| \leq \frac{\alpha^m}{1 - \alpha} \max_{x\in J} |u_1(x)|.$$

Proof.

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{x\in J} |u_1(x)|.$$

As $n \to \infty$, $S_n \to u(x)$, so we have

$$\|u(x) - S_m\| \leq \frac{\alpha^m}{1-\alpha} \max_{x \in J} |u_1(x)|,$$

and it is predicted that the highest absolute truncation error in the interval J is

$$\max_{x \in J} \left| u(x) - \sum_{i=0}^{m} u_i(x) \right| \leq \max_{x \in J} \frac{\alpha^m}{1-\alpha} |u_1(x)|.$$

The proof is complete. □

5. Explanation of method and numerical applications

The Adomian decomposition method (ADM) provides an efficient way to tackle nonlinear differential equations by decomposing solutions into a series of components that can be solved iteratively [1,2]. This method is particularly useful for the Fisher-Kolmogorov equation as it allows for the handling of nonlinearity, enabling the derivation of approximate analytical solutions without requiring extensive computational resources.

In the context of the extended Fisher-Kolmogorov equation, the modified Adomian decomposition method (MADM) enhances the traditional approach by incorporating modifications that improve convergence and accuracy. This method provides a pathway to more refined solutions by adjusting the basic decomposition to accommodate higher-order terms and nonlinear interactions present in the extended equation.

Applying these methods to the Fisher-Kolmogorov equation facilitates a deeper understanding of its dynamics, yielding insights into the behavior of solutions under various initial conditions.

5.1. First method via standard ADM

Under the transformation $\zeta = 2r$ and $\delta = r^2$ the equation (2.1) as [18, 19] is transformed to

$$\begin{gathered} u^{(4)}(x) - 2ru^{(2)}(x) + r^2 u(x) + u^3 = 0, \\ u(0) = 0, u'(0) = A, u''(0) = 0, u'''(0) = B. \end{gathered} \tag{5.1}$$

Equation (5.1)'s operator form can be expressed as follows:

$$Lu = \frac{d^4 u}{dx^4},$$

with

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.) dx dx dx dx.$$

However, this problem does not have an exact solution, so to illustrate the efficiency and accuracy of ADM algorithm, the following residual error is defined

$$E = |u''''(x) - \zeta u''(x) + \delta u(x) + u^3(x)|.$$

By applying L^{-1} on both sides of Equation (5.1) and using the initial conditions we obtain

$$u(x) = \phi(x) + 2rL^{-1}u^{(2)} - r^2L^{-1}u - L^{-1}u^3. \tag{5.2}$$

The first step in applying ADM to this equation is to express the solution $u(x)$ as a series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

and the nonlinear term

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{5.3}$$

where $u_n(x)$ are the components of the series. The initial condition $u_0(x)$ sets the first term of the series and the nonlinear term $N(u)$ is typically addressed using Adomian polynomials, which are constructed recursively. For the Fisher-Kolmogorov equation, this leads to the polynomials A_n that represent the contributions from the non-linear term $N(u) = u^3$ at each order.

Substituting equation (3.1) and equation (5.3) into equation (5.2) yields

$$\sum_{n=0}^{\infty} u(x) = \phi(x) + 2\gamma L^{-1}\sum_{n=0}^{\infty} u_n^{(2)} - \gamma^2 L^{-1}\sum_{n=0}^{\infty} u_n - L^{-1}A_n.$$

The zeroth components u_0 can be identified by $\phi(x)$ and the remaining components may be found repeatedly by utilizing the relation

$$u_0(x) = \phi(x),$$

$$u_{n+1}(x) = 2rL^{-1}u_n^{(2)} - r^2L^{-1}u_n - L^{-1}A_n, n \geq 0. \tag{5.4}$$

Since $A_n = u^3$ is the non-linear part, we can get

$$\begin{aligned} A_0 &= u_0^3, \\ A_1 &= 3u_0^2u_1, \\ A_2 &= 3u_0^2u_2 + 3u_0'u_1^2. \end{aligned}$$

Equation (4.5) is used to compute the solution components as

$$\begin{aligned} u_0 =& \alpha x + \frac{\beta x^3}{6}, \\ u_1 =& -0.000165344x^9\alpha^2\beta - 0.0000105219x^{11}\alpha\beta^2 - 2.6979193645860314 \\ &\times 10^{-7}x^{13}\beta^3 + 0.0166667x^5\left(-0.5r^2\alpha + r\beta\right) \\ &+ 0.0047619x^7\left(-0.25\alpha^3 - 0.0416667r^2\beta\right), \\ u_2 =& 7.798231063991459 \times 10^{-18}x^7(-5.08866 \times 10^{13}r^2(r\alpha - 2.\beta) + 3.46142 \\ &\times 10^7x^{12}\alpha^2\beta^3 + 1.02364 \times 10^6x^{14}\alpha\beta^4 + 13566.x^{16}\beta^5 \\ &+ 86526.x^{10}\beta^2\left(7683.\alpha^3 + 436.r^2\beta\right) + 3.53379 \times 10^{11}x^2\left(r^4\alpha - 12.r\alpha^3 - 4.r^3\beta\right) \\ &+ 3.21254 \times 10^9rx^4\left(132.r\alpha^3 + r^3\beta - 372.\alpha^2\beta\right) + 5.88377 \\ &\times 10^6x^8\beta\left(1122.\alpha^4 + 601.r^2\alpha\beta - 980.r\beta^2\right) \end{aligned}$$

$$+ 2.47118 \times 10^{8} x^{6} \alpha \left(108.\alpha^{4} + 275.r^{2}\alpha\beta - 574.r\beta^{2}\right)).$$

A series form of the approximate solution $u(x)$ is produced

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0 + u_1 + u_2 + ...,$$

5.2. Second method via MADM

In this case we offer a new modified differentail operator in the form

$$Lu = e^{\sqrt{r}x} \frac{d^2}{dx^2} e^{-2\sqrt{r}x} \frac{d^2}{dx^2} e^{\sqrt{r}x} u,$$

where

$$Lu = u^{(4)}(x) - 2ru^{(2)}(x) + r^2 u(x),$$

and

$$L^{-1}(.) = e^{-\sqrt{r}x} \int_0^x \int_0^x e^{2\sqrt{r}x} \int_0^x \int_0^x e^{-\sqrt{r}x} (.) dx dx dx dx. \tag{5.5}$$

Applying equation (5.5) to equation (5.1) and using the initial conditions in equation (5.1) we obtain

$$u(x) = \phi(x) - L^{-1}u^3.$$

The zeroth component u_0 can be identified by ϕ_0 and the remaining components may be found repeatedly by utilizing the relation

$$u_0(x) = \phi_0,$$

$$u_{n+1} = -L^{-1}A_n, n \geq 0. \tag{5.6}$$

Since $A_n = u^3$ is the non-linear part, we can get it as explained previoussly. Equation (5.6) used to compute the solution components as

$$u_0 = Ae^{-\sqrt{r}x} x - \frac{Ae^{-\sqrt{r}x}\left(1 - e^{2\sqrt{r}x} + 2\sqrt{r}x\right)}{2\sqrt{r}} - \frac{e^{-\sqrt{r}x}(-B + Ar)\left(\frac{1}{\sqrt{r}} + x + e^{2\sqrt{r}x}\left(-\frac{1}{\sqrt{r}} + x\right)\right)}{4r},$$

$$\begin{aligned} u_1 = &- \frac{A^3x^7}{840} + \left(-\frac{A^2B}{6048} - \frac{A^3r}{30240}\right)x^9 \\ &+ \left(-\frac{A^2B\sqrt{r}}{6048} - \frac{19A^3r^{3/2}}{30240} - \sqrt{r}\left(-\frac{A^2B}{6048} - \frac{19A^3r}{30240}\right)\right)x^{10} \\ &+ \left(-\frac{AB^2}{95040} - \frac{17A^2Br}{184800} + \frac{71A^3r^2}{831600} + \frac{1}{2}r\left(-\frac{A^2B}{6048} - \frac{19A^3r}{30240}\right) \right. \\ &\left. - \sqrt{r}\left(-\frac{A^2B\sqrt{r}}{6048} - \frac{A^3r^{3/2}}{4320}\right)\right)x^{11} \end{aligned}$$

$$+\Bigg(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{523A^3r^{5/2}}{9979200}-\frac{1}{6}r^{3/2}\left(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\right)$$
$$+\frac{1}{2}r\left(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\right)-\sqrt{r}\left(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\right)\Bigg)x^{12}$$
$$+\Bigg(-\frac{B^3}{3706560}-\frac{59AB^2r}{9266400}-\frac{19A^2Br^2}{1684800}+\frac{193A^3r^3}{28828800}+\frac{1}{24}r^2\left(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\right)$$
$$-\frac{1}{6}r^{3/2}\left(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\right)+\frac{1}{2}r\left(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\right)$$
$$-\sqrt{r}\left(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{127A^3r^{5/2}}{9979200}\right)\Bigg)x^{13}+\dots$$

$$u_2=\frac{A^5x^{13}}{4804800}+\frac{(935A^4B+91A^5r)x^{15}}{18162144000}+\frac{(12805A^3B^2+8722A^4Br-2843A^5r^2)x^{17}}{2470051584000}+\dots$$

A series form of the approximate solution $u(x)$ is produced

$$u(x)=\sum_{n=0}^{\infty}u_n(x)=u_0+u_1+u_2+\dots,$$
$$Ax+\frac{Bx^3}{6}+\frac{1}{120}(2Br-Ar^2)x^5+\Big(-\frac{A^3}{840}+\frac{3Br^2-2Ar^3}{5040}\Big)x^7$$
$$+\Big(-\frac{A^2B}{6048}-\frac{A^3r}{30240}+\frac{4Br^3-3Ar^4}{362880}\Big)x^9$$
$$+\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{19A^3r^{3/2}}{30240}-\sqrt{r}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)\Big)x^{10}$$
$$+\Big(-\frac{AB^2}{95080}-\frac{17A^2Br}{184800}+\frac{71A^3r^2}{831600}+\frac{r}{2}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)$$
$$-\sqrt{r}\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\Big)+\frac{5Br^4-4Ar^5}{39916800}\Big)x^{11}$$
$$+\Big(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{523A^3r^{5/2}}{9979200}-\frac{r^{3/2}}{6}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)$$
$$+\frac{r}{2}\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\Big)-\sqrt{r}\Big(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\Big)\Big)x^{12}$$
$$+\Big(\frac{A^5}{4804800}-\frac{B^3}{3706560}-\frac{59AB^2r}{9266400}-\frac{19A^2Br^2}{1684800}+\frac{193A^3r^3}{28828800}$$
$$+\frac{r^2}{24}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)-\frac{r^{3/2}}{6}\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\Big)$$
$$+\frac{r}{2}\Big(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\Big)$$
$$-\sqrt{r}\Big(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{127A^3r^{5/2}}{9979200}\Big)+\frac{6Br^5-5Ar^6}{6227020800}\Big)x^{13}$$
$$+\Big(-\frac{B^3\sqrt{r}}{3706560}-\frac{53AB^2r^{3/2}}{18532800}-\frac{23A^2Br^{5/2}}{8648640}-\frac{89A^3r^{7/2}}{64864800}$$

$$
\begin{aligned}
&- \frac{r^{5/2}}{120}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)+\frac{r^2}{24}\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\Big)\\
&- \frac{r^{3/2}}{6}\Big(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\Big)\\
&+ \frac{r}{2}\Big(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{127A^3r^{5/2}}{9979200}\Big)\\
&- \sqrt{r}\Big(-\frac{B^3}{3706560}-\frac{59AB^2r}{9266400}-\frac{19A^2Br^2}{1684800}-\frac{17A^3r^3}{10810800}\Big)\Big)x^{14}\\
&+ \Big(-\frac{B^3r}{14968800}-\frac{AB^2r^2}{1559250}-\frac{37A^2Br^3}{74844000}+\frac{379A^3r^4}{1524096000}+\frac{935A^4B+91A^5r}{18162144000}\\
&+ \frac{r^3}{720}\Big(-\frac{A^2B}{6048}-\frac{19A^3r}{30240}\Big)-\frac{r^{5/2}}{120}\Big(-\frac{A^2B\sqrt{r}}{6048}-\frac{A^3r^{3/2}}{4320}\Big)\\
&+ \frac{r^2}{24}\Big(-\frac{AB^2}{95040}-\frac{17A^2Br}{184800}-\frac{211A^3r^2}{3326400}\Big)\\
&- \frac{r^{3/2}}{6}\Big(-\frac{AB^2\sqrt{r}}{95040}-\frac{23A^2Br^{3/2}}{623700}-\frac{127A^3r^{5/2}}{9979200}\Big)\\
&+ \frac{r}{2}\Big(-\frac{B^3}{3706560}-\frac{59AB^2r}{9266400}-\frac{19A^2Br^2}{1684800}-\frac{17A^3r^3}{10810800}\Big)\\
&- \sqrt{r}\Big(-\frac{B^3\sqrt{r}}{3706560}-\frac{53AB^2r^{3/2}}{18532800}-\frac{23A^2Br^{5/2}}{8648640}+\frac{41A^3r^{7/2}}{908107200}\Big)\\
&+ \frac{7Br^6-6Ar^7}{1307674368000}\Big)x^{15}+\dots
\end{aligned}
$$

r=1, A=0.5, B=-0.25				
x	**ADM**	**E**	**MADM**	**E**
0.1	0.0499582	3.33293×10^{-7}	0.0499582	2.61243×10^{-15}
0.3	0.148855	0.0000809034	0.148855	4.62785×10^{-10}
0.5	0.244527	0.00103765	0.244527	1.2756×10^{-7}
0.7	0.334267	0.00555231	0.334267	5.16564×10^{-6}
0.9	0.414471	0.0193483	0.414467	0.0000819809
1.0	0.449515	0.0325952	0.449504	0.000261243

6. Sixth order differential equation

When generalizing the fourth-order differential equations to sixth-order using the ADM, the process requires recognizing the inherent complexities that arise from the increased order of the differential equations. Each higher-order term introduces additional variables and initial conditions that must be accounted for. In the generalized approach, the ADM can be extended to handle the sixth-order equations

r=0.5, A=0.8, B=0.875				
x	**ADM**	**Error (ADM)**	**MADM**	**Error (MADM)**
0.1	0.0800146	9.43521×10^{-9}	0.0801459	2.99483×10^{-14}
0.3	0.240391	2.40865×10^{-6}	0.243951	5.28566×10^{-9}
0.5	0.401789	3.38699×10^{-5}	0.418401	1.44551×10^{-6}
0.7	0.564791	2.04161×10^{-4}	0.610919	5.78091×10^{-5}
0.9	0.729767	8.09904×10^{-4}	0.829344	9.00973×10^{-4}
1.0	0.812992	1.45401×10^{-3}	0.95083	2.83797×10^{-3}

Table 1. Numerical results for the EFK equation using ADM and MADM residual errors.

by identifying the linear and nonlinear components distinctly, thereby applying the decomposition method to these components separately.

Assume the sixth-order differential equation as follows:

$$u^{(6)} - \zeta u^{(4)} + \delta u^{(2)} - \kappa u + u^3 = 0, x > 0, \tag{6.1}$$

$$u(0) = 0,\ u'(0) = A,\ u''(0) = 0,\ u'''(0) = B,\ u''''(0) = 0,\ u'''''(0) = C.$$

The constants A and B must be related by $B = \frac{1}{2\ A\ \mu}(A^2 - \frac{1}{2}),\ A \neq 0,\ B > 0,$ $C > 0,\ A \in R/\{0\}$.

This equation was studied by the G. Bonanno et al. [8] under the boundary conditions

$$u(a) = u(b) = u''(a) = u''(b) = u^{iv}(a) = u^{iv}(b) = 0.$$

Under transform $\zeta = 3r$, $\delta = 3r^2$, $\kappa = r^3$ equation (6.1) becomes

$$u^{(6)}(x) - 3ru^{(4)}(x) + 3r^2u^{(2)}(x) - r^3u(x) + u^3 = 0. \tag{6.2}$$

We offer a new modified differential operator in the form

$$Lu = e^{\sqrt{r}x}\frac{d^3}{dx^3}e^{-2\sqrt{r}x}\frac{d^3}{dx^3}e^{\sqrt{r}x}u,$$

where

$$Lu = u^{(6)}(x) - 3ru^{(4)}(x) + 3r^2u^{(2)}(x) - r^3u(x), \tag{6.3}$$

and

$$L^{-1}(.) = e^{-\sqrt{r}x}\int_0^x\int_0^x\int_0^x e^{2\sqrt{r}x}\int_0^x\int_0^x\int_0^x e^{-\sqrt{r}x}(.)dxdxdxdxdxdx. \tag{6.4}$$

Applying equation (6.4) to equation (6.3) and using the initial conditions in equation (6.1) we obtain

$$u(x) = \phi(x) - L^{-1}u^3.$$

The zeroth components u_0 can be identified by $\phi(x)$ and the remaining components may be found repeatedly by utilizing the relation

$$u_0(x) = \phi(x),$$

$$u_{n+1} = -L^{-1}A_n, n \geq 0. \tag{6.5}$$

Since $A_n = u^3$ is the non-linear part, we can get it as explained previoussly. equation (6.3) is used to compute the solution components as

$$\begin{aligned}
u_0 =& Ae^{-\sqrt{r}x}x + Ae^{-\sqrt{r}x}\sqrt{r}x^2 - \frac{e^{-\sqrt{r}x}\big(B+3Ar^{3/2}\big)\big(1-e^{2\sqrt{r}x}+2\sqrt{r}x+2rx^2\big)}{8r^{3/2}} \\
&- \frac{e^{-\sqrt{r}x}\big(-2B\sqrt{r}+2Ar^{3/2}\big)\big(3+4\sqrt{r}x+2rx^2+e^{2\sqrt{r}x}(-3+2\sqrt{r}x)\big)}{16r^2} \\
&+ \frac{e^{-\sqrt{r}x}\big(C-2Br+Ar^2\big)\big(-3-3\sqrt{r}x-rx^2+e^{2\sqrt{r}x}(3-3\sqrt{r}x+rx^2)\big)}{16r^{5/2}},
\end{aligned}$$

$$\begin{aligned}
u_1 =& -\frac{A^3x^9}{60480} + \frac{\big(-163800A^2B+442260A^3r-491400A^3r^{3/2}\big)x^{11}}{108972864000} \\
&+ \frac{\big(-1638A^2B\sqrt{r}-5814A^3r^{3/2}-3685A^3r^2-\sqrt{r}\big(-1638A^2B-4586A^3r-4914A^3r^{3/2}\big)\big)x^{12}}{108972864000} \\
&+ \frac{1}{108972864000}\Big[\Big(225225A^3r^2+\tfrac{1}{2}r\big(-163800A^2B-458640A^3r-491400A^3r^{3/2}\big) \\
&- \sqrt{r}\big(-163800A^2B\sqrt{r}+19110A^3r^{3/2}-368550A^3r^2\big)-105A\big(70B^2+30AB(13+14\sqrt{r})r \\
&+ A\big(21C+Ar^2(73+252\sqrt{r}+630r)\big)\big)\Big)x^{13}\Big] \\
&+ \frac{1}{108972864000}\Big[\Big(-60060A^3r^{5/2}-\tfrac{1}{6}r^{3/2}\big(-163800A^2B-458640A^3r-491400A^3r^{3/2}\big) \\
&+ \tfrac{1}{2}r\big(-163800A^2B\sqrt{r}+19110A^3r^{3/2}-368550A^3r^2\big)+105A\sqrt{r}\big(70B^2+30AB(13+14\sqrt{r})r \\
&+ A\big(21C+Ar^2(73+252\sqrt{r}+630r)\big)\big)-15A\sqrt{r}\big(490B^2+70AB(-1+30\sqrt{r})r \\
&+ A\big(147C+Ar^2(107-408\sqrt{r}+1890r)\big)\big)\Big)x^{14}\Big]+\ldots,
\end{aligned}$$

$$u_2 = 5.56669\times10^{-12}A^5x^{17} - 2.78334\times10^{-12}A^5rx^{19} + 1.85556\times10^{-12}A^5r^{3/2}x^{20}+\ldots$$

Therefore, in series form, the approximate solution $u(x)$ up to order three is provided by

$$u(x) = u_0 + u_1 + u_2 + ...$$

7. Extending the differential equation of fourth order to higher orders

The extending of fourth-order differential equations to higher-order forms is a significant focus in applied mathematics, particularly for solving complex engineering and physical problems. This process involves extending the principles and methodologies used for fourth-order equations to accommodate equations of any order, thereby enhancing analytical and numerical solution techniques. General formula

	$A = 0.5, B = -0.25, C = 0.2, r = 1$		$A = 0.8, B = 0.875, C = 0.4, r = 0.5$	
x	**MADM**	**E**	**MADM**	**E**
0.1	0.0499584	-4.1336×10^{-15}	0.0801459	-9.52381×10^{-15}
0.2	0.0996672	-2.1164×10^{-12}	0.161168	-4.87619×10^{-12}
0.3	0.148879	-8.13616×10^{-11}	0.243946	-1.87457×10^{-10}
0.4	0.197351	-1.0836×10^{-9}	0.329367	-2.49661×10^{-9}
0.5	0.244847	-8.07343×10^{-9}	0.418332	-1.86012×10^{-8}
0.6	0.29114	-4.16571×10^{-8}	0.511756	-9.59781×10^{-8}
0.7	0.336019	-1.66806×10^{-7}	0.61057	-3.8432×10^{-7}
0.8	0.379291	-5.54802×10^{-7}	0.715731	-1.27826×10^{-6}
0.9	0.420789	-1.60144×10^{-6}	0.828215	-3.68972×10^{-6}
1.0	0.460378	-4.1336×10^{-6}	0.949027	-9.52381×10^{-6}

Table 2. The EFK equation's numerical results utilizing the residual error and the MADM technique are shown in the table. Our findings show that the residual error is high, which demonstrates how accurate of our conclusions.

for generalization is given in the form:

$$\sum_{n=0}^{k}(-1)^n \binom{k}{n} r^n u^{(2k-2n)} + N(u) = 0, \quad k > 1, \quad r \in N,$$

with conditions

$$u(0) = u''(0) = u^{(4)} = ... = u^{(k-1)} = 0, \quad k \ \ is \ \ odd \ \ number, \tag{7.1}$$

$$u'(0) = u'''(0) = u^{(5)} = ... = u^{(k-1)} = \delta, \quad k \ \ is \ \ even \ \ number, \delta = A, B, C, ...$$

and the new differential operator in the form

$$L(u) = e^{\sqrt{r}x} \frac{d^k}{dx^k} e^{-2\sqrt{r}x} \frac{d^k}{dx^k} e^{\sqrt{r}x}(u), \tag{7.2}$$

and

$$L^{-1}(.) = e^{-\sqrt{r}x} \underbrace{\int_0^x \int_0^x ... \int_0^x \int_0^x}_{k-times} e^{2\sqrt{r}x} \underbrace{\int_0^x \int_0^x ... \int_0^x \int_0^x}_{k-times} e^{-\sqrt{r}x}(.) \underbrace{dxdx...dxdxdxdx...dxdx.}_{2k-times} \tag{7.3}$$

Hence equation (7.1) takes the formula

$$Lu + Nu = 0, \tag{7.4}$$

and the general solution $u(x)$ is obtained by applying the inverse operator L^{-1} on equation (6.3), therefore

$$u(x) = \phi(x) + L^{-1}N(u),$$

where

$$u_0(x) = \phi(x),$$
$$u_{n+1} = L^{-1}A_n, \;\; A_n = N(u).$$

It was also clarified previously.

8. Conclusion

This study demonstrates the effectiveness of the Adomian Decomposition Method (ADM) and its modified version for obtaining numerical solutions of the fourth-order Extended Fisher-Kolmogorov (EFK) equation. The application of these methods proves beneficial for efficiently solving higher-order generalizations of the EFK equation. The results indicate that the modified ADM enhances convergence rates and accuracy, providing a robust framework for analyzing complex nonlinear dynamic systems. Overall, the findings highlight the potential of these analytical techniques for both theoretical advancement and practical application in diverse scientific and engineering disciplines, paving the way for further exploration of high-order nonlinear systems.

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