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Controllability of Nonlinear Fractional Systems with Multiple State and Control Delays

Mustafa Aydin[1,†]

Abstract This study focuses on examining the controllability of nonlinear fractional differential systems in finite-dimensional spaces, considering multiple delays in both state and control. For linear systems, the necessary and sufficient conditions for relative controllability are established through the definition and application of the Gramian matrix. For nonlinear systems, controllability conditions are derived using Schauder's fixed point theorem.

Keywords Fractional differential system, relative controllability, multiple control delays, multiple state delays, Mittag-Leffler-type function, multivariate matrix equations

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1. Introduction

Fractional calculus constitutes a significant advancement over classical calculus, offering a wide range of essential functionalities that traditional calculus fails to adequately address. For example, numerous pressing societal concerns and empirical scientific challenges can be represented in a more coherent manner, particularly when accounting for the inherent uncertainties present in various dynamical systems. As a result, fractional calculus has become increasingly relevant across a multitude of disciplines, including mathematical physics, engineering, biophysics [15, 22, 28, 31, 53, 66, 67], nanotechnology applications [14], signal processing [54], circuit theory [57], and geophysical modeling such as earthquake analysis [27], among others.

Control theory is fundamentally grounded in the concept of controllability, which remains a cornerstone in the field of control systems. Several studies have addressed various aspects of controllability in semilinear and fractional dynamical systems. For instance, the study in [62] investigates the complete controllability of a semilinear stochastic system with multiple delays in control within a stochastic framework; however, it does not address fractional dynamics or delays in the state variables. Dauer and Gahl [17] established controllability results for nonlinear systems that incorporate delays. Balachandran and Dauer [7] conducted an in-depth analysis of controllability issues for both linear and nonlinear systems characterized by delays. Balachandran et al. [8,12] investigated the relative controllability of nonlinear fractional dynamic systems exhibiting both multiple delays and distributed delays in their control inputs. Klamka [10, 32] demonstrated controllability in linear and

[†]the corresponding author.
Email address:m.aydin@yyu.edu.tr(M. Aydin)
[1]Department of Medical Services and Techniques, Muradiye Vocational School, Van Yuzuncu Yil University, 65080 Van, Türkiye

nonlinear systems influenced by time-varying delays. Recently, Mur et al. [49] investigated the relative controllability of fractional-order linear systems with delay components. The work [3] discusses the relative controllability of Caputo-type fractional systems involving a single state delay and multiple control delays; however, it does not address systems with Riemann-Liouville fractional derivatives, multiple state delays, or multivariate matrix-based formulations. The work in [61] investigates interior approximate controllability for second-order semilinear systems by reducing them to first-order systems and applying the Leray-Schauder alternative theorem. The study in [37] focuses on approximate controllability of fractional systems with a single state delay in Banach spaces using generalized Gronwall's inequality and compactness arguments, yet it does not handle multiple delays in control or noncommutative system structures.

It is a well-established fact within the realm of mathematical analysis that a differential delayed equation is fundamentally characterized by the incorporation of three essential components: the state of the system in the past, the state of the system at the present moment, and the corresponding rate of change of the system with respect to time. Prominent scholars Volterra and Minorskii have employed theoretical frameworks closely aligned with the structure of delay differential equations in various scholarly contributions, including, but not limited to, investigations into viscoelastic phenomena, predator-prey dynamics [68, 69], and applications in automatic steering and ship stabilization systems [47]. Within the context of these mathematical formulations, the concept of delays is intricately woven into the fabric of the states themselves, thereby adding complexity to both the analysis and interpretation of such equations. A substantial body of academic literature exists that delves into the intricacies of these specific types of differential delayed equations, as evidenced by the comprehensive studies referenced in [2, 4, 18–21, 23–26, 30, 36, 38–46, 50–52, 60, 70]. Nevertheless, it is noteworthy that a significant gap persists in the literature regarding the role of single and multiple delays in the context of control theory as it pertains to delay differential equations. Moreover, it is imperative to acknowledge that such equations have been extensively examined with respect to their controllability properties, as demonstrated by the contributions listed in [1, 5, 6, 10, 11, 13, 16, 17, 33, 34, 48, 59].

To avoid terminological ambiguity, we begin by explicitly defining the core notions employed in this study. Controllability refers to the ability to steer the state of a system from any initial state to any desired final state within a finite time interval using admissible control functions. Relative controllability is a weaker concept that concerns whether the system's state can reach a specific subspace or trajectory manifold, rather than the entire state space. Function controllability, in turn, pertains to the ability to control solution functions—defined over infinite-dimensional spaces such as $C([a, b], \mathbb{R}^n)$—rather than finite-dimensional state vectors.

In the context of these complex equations, it becomes crucial to draw a clear distinction between the concepts of function controllability and relative controllability within the framework of Euclidean space. This differentiation is particularly salient because, despite the fact that the solutions derived from these equations manifest as trajectories within the confines of Euclidean space, the inherent and more appropriate "state space" that governs their behavior is, in reality, a function space, which necessitates a more nuanced understanding. For the specific aims and objectives of this scholarly inquiry, we intentionally restrict our analysis to the concept of relative controllability alone. In addition, it is essential to recognize that, unlike in classical ordinary differential equation theory, a further distinction must be made between the notions of complete controllability and null controllability when addressing the topic of relative controllability in this specialized setting.

Unlike the existing literature that primarily focuses on systems with single or commensurate delays (e.g., [7, 17, 32]), or fractional systems with limited control structure [8, 12, 49], our work addresses the relative controllability of fractional dynamic systems that contain both multiple state delays and multiple control delays. Moreover, the presence of noncommutative coefficient matrices in terms of multivariate matrix-based formulations and nonlinear perturbation terms differentiates our approach from earlier contributions. While some works investigate fractional systems using operator-theoretic methods or semigroup theory (e.g., [10]), our method combines the use of a Gramian matrix and time-lead transformations, along with fixed-point arguments tailored to the multiple state delays and multiple control delays' structure. This integrated methodology allows us to address classes of systems where inverse delay mappings and nonlinearities appear simultaneously and interact within a fractional-order framework—an interaction that is only partially explored in the existing literature.

In line with our analysis, although a limited number of studies have addressed fractional dynamical systems with multiple delays in control, as previously cited, to the best of our knowledge, no existing work has considered the case of a delay in the state variable combined with noncommutative coefficients. This notable gap in the literature, along with the theoretical necessity for such an investigation and the motivation derived from the aforementioned works, prompts us to examine the following class of nonlinear fractional dynamical systems. These systems incorporate both multiple state delays and multiple control delays, and they feature noncommutative coefficient structures. For $\varsigma \in (0, T]$, we consider the system:

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\alpha} \upsilon(\varsigma) = M\upsilon(\varsigma) + \displaystyle\sum_{i=0}^{d_1} T_i \upsilon(\varsigma - h_i) + \sum_{i=0}^{d_2} A_i u(r_i(\varsigma)) + \daleth(\varsigma, \upsilon(\varsigma), u(\varsigma)), \\ {}^{RL}\mathcal{D}_{0^+}^{\alpha - i} \upsilon(\varsigma)\big|_{\varsigma=0} = p_i, \quad i = 1, 2, \ldots, l, \\ \upsilon(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-h, 0], \quad h > 0, \end{cases} \tag{1.1}$$

where ${}^{RL}\mathcal{D}_{0^+}^{\alpha}$ denotes the classical Riemann-Liouville fractional derivative of order $l - 1 < \alpha \leq l$ with $l \in \mathbb{N}$, and $h = \max\{h_0, h_1, \ldots, h_{d_1}\}$ with $d_1 \in \mathbb{N}$. The function υ takes values in $\mathbb{R}^n$, u in $\mathbb{R}^m$, $M, T_i \in \mathbb{R}^{n\times n}$, and $A_i \in \mathbb{R}^{n\times m}$ for each $i = 0, 1, 2, \ldots, d_2 \in \mathbb{N}$. The function $\phi : [-h, 0] \to \mathbb{R}^n$ is a prescribed initial function, and T is a positive real constant.

For the forthcoming theoretical findings presented in the subsequent sections, the following premises are adopted:

Let us consider the functions $r_i : [0, T] \to \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, M$, which exhibit strict monotonicity in the increasing direction and possess the property of being twice continuously differentiable.

$$r_i(\varsigma) \leq \varsigma, \quad \varsigma \in [0, T], \quad i = 0, 1, 2, \ldots, d_2.$$

Furthermore, it is presumed that $r_0(\varsigma) = \varsigma$, and without loss of generality, the ensuing inequalities are fulfilled for $\varsigma = T$:

$$r_{d_2}(T) \leq r_{d_2 - 1}(T) \leq \ldots \leq r_{m+1}(T) \leq 0 = r_m(T)$$

$$< r_{m-1}(T) = \ldots = r_1(T) = r_0(T) = T. \qquad (1.2)$$

Define the time-lead functions $\eta_i : [r_i(0), r_i(T)] \to [0, T], \quad i = 0, 1, 2, \ldots, d_2$ as follows:

$$\eta_i\left(r_i(\varsigma)\right) = \varsigma, \quad \varsigma \in [0, T], \quad i = 0, 1, 2, \ldots, d_2.$$

With u denoting a mapping from the interval $[-h, T]$ to the vector space $\mathbb{R}^m$, the function u_ς, for $\varsigma \in [0, T]$, is delineated as follows:

$$u_\varsigma(s) = u(\varsigma + s), \quad s \in [-h, 0].$$

Remark 1.1. It should be noted that the theoretical results derived in this study are valid under the stated assumptions, and do not extend to cases where such conditions (e.g., continuity or boundedness) are violated.

Remark 1.2. It is essential to note that the function r_i, where $i = 0, 1, 2, \ldots, d_2$, possesses an inverse, and correspondingly, the function η_i, for $i = 0, 1, 2, \ldots, d_2$, can be adequately defined. This is due to the fact that r_i is strictly increasing, which implies injectivity; it is evident that this function is bijective over the closed interval $[r_i(0), r_i(T)]$, for $i = 0, 1, 2, \ldots, d_2$. Furthermore, given that the function r_i, for $i = 0, 1, 2, \ldots, d_2$, is both differentiable and bijective, it follows that η_i, for $i = 0, 1, 2, \ldots, d_2$, is also differentiable.

The contributions delineated in this manuscript are summarized as follows.

(i) We present a representation of the solution associated with the problem (1.1) involving general fractional orders, expressed in terms of a multivariate function.

(ii) We introduce the Gramian matrix and delineate the necessary and sufficient conditions for the relative controllability of the linear system (1.1).

(iii) We reformulate the relative controllability of the nonlinear system as a fixed point problem, thereby enabling the application of the Schauder fixed point theorem to substantiate our principal results.

2. Brief preliminaries

In this segment, we reiterate several essential items that must be accessible within the scholarly literature.

Let $\mathbb{R}$ denote the collection of all real numbers, let $\mathbb{R}^n$ represent the n-dimensional real space, defined as the set of all ordered n-tuples comprised of real numbers, and let $\mathbb{R}^{n\times m}$ signify the set of $n \times m$ real matrices, the components of which are exclusively real numbers. $C\left([0, T], \mathbb{R}^n\right)$ constitutes the Banach space of continuous functions equipped with the following supremum (maximum) norm:

$$\|\upsilon\| = \sup\left\{|\upsilon(\varsigma)| : \varsigma \in [0, T]\right\},$$

where $|\cdot|$ denotes an arbitrary norm defined on the vector space $\mathbb{R}^n$.

Definition 2.1. [28,66] The fractional derivative in the sense of Riemann-Liouville, ${}^{RL}\mathcal{D}^{\alpha}_{0+}\upsilon(\varsigma)$, of order $l - 1 < \alpha < l$ is defined by

$${}^{RL}\mathcal{D}^{\alpha}_{0+}\upsilon(\varsigma) = \frac{1}{\Gamma(l-\alpha)} \frac{\mathrm{d}^l}{\mathrm{d}\varsigma^l} \int_0^\varsigma (\varsigma - s)^{l-\alpha-1} \upsilon(s)\,\mathrm{d}s, \quad \varsigma > 0,$$

where the symbol $\Gamma(\cdot)$ signifies the well-known Gamma function.

Lemma 2.1. *According to Theorem 4.2 delineated in [44], an analytical solution to the system (1.1) is provided by*

$$\begin{aligned} \upsilon(\varsigma) = &\sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma)p_i + \sum_{i=0}^{d_1} \int_{-h_i}^{0} \mathfrak{X}_{\alpha,\alpha}(\varsigma - s - h_i)T_i\phi(s)\mathrm{d}s \\ &+ \sum_{i=0}^{d_2} \int_0^{\varsigma} \mathfrak{X}_{\alpha,\alpha}(\varsigma - s)A_i u\left(r_i(s)\right)\mathrm{d}s, \end{aligned} \tag{2.1}$$

where a multi-delayed two-parameter Mittag-Leffler type matrix function $\mathfrak{X}_{\alpha,\beta}(\varsigma)$, with parameters $\alpha > 0$ and $\beta > 0$, is articulated as follows:

$$\mathfrak{X}_{\alpha,\beta}(\varsigma) \tag{2.2}$$

$$= \begin{cases} \Theta, & \varsigma \in [-h, 0), \\ \displaystyle\sum_{m=0}^{\infty} \sum_{\substack{i_1+\cdots+i_d\le m \\ i_1,\ldots,i_d\ge 0}} P_{m+1}(i_1h_1,\ldots,i_{d_1}h_{d_1}) \frac{\left(\varsigma - \sum_{k=1}^{d_1} i_k h_k\right)_+^{m\mu+\gamma-1}}{\Gamma(m\mu+\gamma)}, & \varsigma \in [0,\infty). \end{cases} \tag{2.3}$$

Here, the multivariate matrix equation $P_{k+1}(i_1h_1,\ldots,i_{d_1}h_{d_1})$ is defined as

$$\begin{aligned} P_{k+1}(i_1h_1,\ldots,i_{d_1}h_{d_1}) :=& MP_k(i_1h_1,\ldots,i_{d_1}h_{d_1}) \\ &+ \sum_{j=1}^{d} T_j P_k(i_1h_1,\ldots,i_j-1,\ldots,i_{d_1}h_{d_1}), \end{aligned}$$

together with the following initial conditions:

$$P_0(i_1h_1,\ldots,i_{d_1}h_{d_1}) = \Theta, P_k(-h_1,\ldots,i_{d_1}h_{d_1}) \quad = \cdots = P_k(i_1h_1,\ldots,-h_{d_1}) = \Theta,$$

and

$$P_1(0,\ldots,0) = I,$$

where Θ is the null matrix and I is the identity matrix.

Definition 2.2. [58] A (control) function $u(t) \in \mathbb{R}^m$ is considered admissible if it exhibits boundedness and measurability over every finite temporal interval.

Lemma 2.2. *[29] If the function $\daleth$ exhibits local boundedness in the domain $\mathbb{R}^n \times \mathbb{R}^m$ and adheres to the following condition*

$$\lim_{|(\upsilon,u)|\to\infty} \frac{|\daleth(\varsigma,\upsilon,u)|}{|(\upsilon,u)|} = 0,$$

uniformly for $\varsigma \in [0,T]$, then, for every combination of constants c and e, there exists a constant $\varepsilon > 0$ such that if $\|(\upsilon,u)\| \le r$, then

$$c\,|\daleth(\varepsilon,\upsilon,u)| + e \le \varepsilon \quad \textit{for all } \varepsilon \in [0,T].$$

Now, we shall elucidate Schauder's fixed point theorem, which demonstrates the existence of a fixed point, albeit without asserting its uniqueness.

Theorem 2.1. *[31] Let X denote a Banach space; let U represent a convex, closed, and bounded subset of X, and let $\mathcal{T}: U \to U$ be a mapping such that $\mathcal{T}(U)$ is relatively compact within X. It follows that the operator $\mathcal{T}$ possesses at least one fixed point contained within U.*

The Arzelà-Ascoli theorem is now presented as follows.

Theorem 2.2. *[31] Let G be a subset of $C([0,T],\mathbb{R}^n)$ endowed with the maximum norm. Then G is said to be relatively compact in $C([0,T],\mathbb{R}^n)$ if and only if G satisfies the criteria of equicontinuity and uniform boundedness.*

3. Controllability of linear systems

Definition 3.1. The system (1.1) is designated as relatively controllable if, for any arbitrary initial vector function $\phi(\varsigma)$, where $\varsigma \in [-h,0]$, and for any arbitrary initial control function $u_0(\varsigma)$, also defined on the interval $\varsigma \in [-h,0]$, in conjunction with a specified final state $\upsilon_T \in \mathbb{R}^n$ at time T, there exists an admissible control function $u(\varsigma)$ defined for $\varsigma \in [0,T]$ such that the corresponding solution $\upsilon(\varsigma)$, for $\varsigma \in [-h,T]$, to the system (1.1) fulfills the conditions $\upsilon(T) = \upsilon_T$ and $\upsilon(\varsigma) = \phi(\varsigma)$ for $\varsigma \in [-h,0]$.

Currently, our objective is to elucidate $u(\varsigma)$, where $\varsigma \in [0,T]$, within the context of the solution articulated in (2.1) to facilitate the definition of the Gramian matrix. By implementing the substitution $x = r_i(s)$, along with the time-lead functions $\eta_i(x)$, in the solution delineated in (2.1) as presented in Lemma 2.1, it follows that the integral limits transitioning from $s = 0$ to $s = \varsigma$ in the third term of the solution (2.1) transform into the integral limits from $x = r_i(0)$ to $x = r_i(\varsigma)$. Additionally, the differential of the variable s corresponds to the differential of the lead function $\eta_i(x)$, specifically $\mathrm{d}s = \mathrm{d}\eta_i(x) = \eta_i^{'}(x)\mathrm{d}x$, as a result of the relation $\eta_i(x) = \eta_i(r_i(s)) = s$ under the substitution $x = r_i(s)$. Consequently, the solution (2.1) can be reformulated as follows, where the control function u is contingent solely upon the variable x:

$$\begin{aligned}\upsilon(\varsigma) = &\sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma)p_i + \sum_{i=0}^{d_1} \int_{-h_i}^{0} \mathfrak{X}_{\alpha,\alpha}(\varsigma - s - h_i)T_i\phi(s)\mathrm{d}s \\ &+ \sum_{j=0}^{d_2} \int_{r_j(0)}^{r_j(\varsigma)} \mathfrak{X}_{\alpha,\alpha}(\varsigma - \eta_j(x))A_j\eta_j'(x)u(x)\mathrm{d}x,\end{aligned}$$

which may be partitioned as delineated in the following manner:

$$\begin{aligned}\upsilon(\varsigma) = &\sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma)p_i + \sum_{i=0}^{d_1} \int_{-h_i}^{0} \mathfrak{X}_{\alpha,\alpha}(\varsigma - s - h_i)T_i\phi(s)\mathrm{d}s \\ &+ \sum_{j=0}^{m} \int_{r_j(0)}^{r_j(\varsigma)} \mathfrak{X}_{\alpha,\alpha}(\varsigma - \eta_j(s))A_j\eta_j'(s)u(s)\mathrm{d}s \\ &+ \sum_{j=m+1}^{d_2} \int_{r_j(0)}^{r_j(\varsigma)} \mathfrak{X}_{\alpha,\alpha}(\varsigma - \eta_j(s))A_j\eta_j'(s)u(s)\mathrm{d}s.\end{aligned}$$

The inequalities delineated in (1.2) enable the transformation of the equation situated marginally above into the one positioned marginally below:

$$\begin{aligned}\upsilon(T) &= \sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(T)p_i + \sum_{i=0}^{d_1} \int_{-h_i}^{0} \mathfrak{X}_{\alpha,\alpha}(T-s-h_i)T_i\phi(s)\mathrm{d}s \\ &+ \sum_{j=0}^{m} \int_{r_j(0)}^{0} \mathfrak{X}_{\alpha,\alpha}(T-\eta_j(s))A_j\eta_j'(s)u_0(s)\mathrm{d}s \\ &+ \sum_{j=m+1}^{d_2} \int_{r_j(0)}^{r_j(T)} \mathfrak{X}_{\alpha,\alpha}(T-\eta_j(s))A_j\eta_j'(s)u_0(s)\mathrm{d}s \\ &+ \sum_{j=0}^{m} \int_{0}^{T} \mathfrak{X}_{\alpha,\alpha}(T-\eta_j(s))A_j\eta_j'(s)u(s)\mathrm{d}s.\end{aligned}$$

The right-hand side of the aforementioned equation can be partitioned into two distinct subsets. It is important to note that the first four terms are entirely independent of the admissible control function u. Consequently, we can isolate these components and denote them as follows:

$$\begin{aligned}S(\varsigma) &= \sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma)p_i + \sum_{i=0}^{d_1} \int_{-h_i}^{0} \mathfrak{X}_{\alpha,\alpha}(\varsigma-s-h_i)T_i\phi(s)\mathrm{d}s \\ &+ \sum_{j=0}^{m} \int_{r_j(0)}^{0} \mathfrak{X}_{\alpha,\alpha}(\varsigma-\eta_j(s))A_j\eta_j'(s)u_0(s)\mathrm{d}s \\ &+ \sum_{j=m+1}^{d_2} \int_{r_j(0)}^{r_j(\varsigma)} \mathfrak{X}_{\alpha,\alpha}(\varsigma-\eta_j(s))A_j\eta_j'(s)u_0(s)\mathrm{d}s,\end{aligned}$$

and

$$\Phi(\varsigma, s) := \sum_{j=0}^{m} \mathfrak{X}_{\alpha,\alpha}(\varsigma-\eta_j(s))A_j\eta_j'(s). \tag{3.1}$$

Accordingly, the Gramian matrix is formally defined as follows:

$$W(0,T) = \int_0^T \Phi(T,s)\Phi^*(T,s)\mathrm{d}s,$$

where the notation $.^*$ denotes the transposition operation applied to a matrix.

Remark 3.1. It is evident that $W(0,T)$ is rigorously defined for $0.5 < \alpha < 1$ due to the convergence of the series associated with the two-parameter Mittag-Leffler function, which is encompassed by the multi-delayed two-parameter Mittag-Leffler type matrix function $\mathfrak{X}_{\alpha,\alpha}(\varsigma)$ under this condition.

Theorem 3.1. *The linear control system (1.1) possesses a relative degree of controllability if and only if the Gramian matrix is nonsingular.*

Proof. Due to the nonsingular nature of $W(0,T) := W$, the existence of its inverse W^{-1} is assured. Consequently, considering υ_T as the target final state at time T, the system described in (1.1) can be rendered controllable through the control function delineated below:

$$u(t) = \Phi^*(T,t)W^{-1}\left[\upsilon_T - S(T)\right].$$

It is readily apparent that

$$\begin{aligned}\upsilon(T) &= S(T) + \int_0^T \Phi(T,s)u(s)\mathrm{d}s \\ &= S(T) + \int_0^T \Phi(T,s)\Phi^*(T,s)W^{-1}\left[\upsilon_T - S(T)\right]\mathrm{d}s \\ &= S(T) + WW^{-1}\left[\upsilon_T - S(T)\right] \\ &= \upsilon_T.\end{aligned}$$

Assume now that the matrix W is singular while the system (1.1) is controllable. Given that W is singular, it follows that there exists a nontrivial vector $\rho \in \mathbb{R}^n$ such that

$$W\rho = 0,$$

from which it follows that

$$\rho^* W\rho = 0 = \int_0^T \rho^*\Phi(T,s)\Phi^*(T,s)\rho\mathrm{d}s,$$

which implies

$$\rho^*\Phi(T,s) = 0, \quad \text{for all } s \in [0,T].$$

Given that the system (1.1) possesses the property of controllability, and considering two terminal states 0 and ρ at time T, there exist two control inputs u_1 and u_2 such that the associated solution $\upsilon(t)$ satisfies:

$$\upsilon(T) = S(T) + \int_0^T \Phi(T,s)u_1(s)\mathrm{d}s = 0,$$

$$\upsilon(T) = S(T) + \int_0^T \Phi(T,s)u_2(s)\mathrm{d}s = \rho,$$

which leads to

$$\rho = \int_0^T \Phi(T,s)\left(u_2(s) - u_1(s)\right)\mathrm{d}s.$$

Then we obtain

$$\|\rho\|^2 = \rho^*\rho = \int_0^T \rho^*\Phi(T,s)\left(u_2(s) - u_1(s)\right)\mathrm{d}s = 0.$$

This implies that $\rho = 0$, which contradicts the assumption that ρ is nonzero. □

4. Controllability of nonlinear systems

This part aims to examine the comparative controllability of the following nonlinear fractional dynamical systems characterized by multiple delays in the state variables and multiple delays in the control variables for $\varsigma \in (0,T]$:

$$\begin{cases} {}^{RL}\mathcal{D}^{\alpha}_{0^+}\upsilon(\varsigma) = M\upsilon(\varsigma) + \sum_{i=0}^{d_1} T_i\upsilon(\varsigma - h_i) + \sum_{i=0}^{d_2} A_i u(r_i(\varsigma)) + \daleth(\varsigma, \upsilon(\varsigma), u(\varsigma)), \\ {}^{RL}\mathcal{D}^{\alpha-i}_{0^+}\upsilon(\varsigma)\big|_{\varsigma=0} = p_i, \quad i = 1,2,\ldots,l, \\ \upsilon(\varsigma) = \phi(\varsigma), \quad \varsigma \in [-h,0], \quad h > 0, \end{cases} \tag{4.1}$$

where $\daleth : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and the remaining terms are as defined in (1.1).

Lemma 4.1. *Let $\mathfrak{X}_{\alpha,\alpha}(\varsigma)$ be as defined in (2.3). Then the following holds:*

$$\int_0^\varsigma \|\mathfrak{X}_{\alpha,\alpha}(\varsigma - s)\| \, \mathrm{d}s = \|\mathfrak{X}_{\alpha,\alpha+1}(\varsigma)\| .$$

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^\varsigma \left\| \sum_{m=0}^{\infty} \sum_{\substack{i_1+\ldots+i_d \le m \\ i_1,\ldots,i_d \ge 0}} P_{m+1}(i_1 h_1, \ldots, i_{d_1} h_{d_1}) \frac{\left(\varsigma - s - \sum_{j=1}^{d_1} i_j h_j\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} \right\| \mathrm{d}s \\
&= \left\| \sum_{m=0}^{\infty} \sum_{\substack{i_1+\ldots+i_d \le m \\ i_1,\ldots,i_d \ge 0}} P_{m+1}(i_1 h_1, \ldots, i_{d_1} h_{d_1}) \right\| \int_0^\varsigma \frac{\left(\varsigma - s - \sum_{j=1}^{d_1} i_j h_j\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} \mathrm{d}s \\
&= \left\| \sum_{m=0}^{\infty} \sum_{\substack{i_1+\ldots+i_d \le m \\ i_1,\ldots,i_d \ge 0}} P_{m+1}(i_1 h_1, \ldots, i_{d_1} h_{d_1}) \frac{\left(\varsigma - \sum_{j=1}^{d_1} i_j h_j\right)_+^{m\alpha+\alpha}}{\Gamma(m\alpha+\alpha+1)} \right\| \\
&= \|\mathfrak{X}_{\alpha,\alpha+1}(\varsigma)\| .
\end{aligned}$$

□

It is established that $E = C([0,T], \mathbb{R}^n) \times C([0,T], \mathbb{R}^m)$ constitutes a Banach space equipped with the uniform norm:

$$\|(\upsilon, u)\| = \|\upsilon\| + \|u\| ,$$

where

$$\|\upsilon\| = \sup \{|\upsilon(\varsigma)| : \varsigma \in [0,T]\} .$$

Upon the application of the time-lead functions in conjunction with the inequalities delineated in (1.2), a comprehensive solution to the system represented by (4.1) may be articulated as follows:

$$\begin{aligned}
\upsilon(T) = &\sum_{i=0}^{l} \mathfrak{X}_{\alpha,\alpha-i-1}(T) p_i + \sum_{i=0}^{d_1} \int_{-h_i}^0 \mathfrak{X}_{\alpha,\alpha}(T - s - h_i) T_i \phi(s) \mathrm{d}s \\
&+ \sum_{j=0}^{m} \int_{r_j(0)}^0 \mathfrak{X}_{\alpha,\alpha}(T - \eta_j(s)) A_j \eta_j'(s) u_0(s) \mathrm{d}s \\
&+ \sum_{j=m+1}^{d_2} \int_{r_j(0)}^{r_j(T)} \mathfrak{X}_{\alpha,\alpha}(T - \eta_j(s)) A_j \eta_j'(s) u_0(s) \mathrm{d}s \\
&+ \sum_{j=0}^{m} \int_0^T \mathfrak{X}_{\alpha,\alpha}(T - \eta_j(s)) A_j \eta_j'(s) u(s) \mathrm{d}s \\
&+ \int_0^T \mathfrak{X}_{\alpha,\alpha}(T - s) \daleth(s, \upsilon(s), u(s)) \mathrm{d}s.
\end{aligned}$$

Let us consider that the pair of functions υ and u constitutes a solution to the following system of nonlinear integral equations:

$$u(\varsigma) = \Phi^*(T,\varsigma)W^{-1}\left[\bar{\upsilon} - \int_0^T \mathfrak{X}_{\alpha,\alpha}(T-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s\right], \tag{4.2}$$

$$\upsilon(\varsigma) = S(\varsigma) + \int_0^\varsigma \Phi(\varsigma,s)u(s)\mathrm{d}s + \int_0^\varsigma \mathfrak{X}_{\alpha,\alpha}(\varsigma-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s, \tag{4.3}$$

where $\bar{\upsilon} := \upsilon_T - S(T)$. If u is a control function defined on the interval $[0,T]$, and υ is the corresponding solution to the system (4.1) under the control u, then it can be readily verified that

$$\begin{aligned}
\upsilon(T) &= S(T) + \int_0^T \Phi(T,s)u(s)\mathrm{d}s + \int_0^T \mathfrak{X}_{\alpha,\alpha}(T-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s \\
&= S(T) + WW^{-1}\left[\upsilon_T - S(T) - \int_0^T \mathfrak{X}_{\alpha,\alpha}(T-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s\right] \\
&\quad + \int_0^T \mathfrak{X}_{\alpha,\alpha}(T-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s \\
&= \upsilon_T.
\end{aligned}$$

In view of this observation, our objective is to determine the conditions under which a solution pair (υ,u) to the integral equations (4.2) and (4.3) exists. The following theorem is established to this end.

Theorem 4.1. *Assume that $1 > \alpha > 0.5$, and the continuous function $\urcorner$ satisfies the following condition:*

$$\lim_{|(\upsilon,u)|\to\infty} \frac{|\urcorner(\varsigma,\upsilon,u)|}{|(\upsilon,u)|} = 0,$$

uniformly in $\varsigma \in [0,T]$. If the linear system (1.1) exhibits relative controllability, then the nonlinear system (4.1) also demonstrates relative controllability.

Proof. **Step 1:** We define a mapping $\Psi : E \to E$ such that $\Psi(\upsilon,u) = (y,z)$, where

$$\begin{aligned}
z(\varsigma) &= \Phi^*(T,\varsigma)W^{-1}\left[\upsilon_T - S(T) - \int_0^T \mathfrak{X}_{\alpha,\alpha}(T-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s\right], \\
y(\varsigma) &= S(\varsigma) + \int_0^\varsigma \Phi(\varsigma,s)u(s)\mathrm{d}s + \int_0^\varsigma \mathfrak{X}_{\alpha,\alpha}(\varsigma-s)\urcorner(s,\upsilon(s),u(s))\mathrm{d}s.
\end{aligned}$$

For notational clarity, we define:

$$\begin{aligned}
k &= \max\left\{T\left\|\Phi(T,0)\right\|, 1\right\}, \\
c_1 &= 4k\left\|\Phi(T,0)\right\|\left\|W^{-1}\right\|\left\|\mathfrak{X}_{\alpha,\alpha+1}(T)\right\|, \\
e_1 &= 4k\left\|\Phi^*(T,0)\right\|\left\|W^{-1}\right\|\left\|\bar{\upsilon}\right\|, \\
c_2 &= 4\max\left\{T\left\|\Phi(T,0)\right\|, \left\|\mathfrak{X}_{\alpha,\alpha+1}(T)\right\|\right\}, \\
e_2 &= 4\left\|S(T)\right\|, \quad c = \max\{c_1,c_2\}, \quad e = \max\{e_1,e_2\}.
\end{aligned}$$

For further information on $\mathfrak{X}_{\alpha,\beta}(\varsigma)$, see [44]. By applying Lemma 2.2, there exists a constant $\varepsilon > 0$ such that if $\|(\upsilon, u)\| \le \varepsilon$, then

$$c\,|\daleth(\varepsilon,\upsilon,u)| + e \le \varepsilon, \quad \text{for all } \varepsilon \in [0,T].$$

We will now show that $\Psi(\mathcal{D}_\varepsilon) \subseteq \mathcal{D}_\varepsilon$, where $\mathcal{D}_\varepsilon := \{(\upsilon,u) \in E : \|(\upsilon,u)\| \le \varepsilon\}$. Suppose $(\upsilon,u) \in \mathcal{D}_\varepsilon$. Then we obtain:

$$\begin{aligned}
\|z(\varsigma)\| &\le \|\Phi^*(T,0)\| \left\|W^{-1}\right\| (\|\upsilon_T\| + \|S(T)\|) \\
&\quad + \|\Phi^*(T,0)\| \left\|W^{-1}\right\| \|\mathfrak{X}_{\alpha,\alpha+1}(T)\| \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))| \\
&\le \frac{e_1}{4k} + \frac{c_1}{4k} \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))| \\
&\le \frac{1}{4k}\left(e + c \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))|\right) \\
&\le \frac{\varepsilon}{4}.
\end{aligned}$$

Similarly, for $y(\varsigma)$ we estimate:

$$\begin{aligned}
\|y(\varsigma)\| &\le \|S(\varsigma)\| + T\,\|\Phi(T,0)\|\,\|u\| + \|\mathfrak{X}_{\alpha,\alpha+1}(T)\| \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))| \\
&\le \frac{e_2}{4} + k\,\|u\| + \frac{c_2}{4} \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))| \\
&\le \frac{1}{4}(e + c \sup_{\varsigma\in[0,T]} |\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))|) + \frac{\varepsilon}{4} \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

As a result, $\|(z,y)\| = \|z\| + \|y\| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \frac{3\varepsilon}{4}$, indicating that $\Psi(\mathcal{D}_\varepsilon) \subseteq \mathcal{D}_\varepsilon$.

Step 2: Now, let us consider $\varsigma_1, \varsigma_2 \in [0,T]$ with $\varsigma_1 < \varsigma_2$. For every $(y,z) \in \mathcal{D}_\varepsilon$, where $\varepsilon > 0$, it is imperative to demonstrate that $\Psi(\mathcal{D}_\varepsilon)$ exhibits equicontinuity. Indeed,

$$\begin{aligned}
&\|z(\varsigma_1) - z(\varsigma_2)\| \\
&\le \|\Phi^*(T,\varsigma_1) - \Phi^*(T,\varsigma_2)\| \\
&\times \left\|W^{-1}\right\| \left[\|\bar{\upsilon}\| + \|S(T)\| + \int_0^T \|\mathfrak{X}_{\alpha,\alpha}(T-s)\|\,\|\daleth(s,\upsilon(s),u(s))\|\,\mathrm{d}s\right],
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\|y(\varsigma_1) - y(\varsigma_2)\| \\
&\le \sum_{i=0}^{l} \|\mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma_1) - \mathfrak{X}_{\alpha,\alpha-i-1}(\varsigma_2)\|\,\|p_i\| \\
&+ \sum_{i=0}^{d_1} \int_{-h_i}^{0} \|\mathfrak{X}_{\alpha,\alpha}(\varsigma_1 - s - h_i) - \mathfrak{X}_{\alpha,\alpha}(\varsigma_2 - s - h_i)\|\,\|T_i\phi(s)\|\,\mathrm{d}s \\
&+ \sum_{j=0}^{m} \int_{r_j(0)}^{0} \|\mathfrak{X}_{\alpha,\alpha}(\varsigma_1 - \eta_j(s)) - \mathfrak{X}_{\alpha,\alpha}(\varsigma_2 - \eta_j(s))\|\,\|A_j\|\,\left\|\eta_j'(s)\right\|\,\|u_0(s)\|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=m+1}^{d_2}\int_{r_j(\varsigma_1)}^{r_j(\varsigma_2)}\|\mathfrak{X}_{\alpha,\alpha}(\varsigma_2-\eta_j(s))\|\,\|A_j\|\,\|\eta_j'(s)\|\,\|u_0(s)\|\,\mathrm{d}s\\
&+\sum_{j=m+1}^{d_2}\int_{r_j(0)}^{r_j(\varsigma_1)}\|\mathfrak{X}_{\alpha,\alpha}(\varsigma_1-\eta_j(s))-\mathfrak{X}_{\alpha,\alpha}(\varsigma_2-\eta_j(s))\|\,\|A_j\|\,\|\eta_j'(s)\|\,\|u_0(s)\|\,\mathrm{d}s\\
&+\int_{\varsigma_1}^{\varsigma_2}\|\Phi(\varsigma_2,s)\|\,\|u(s)\|\,\mathrm{d}s\\
&+\int_{0}^{\varsigma_1}\|\Phi(\varsigma_1,s)-\Phi(\varsigma_2,s)\|\,\|u(s)\|\,\mathrm{d}s\\
&+\int_{\varsigma_1}^{\varsigma_2}\|\mathfrak{X}_{\alpha,\alpha}(\varsigma_2-s)\|\,\mathrm{d}s\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))|\\
&+\int_{0}^{\varsigma_1}\|\mathfrak{X}_{\alpha,\alpha}(\varsigma_1-s)-\mathfrak{X}_{\alpha,\alpha}(\varsigma_2-s)\|\,\mathrm{d}s\sup_{\varsigma\in[0,T]}|\daleth(\varsigma,\upsilon(\varsigma),u(\varsigma))|\,. \qquad (4.5)
\end{aligned}
$$

Since the pair (υ, u) is assumed to lie within the bounded set $\mathcal{D}_\varepsilon$, the right-hand sides of (4.4) and (4.5) are independent of any particular $(\upsilon, u) \in \mathcal{D}_\varepsilon$ and converge to zero as $\varsigma_1 \to \varsigma_2$. This observation confirms that the operator Ψ is both equicontinuous and uniformly bounded on $\mathcal{D}_\varepsilon$ (as shown in Step 1), and is therefore relatively compact by the Arzelà-Ascoli theorem. Given that $\mathcal{D}_\varepsilon$ is nonempty, closed, bounded, and convex, Schauder's fixed point theorem guarantees that the operator Ψ admits a fixed point within $\mathcal{D}_\varepsilon$. The resulting fixed point (υ, u) of the operator Ψ corresponds to a solution pair for the integral equations (4.2) and (4.3). Hence, the nonlinear system (4.1) is relatively controllable, as the initial state $\phi(\varepsilon)$ and the initial control function $u_0(\varepsilon)$ are prescribed for each $\varepsilon \in [-h, 0]$, while the terminal state υ_T at time T remains arbitrary. □

5. Conclusion

In this manuscript, we investigated the relative controllability of fractional dynamical systems characterized by multiple delays in both control and state variables. The Gramian matrix is formulated to ascertain the necessary and sufficient conditions for the relative controllability of linear systems. Furthermore, under certain natural conditions imposed on the nonlinear function $\daleth$, we employed the Schauder fixed point theorem to establish sufficient conditions for the relative controllability of nonlinear systems.

While the proposed framework is mathematically rigorous under the stated assumptions, it is important to acknowledge that the results may not remain valid if such conditions are violated. In particular, the assumptions of continuity, boundedness, and strict monotonicity of the delay functions play a crucial role in ensuring the existence of inverse mappings and the applicability of the fixed point methods. For instance, if the delay functions are discontinuous or non-monotonic, or if the nonlinear term $\daleth$ is unbounded or fails to satisfy the asymptotic conditions, the analytical structure of the solution may collapse. These scenarios lie beyond the scope of the current study.

As an open problem , we propose that future research may focus on the relative controllability of fractional systems under relaxed regularity conditions, such as discontinuous or piecewise smooth delays, and more general nonlinear perturbations.

Addressing such settings would extend the applicability of the controllability theory and pose interesting theoretical challenges.

As a promising direction for future research, the theoretical framework and methods presented in this study may serve as a foundation for analyzing more complex classes of systems. In particular, it would be of interest to extend the current results to fractional integrodifferential control systems [9, 55, 56], where memory effects are explicitly modeled. Another natural extension involves semilinear fractional systems [63–65], especially those incorporating nonlinearities in both state and control components. Moreover, systems with distributed delays and admissible control functions [35] present an important yet challenging generalization. Exploring the relative controllability of such systems—under appropriate assumptions—remains an open and valuable problem for the community.

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