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Three-Dimensional Polynomial Differential Systems with an Isolated Compact Invariant Algebraic Surface*

Dongmei Xiao[1], Shengnan Yin[2] and Chenwan Zhou[1,†]

Abstract The aim of this paper is to characterize the simplest three-dimensional polynomial differential system having an equilibrium and a 2-dimensional orientable smooth compact manifold with genus $g \leq 1$ in $\mathbb{R}^3$, where the 2-dimensional orientable smooth compact manifold is sphere $\mathbb{E}^2$ or torus $\mathbb{T}^2$. We first look for the smallest degree of polynomial differential systems with both an equilibrium and an isolated compact invariant algebraic surface $\mathbb{E}^2$ or $\mathbb{T}^2$. It is shown that the smallest degree of the system depends on the relative position between the equilibrium and the compact invariant algebraic surface in $\mathbb{R}^3$. Furthermore, the sufficient and necessary algebraic conditions are given for the smallest order three-dimensional polynomial differential system having both an equilibrium and an isolated compact invariant algebraic surface. Lastly, we discuss the influence of the coexistence of an isolated compact invariant algebraic surface and an equilibrium on dynamics of the three-dimensional polynomial differential system.

Keywords Three-dimensional, polynomial differential systems, isolated, invariant, compact algebraic surface

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1. Introduction

Three-dimensional polynomial differential systems are widely used as some approximations of mathematical models from physics, biology, chemistry and egineering, for instance, Lorenz system [14, 15], Kolmogorov system [1, 2, 9] and Chua system [3–5], whose dynamics plays an important role in understanding complex nonlinear phenomena such as chaos, strange attractors and turbulence. The occurrence of complex phenomena is related to some invariant sets of the three-dimensional polynomial differential system. The invariant set is usually composed of equilibrium points, compact nontrivial orbits and some noncompact orbits whose limit sets are either the equilibrium points or the compact orbits of the system. A natural question is raised: can an invariant set of three-dimensional polynomial differential systems become an isolated 2-dimensional compact invariant manifold embedded

†The corresponding author.
Email address: xiaodm@sjtu.edu.cn (Dongmei Xiao), snyin06@163.com (Shengnan Yin), daydayupzcw@sjtu.edu.cn (Chenwan Zhou)
[1]School of Mathematical Sciences, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai 200240, China
[2]Huantai No.1 Middle School, 256400, Shandong, China
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in $\mathbb{R}^3$? The definition of an invariant manifold can be found in [7]. To discuss the question, we consider the following three-dimensional polynomial differential systems

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3),\ i = 1, 2, 3, \tag{1.1}$$

where $f_i(x_1, x_2, x_3)$ is a polynomial in the variables x_1, x_2 and x_3 with degree m_i, denoted by $f_i \in \mathbb{R}[x_1, x_2, x_3]$, $i = 1, 2, 3$. Here $\mathbb{R}[x_1, x_2, x_3]$ is the ring of the polynomials in the variables x_1, x_2 and x_3 with coefficients in $\mathbb{R}$. We say that

$$n = \max_{i=1,2,3} \{\deg f_i(x_1, x_2, x_3)\} = \max \{m_1, m_2, m_3\}$$

is *the order (or degree)* of system (1.1). The existence of invariant algebraic surfaces for system (1.1) or a kind of system (1.1) (e.g. Kolmogorov system) and its dynamics have been studied by many mathematicians, see [8, 10–13, 16, 17] and references therein.

In the paper, we are interested in the case that system (1.1) has at least one equilibrium in $\mathbb{R}^3$. Without loss of generality, it can be assumed that the equilibrium is at the origin $O(0, 0, 0)$. Then the n-order system (1.1) with an equilibrium at $O(0, 0, 0)$ can be rewritten as

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3) = \sum_{k=1}^{n} f_i^{(k)}(x_1, x_2, x_3),\ i = 1, 2, 3, \tag{1.2}$$

where $f_i^{(k)}(x_1, x_2, x_3)$ is a homogeneous polynomial in the variables x_1, x_2 and x_3 with degree k, $1 \le k \le n$ and $i = 1, 2, 3$. Hence, the vector field $(f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))$ associated with system (1.2) has no constant terms.

Let M be a smooth closed orientable surface of genus g in $\mathbb{R}^3$. Then the simplest 2-dimensional orientable smooth compact manifolds with genus $g \le 1$ in $\mathbb{R}^3$ are ellipsoid and tori. Note that the 2-dimensional ellipsoid surface can be transformed into a unit sphere in $\mathbb{R}^3$ by an affine transformation. Inspired by Llibre et al. [13], we consider if the n-order system (1.2) has an invariant sphere with the following form

$$\mathbb{E}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = 1, a, b, c \in \mathbb{R}\},$$

and an invariant torus in the form

$$\mathbb{T}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2 - r^2)^2 + x_3^2 = 1, r > 0\}.$$

Our aim is to look for the smallest degree of system (1.2) such that system (1.2) has an isolated invariant sphere $\mathbb{E}^2$ (or torus $\mathbb{T}^2$), and give the sufficient and necessary algebraic conditions for system (1.2) with the smallest degree having an isolated invariant sphere $\mathbb{E}^2$ (or torus $\mathbb{T}^2$). We say $\mathbb{E}^2$ ($\mathbb{T}^2$) is *invariant* for system (1.2) if the orbit of system (1.2) passing through any a point in $\mathbb{E}^2$ ($\mathbb{T}^2$, resp.) is completely contained in $\mathbb{E}^2$ ($\mathbb{T}^2$, resp.). Note that $\mathbb{E}^2$ and $\mathbb{T}^2$ are quadratic and quartic algebraic surfaces, respectively. One of the important tools used to study the invariance of algebraic surfaces for polynomial differential systems is Darboux theory founded by Darboux in [6]. Assume that $H(x_1, x_2, x_3)$ is a real polynomial with degree m, $m \ge 1$. Then the algebraic surface $H(x_1, x_2, x_3) = 0$ is invariant for the n-order polynomial differential system (1.2) if there exists a real polynomial $K(x_1,x_2,x_3)$ satisfying the following equality

$$\sum_{i=1}^{3}\frac{\partial H(x_1,x_2,x_3)}{\partial x_i}f_i(x_1,x_2,x_3)=H(x_1,x_2,x_3)K(x_1,x_2,x_3). \tag{1.3}$$

The polynomial $K(x_1,x_2,x_3)$ is called the *cofactor* of the invariant algebraic surface $H(x_1,x_2,x_3)=0$. Obviously, the degree of $K(x_1,x_2,x_3)$ is less than $n-1$ by (1.3). If $K(x_1,x_2,x_3)\equiv 0$, then $H(x_1,x_2,x_3)$ is a *first integral* of system (1.2). Hence, $H(x_1,x_2,x_3)=c$, for any $c\in\mathbb{R}$, is an invariant algebraic surface of system (1.2). This implies that $H(x_1,x_2,x_3)=0$ is not isolated in $\mathbb{R}^3$. Therefore, $H(x_1,x_2,x_3)=0$ is an *isolated invariant algebraic surface* if and only if $K(x_1,x_2,x_3)\not\equiv 0$.

Note that the relative position between the center $S(a,b,c)$ of the sphere $\mathbb{E}^2$ and the equilibrium $O(0,0,0)$ will determine the relative position between $O(0,0,0)$ and $\mathbb{E}^2$. In general, there are three possible relative positions between $\mathbb{E}^2$ and $O(0,0,0)$: (i). the equilibrium $O(0,0,0)$ is exactly the center $S(a,b,c)$ of the sphere $\mathbb{E}^2$ iff $(a,b,c)=(0,0,0)$; (ii). $O(0,0,0)\in\mathbb{E}^2$ iff $a^2+b^2+c^2=1$; (iii). $O(0,0,0)\notin\mathbb{E}^2\cup\{(a,b,c)\}$ iff $0<a^2+b^2+c^2\neq 1$. Since $r>0$ in the expression of torus $\mathbb{T}^2$, there exist two possible relative positions between the torus $\mathbb{T}^2$ and the equilibrium $O(0,0,0)$: (I). $O(0,0,0)\in\mathbb{T}^2$ iff $r=1$; (II). $O(0,0,0)\notin\mathbb{T}^2$ iff $0<r\neq 1$. We characterize the smallest degree of system (1.2) with an isolated invariant sphere $\mathbb{E}^2$ (torus $\mathbb{T}^2$) in three relative positions (in two relative positions, resp.), and give the sufficient and necessary algebraic conditions for system (1.2) with the smallest degree having an isolated invariant $\mathbb{E}^2$ ($\mathbb{T}^2$) in the corresponding relative positions, which provide implicit three-dimensional polynomial differential systems with an isolated invariant $\mathbb{E}^2$ ($\mathbb{T}^2$), respectively.

The structure of the paper is as follows. In Section 2, we first study the smallest degree of system (1.2) such that system (1.2) has both an isolated equilibrium O and an isolated sphere $\mathbb{E}^2$; then we characterize this three dimensional polynomial differential systems with the smallest degree and obtain the sufficient and necessary algebraic conditions under three different relative positions between the equilibrium O and $\mathbb{E}^2$. In Section 3, we study the smallest degree of system (1.2) such that system (1.2) has both an isolated equilibrium O and an isolated torus $\mathbb{T}^2$ in two different relative positions between the equilibrium O and $\mathbb{T}^2$, and characterize this three dimensional polynomial differential systems with the smallest degree. In the last section, we discuss the influence of the coexistence of the isolated invariant $\mathbb{E}^2$ (or $\mathbb{T}^2$) and an equilibrium on dynamics of the three dimensional polynomial differential system.

2. Three-dimensional polynomial differential systems having an isolated invariant sphere

In the section, we investigative the smallest degree (or order) of the n-order three-dimensional polynomial differential system (1.2) having an isolated invariant sphere $\mathbb{E}^2$ in three different relative positions. Then we discuss the sufficient and necessary algebraic conditions for the smallest order system (1.2) with the invariant $\mathbb{E}^2$ in the corresponding three different relative positions.

We first study the relative position between the equilibrium $O(0,0,0)$ and the invariant sphere $\mathbb{E}^2$: the center $S(a,b,c)$ of $\mathbb{E}^2$ is exactly at $O(0,0,0)$, in other words, $(a,b,c) = (0,0,0)$. We denote this sphere by $\mathbb{E}_0^2$.

Theorem 2.1. *If the n-order system* (1.2) *has an isolated invariant sphere* $\mathbb{E}_0^2$ *with* $a^2+b^2+c^2=0$*, then the smallest degree of system* (1.2) *is three, that is,* $n=3$*. Furthermore, a cubic polynomial differential system in the form*

$$\begin{cases} \frac{dx_1}{dt} = \sum_{i=1}^{3} a_i x_i + \sum_{1\le i\le j\le 3} a_{ij}x_ix_j + \sum_{1\le i\le j\le k\le 3} a_{ijk}x_ix_jx_k, \\ \frac{dx_2}{dt} = \sum_{i=1}^{3} b_i x_i + \sum_{1\le i\le j\le 3} b_{ij}x_ix_j + \sum_{1\le i\le j\le k\le 3} b_{ijk}x_ix_jx_k, \\ \frac{dx_3}{dt} = \sum_{i=1}^{3} c_i x_i + \sum_{1\le i\le j\le 3} c_{ij}x_ix_j + \sum_{1\le i\le j\le k\le 3} c_{ijk}x_ix_jx_k, \end{cases} \tag{2.1}$$

where $a_i, b_i, c_i, a_{ij}, b_{ij}, c_{ij}, a_{ijk}, b_{ijk}$ *and* c_{ijk} *are real parameters for any* $i,j,k \in \{1,2,3\}$*, has an isolated invariant sphere* $\mathbb{E}_0^2$ *in* $\mathbb{R}^3$ *if and only if*

(i) all coefficients of quadratic homogeneous terms of the vector field associated with (2.1) *are zero, that is,*

$$a_{ij} = b_{ij} = c_{ij} = 0, \ \forall i,j \in \{1,2,3\}; \tag{2.2}$$

(ii) the coefficients of linear homogeneous terms of the vector field associated with (2.1) *satisfy*

$$a_1^2 + b_2^2 + c_3^2 + (a_2+b_1)^2 + (a_3+c_1)^2 + (b_3+c_2)^2 \neq 0; \tag{2.3}$$

(iii) the coefficients of linear homogeneous terms and cubic homogeneous terms of the vector field associated with (2.1) *satisfy the following equalities*

$$\begin{cases} a_1 = -a_{111}, \ b_2 = -b_{222}, \ c_3 = -c_{333}, \\ a_2 + b_1 + a_{112} + b_{111} = 0, \\ a_3 + c_1 + a_{113} + c_{111} = 0, \\ b_2 + a_1 + a_{122} + b_{112} = 0, \\ c_3 + a_1 + a_{133} + c_{113} = 0, \\ a_2 + b_1 + a_{222} + b_{122} = 0, \\ a_3 + c_1 + a_{333} + c_{133} = 0, \\ b_3 + c_2 + b_{223} + c_{222} = 0, \\ c_3 + b_2 + b_{233} + c_{223} = 0, \\ b_3 + c_2 + b_{333} + c_{233} = 0, \\ b_3 + c_2 + a_{123} + b_{113} + c_{112} = 0, \\ a_3 + c_1 + a_{223} + b_{123} + c_{122} = 0, \\ a_2 + b_1 + a_{233} + b_{133} + c_{123} = 0. \end{cases} \tag{2.4}$$

Proof. If $\mathbb{E}_0^2$ is an invariant sphere of system (1.2), then let $H_0(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 - 1$. The derivative of $H_0(x_1,x_2,x_3)$ along the flow of system (1.2) should be $H_0(x_1,x_2,x_3)K(x_1,x_2,x_3)$ by Darboux theory, that is

$$\frac{dH_0(x_1,x_2,x_3)}{dt}\Big|_{(1.2)} = \sum_{i=1}^{3} \frac{\partial H_0}{\partial x_i} f_i(x_1,x_2,x_3) = H_0(x_1,x_2,x_3)K(x_1,x_2,x_3), \tag{2.5}$$

where $K(x_1, x_2, x_3)$ is a cofactor of the invariant sphere $\mathbb{E}_0^2$, whose degree is at most $n-1$. Assume that

$$K(x_1, x_2, x_3) = \sum_{k=0}^{n-1} K^{(k)}(x_1, x_2, x_3),$$

where $K^{(k)}(x_1, x_2, x_3)$ is the homogeneous polynomial terms with degree k of $K(x_1, x_2, x_3)$. By calculating both sides of equality (2.5) respectively, we have

$$\begin{aligned}\sum_{i=1}^{3} 2x_i f_i(x_1, x_2, x_3) =& - K(x_1, x_2, x_3) + (x_1^2 + x_2^2 + x_3^2) K(x_1, x_2, x_3)\\ =& - \sum_{k=0}^{n-1} K^{(k)}(x_1, x_2, x_3)\\ &+ (x_1^2 + x_2^2 + x_3^2)\left(\sum_{k=0}^{n-1} K^{(k)}(x_1, x_2, x_3)\right).\end{aligned} \tag{2.6}$$

Since $f_i(x_1, x_2, x_3)$ has no constant term for $i = 1, 2, 3$, the left side of (2.6) has no constant term and linear terms. Therefore, the $K^{(0)}(x_1, x_2, x_3)$ and $K^{(1)}(x_1, x_2, x_3)$ should be null by (2.6), i.e.

$$K^{(0)}(x_1, x_2, x_3) \equiv K^{(1)}(x_1, x_2, x_3) \equiv 0. \tag{2.7}$$

Note that $K(x_1, x_2, x_3) \not\equiv 0$ if $\mathbb{E}_0^2$ is an isolated invariant sphere of system (1.2). Hence. the equality (2.6) holds only if the degree of $K(x_1, x_2, x_3)$ is greater than two. This leads to the degree n of system (1.2) being greater than three, i.e. $n \geq 3$.

We now consider system (1.2) as cubic polynomial differential systems, that is, we consider system (2.1).

In the following we focus on whether system (2.1) has an isolated invariant sphere $\mathbb{E}_0^2$ or not, and characterize system (2.1) if it has an isolated invariant sphere $\mathbb{E}_0^2$.

Let us first assume that $\mathbb{E}_0^2$ is an isolated invariant sphere of system (2.1). Then by Darboux theory, we derive that there exists a non-zero polynomial $K_0(x_1, x_2, x_3)$ with degree less than two such that

$$\frac{dH_0(x_1, x_2, x_3)}{dt}|_{(2.1)} = H_0(x_1, x_2, x_3) K_0(x_1, x_2, x_3). \tag{2.8}$$

Note that $K_0(x_1, x_2, x_3)$ is a quadratic homogeneous polynomial due to (2.7), i.e. $K_0(x_1, x_2, x_3) = K_0^{(2)}(x_1, x_2, x_3)$. By directly computing both sides of equality (2.8), we have

$$\begin{aligned}&(x_1^2 + x_2^2 + x_3^2 - 1) K_0^{(2)}(x_1, x_2, x_3)\\ =&2x_1 \sum_{i=1}^{3} a_i x_i + 2x_2 \sum_{i=1}^{3} b_i x_i + 2x_3 \sum_{i=1}^{3} c_i x_i\\ &+ 2x_1 \sum_{1\leq i\leq j\leq 3} a_{ij} x_i x_j + 2x_2 \sum_{1\leq i\leq j\leq 3} b_{ij} x_i x_j + 2x_3 \sum_{1\leq i\leq j\leq 3} c_{ij} x_i x_j\\ &+ 2x_1 \sum_{1\leq i\leq j\leq k\leq 3} a_{ijk} x_i x_j x_k + 2x_2 \sum_{1\leq i\leq j\leq k\leq 3} b_{ijk} x_i x_j x_k\end{aligned} \tag{2.9}$$

$$+ 2x_3 \sum_{1\le i\le j\le k\le 3} c_{ijk}x_ix_jx_k.$$

Since the left hand of equality (2.9)

$$(x_1^2+x_2^2+x_3^2-1)K_0^{(2)}(x_1,x_2,x_3) = -K_0^{(2)}(x_1,x_2,x_3)+(x_1^2+x_2^2+x_3^2)K_0^{(2)}(x_1,x_2,x_3),$$

is composed of quadratic homogeneous terms and fourth-order homogeneous terms in the variables x_1, x_2 and x_3, the cubic homogeneous terms of the variables x_1, x_2 and x_3 on the right hand of (2.9) should be null, that is,

$$2x_1 \sum_{1\le i\le j\le 3} a_{ij}x_ix_j + 2x_2 \sum_{1\le i\le j\le 3} b_{ij}x_ix_j + 2x_3 \sum_{1\le i\le j\le 3} c_{ij}x_ix_j \equiv 0.$$

This leads to $a_{ij} = b_{ij} = c_{ij} = 0$, $\forall i,j \in \{1,2,3\}$, that is, all coefficients of quadratic homogeneous terms of the vector field associated with system (2.1) are zero. So the conclusion (i) in the theorem is true if system (2.1) has an isolated invariant sphere $\mathbb{E}_0^2$. Further, we obtain the following equalities by comparing the quadratic homogeneous terms and fourth-order homogeneous terms on both sides of equality (2.9), respectively

$$K_0^{(2)}(x_1,x_2,x_3) = -(2x_1 \sum a_ix_i + 2x_2 \sum b_ix_i + 2x_3 \sum c_ix_i), \tag{2.10}$$

$$\begin{aligned} &(x_1^2+x_2^2+x_3^2)K_0^{(2)}(x_1,x_2,x_3) \\ =&2x_1 \sum_{1\le i\le j\le k\le 3} a_{ijk}x_ix_jx_k \\ &+ 2x_2 \sum_{1\le i\le j\le k\le 3} b_{ijk}x_ix_jx_k + 2x_3 \sum_{1\le i\le j\le k\le 3} c_{ijk}x_ix_jx_k. \end{aligned} \tag{2.11}$$

Pluging (2.10) to (2.11), we have

$$\begin{aligned} &-(x_1^2+x_2^2+x_3^2)(x_1\sum_{i=1}^3 a_ix_i + x_2\sum_{i=1}^3 b_ix_i + x_3\sum_{i=1}^3 c_ix_i) \\ =&x_1 \sum_{1\le i\le j\le k\le 3} a_{ijk}x_ix_jx_k + x_2 \sum_{1\le i\le j\le k\le 3} b_{ijk}x_ix_jx_k + x_3 \sum_{1\le i\le j\le k\le 3} c_{ijk}x_ix_jx_k. \end{aligned} \tag{2.12}$$

Comparing the corresponding coefficients of the same polynomial term on the left and right hands of (2.12), we obtain the equalities (2.4) in (iii) of the theorem. And $K_0^{(2)}(x_1,x_2,x_3) \not\equiv 0$, which leads that (2.3) in (ii) of the theorem holds by (2.10).

On the other hand, if the parameters of system (2.1) satisfy conditions (2.2), (2.3) and (2.4), then it can be checked that

$$\frac{dH_0(x_1,x_2,x_3)}{dt}|_{(2.1)} = 2H_0(x_1,x_2,x_3)K_0(x_1,x_2,x_3),$$

where

$$K_0(x_1,x_2,x_3) = -a_1x_1^2-b_2x_2^2-c_3x_3^2-(a_2+b_1)x_1x_2-(a_3+c_1)x_1x_3-(b_3+c_2)x_2x_3 \not\equiv 0.$$

Thus, $\mathbb{E}^2$ is an isolated invariant sphere of system (2.1). The theorem is proved. □

We now consider the second relative position between the equilibrium $O(0,0,0)$ and the invariant sphere $\mathbb{E}^2$: $O(0,0,0) \in \mathbb{E}^2$, in other words, $a^2+b^2+c^2=1$. We denote this sphere by $\mathbb{E}_1^2$.

Theorem 2.2. *If the n-order system* (1.2) *has an isolated invariant sphere $\mathbb{E}_1^2$ with $a^2+b^2+c^2=1$, then the smallest degree of system* (1.2) *is two, that is, $n=2$. Furthermore, a quadratic polynomial differential system*

$$\begin{cases} \frac{dx_1}{dt} = \sum_{i=1}^{3} a_i x_i + \sum_{1\le i\le j\le 3} a_{ij} x_i x_j, \\ \frac{dx_2}{dt} = \sum_{i=1}^{3} b_i x_i + \sum_{1\le i\le j\le 3} b_{ij} x_i x_j, \\ \frac{dx_3}{dt} = \sum_{i=1}^{3} c_i x_i + \sum_{1\le i\le j\le 3} c_{ij} x_i x_j, \end{cases} \tag{2.13}$$

where a_i, b_i, c_i, a_{ij}, b_{ij} and c_{ij} are real parameters for any $i,j \in \{1,2,3\}$, has an isolated invariant sphere $\mathbb{E}_1^2$ in $\mathbb{R}^3$ if and only if there exists a constant k such that the following equalities and inequality all hold.

$$\begin{cases} aa_1+bb_1+cc_1 = ak, \\ aa_2+bb_2+cc_2 = bk, \\ aa_3+bb_3+cc_3 = ck, \\ a^2+b^2+c^2 = 1; \end{cases} \tag{2.14}$$

$$\begin{cases} k-2aa_{11} = 2a_1-2bb_{11}-2cc_{11}, \\ k-2bb_{22} = 2b_2-2aa_{22}-2cc_{22}, \\ k-2cc_{33} = 2c_3-2aa_{33}-2bb_{33}, \\ -2ab_{22}-2ba_{11} = a_2+b_1-aa_{12}-bb_{12}-cc_{12}, \\ -2ac_{33}-2ca_{11} = a_3+c_1-aa_{13}-bb_{13}-cc_{13}, \\ -2bc_{33}-2cb_{22} = b_3+c_2-aa_{23}-bb_{23}-cc_{23}; \end{cases} \tag{2.15}$$

$$\begin{cases} 2a_{11} = 2a_{22}+2b_{12} = 2a_{33}+2c_{13}, \\ 2a_{12}+2b_{11} = 2b_{22} = 2b_{33}+2c_{23}, \\ 2a_{13}+2c_{11} = 2b_{23}+2c_{22} = 2c_{33}, \\ a_{23}+b_{13}+c_{12} = 0; \end{cases} \tag{2.16}$$

and

$$\sum_{i=1}^{3}(aa_i+bb_i+cc_i)^2 + a_{11}^2 + b_{22}^2 + c_{33}^2 \neq 0. \tag{2.17}$$

To prove Theorem 2.2, we first give a lemma, which shows that system (1.2) can not have an isolated invariant sphere $\mathbb{E}_1^2$ if it is a linear differential system.

Lemma 2.1. *Assume that system* (1.2) *is a linear differential system in the form*

$$\begin{cases} \frac{dx_1}{dt} = \sum_{i=1}^{3} a_i x_i, \\ \frac{dx_2}{dt} = \sum_{i=1}^{3} b_i x_i, \\ \frac{dx_3}{dt} = \sum_{i=1}^{3} c_i x_i, \end{cases} \tag{2.18}$$

where a_i, b_i, c_i are real parameters for $i=1,2,3.$, which has an invariant sphere $\mathbb{E}_1^2$ in $\mathbb{R}^3$. Then $\mathbb{E}_1^2$ is not isolated, that is, system (2.18) *has a first integral $(x_1-a)^2 + (x_2-b)^2 + (x_3-c)^2 - 1$ with $a^2+b^2+c^2=1$.*

Proof. Assume that system (2.18) has an invariant sphere $\mathbb{E}_1^2$ in $\mathbb{R}^3$. Then let

$$H_1(x_1, x_2, x_3) = (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - 1,$$

where a, b and c satisfy $a^2 + b^2 + c^2 = 1$. We have

$$\frac{dH_1(x_1, x_2, x_3)}{dt}|_{(2.18)} = H_1(x_1, x_2, x_3)k_0,$$

where k_0 is a constant. Hence,

$$\begin{aligned} &((x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - 1)k_0 \\ =&2(x_1 - a)(\sum_{i=1}^{3} a_i x_i) + 2(x_2 - b)(\sum_{i=1}^{3} b_i x_i) + 2(x_3 - c)(\sum_{i=1}^{3} c_i x_i). \end{aligned} \tag{2.19}$$

Since the coefficients of the same polynomial terms on both sides of (2.19) should be equal, we obtain

$$\begin{cases} -2ak_0 = -2aa_1 - 2bb_1 - 2cc_1, \\ -2bk_0 = -2aa_2 - 2bb_2 - 2cc_2, \\ -2ck_0 = -2aa_3 - 2bb_3 - 2cc_3, \end{cases} \tag{2.20}$$

and

$$\begin{cases} k_0 = 2a_1, \quad a_1 = b_2 = c_3, \\ a_2 + b_1 = 0, a_3 + c_1 = 0, c_2 + b_3 = 0. \end{cases} \tag{2.21}$$

Pulging (2.21) into (2.20), one gets

$$\begin{cases} aa_1 + ba_2 + ca_3 = 0, \\ ba_1 - aa_2 - cc_2 = 0, \\ ca_1 - aa_3 + bc_2 = 0. \end{cases} \tag{2.22}$$

Note that $a^2 + b^2 + c^2 = 1$. There are three cases for a, b and c. Case 1: $abc \neq 0$; Case 2: there is only one of a, b and c being zero; Case 3: there are only two of a, b and c being zero. Multiplying some equations of system (2.22) by an appropriate nonzero a, b and c, and then adding them together, we obtain $a_1 = 0$ in the three cases. Hence, from (2.21) we have $k_0 = 0$. This implies

$$\frac{dH_1(x_1, x_2, x_3)}{dt}|_{(2.18)} \equiv 0.$$

Therefore, $H_1(x_1, x_2, x_3)$ is a first integral of system (2.18). And so the invariant sphere $\mathbb{E}_1^2$ is not isolated in $\mathbb{R}^3$. We finish the proof.

□

We are now in the position to prove Theorem 2.2.

Proof. [Proof of Theorem 2.2] If system (2.1) has an isolated invariant sphere $\mathbb{E}_1^2$, then the degree of system (2.1) is greater than two according to Lemma 2.1, that is, $n \geq 2$. We now consider system (1.2) as quadratic polynomial differential systems (2.13), and consider whether system (2.13) has an isolated invariant sphere $\mathbb{E}_1^2$ or not. We first assume that system (2.13) has an isolated invariant sphere $\mathbb{E}_1^2$ in $\mathbb{R}^3$.

Then by Darboux theory, there exists a nonzero polynomial with degree less than one, denoted by $K_1(x_1, x_2, x_3)$, such that

$$\frac{dH_1(x_1, x_2, x_3)}{dt}|_{(2.13)} = H_1(x_1, x_2, x_3)K_1(x_1, x_2, x_3), \tag{2.23}$$

where $H_1(x_1, x_2, x_3) = (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 - 1$ with $a^2 + b^2 + c^2 = 1$, $K_1(x_1, x_2, x_3) = k + k_1x_1 + k_2x_2 + k_3x_3$, in which $k, k_l \in \mathbb{R}$ and $l \in \{1, 2, 3\}$. By a direct computation of (2.23), we have

$$\begin{aligned}
&(x_1^2 + x_2^2 + x_3^2 - 2ax_1 - 2bx_2 - 2cx_3)(k + \sum_{l=1}^{3} k_l x_l)\\
=&2(x_1 - a)(\sum_{i=1}^{3} a_i x_i + \sum_{1\le i\le j\le 2} a_{ij}x_ix_j)\\
+&2(x_2 - b)(\sum_{i=1}^{3} b_i x_i + \sum_{1\le i\le j\le 2} b_{ij}x_ix_j)\\
+&2(x_3 - c)(\sum_{i=1}^{3} c_i x_i + \sum_{1\le i\le j\le 2} c_{ij}x_ix_j).
\end{aligned} \tag{2.24}$$

Hence, the coefficients of the same polynomial terms on both sides of (2.24) should be equal. By comparing the coefficients of the same polynomial terms with degree one, two and three on both sides of (2.24), respectively, we get the condition (2.14),

$$\begin{cases}
k - 2ak_1 = 2a_1 - 2aa_{11} - 2bb_{11} - 2cc_{11},\\
k - 2bk_2 = 2b_2 - 2aa_{22} - 2bb_{22} - 2cc_{22},\\
k - 2ck_3 = 2c_3 - 2aa_{33} - 2bb_{33} - 2cc_{33},\\
-ak_2 - bk_1 = a_2 + b_1 - aa_{12} - bb_{12} - cc_{12},\\
-ak_3 - ck_1 = a_3 + c_1 - aa_{13} - bb_{13} - cc_{13},\\
-bk_3 - ck_2 = b_3 + c_2 - aa_{23} - bb_{23} - cc_{23},
\end{cases} \tag{2.25}$$

and

$$\begin{cases}
k_1 = 2a_{11} = 2a_{22} + 2b_{12} = 2a_{33} + 2c_{13},\\
k_2 = 2a_{12} + 2b_{11} = 2b_{22} = 2b_{33} + 2c_{23},\\
k_3 = 2a_{13} + 2c_{11} = 2b_{23} + 2c_{22} = 2c_{33},\\
a_{23} + b_{13} + c_{12} = 0.
\end{cases} \tag{2.26}$$

Plugging (2.26) and (2.14) into (2.25), we derive (2.15). And condition (2.26) is the condition (2.16). If $K_1(x_1, x_2, x_3) \not\equiv 0$, then condition (2.17) should hold. Thus, these conditions are necessary.

On the other hand, assume that the parameters of system (2.13) satisfy conditions (2.14)-(2.16). Then one can check that there is a nonzero polynomial $K(x_1, x_2, x_3) = k + 2a_{11}x_1 + 2b_{22}x_2 + 2c_{33}x_3$, where k satisfies (2.14) and (2.15), such that

$$\frac{dH_1(x_1, x_2, x_3)}{dt}|_{(2.13)} = 2H_1(x_1, x_2, x_3)K(x_1, x_2x_3). \tag{2.27}$$

Therefore, system (2.13) has an isolated invariant sphere $\mathbb{E}_1^2$ in $\mathbb{R}^3$.

We finish the proof. □

Last we consider the third relative position between the equilibrium $O(0,0,0)$ and the invariant sphere $\mathbb{E}^2$: $O(0,0,0) \notin \mathbb{E}^2 \cup \{(0,0,0)\}$, in other words, $0 < a^2+b^2+c^2 \neq 1$. We denote this sphere by $\mathbb{E}^2_2$. Using the arguments similar to those in proofs of Theorem 2.1 and Theorem 2.2, we obtain the following conclusion. To save the space, we leave its proof to readers.

Theorem 2.3. *If the n-order system* (1.2) *has an isolated invariant sphere* $\mathbb{E}^2_2$ *with* $0 < a^2+b^2+c^2 \neq 1$, *then the smallest degree of system* (1.2) *is two, that is,* $n = 2$. *Furthermore, the quadratic polynomial differential system* (2.13) *has an isolated invariant sphere* $\mathbb{E}^2_2$ *in* $\mathbb{R}^3$ *if and only if the following equalities and inequality all hold.*

$$\begin{cases} a_{11} = a_{33} + c_{13} = a_{22} + b_{12} = -(a^2+b^2+c^2)^{-1}(aa_1 + bb_1 + cc_1), \\ a_{12} + b_{11} = b_{22} = b_{33} + c_{23} = -(a^2+b^2+c^2)^{-1}(aa_2 + bb_2 + cc_2), \\ a_{13} + c_{11} = b_{23} + c_{22} = c_{33} = -(a^2+b^2+c^2)^{-1}(aa_3 + bb_3 + bc_3), \\ a_{23} + b_{13} + c_{12} = 0; \end{cases} \tag{2.28}$$

$$\begin{cases} a_1 + aa_{11} = bb_{11} + cc_{11}, \\ a_2 + b_1 + a(a_{12} + 2b_{11}) + b(b_{12} + 2a_{22}) = cc_{12}, \\ a_3 + c_1 + a(a_{13} + 2c_{11}) + c(c_{13} + 2a_{33}) = bb_{13}, \\ b_2 + bb_{22} = aa_{22} + cc_{22}, \\ b_3 + c_2 + b(b_{23} + 2c_{22}) + c(c_{23} + 2b_{33}) = aa_{23}, \\ c_3 + cc_{33} = aa_{33} + bb_{33}; \end{cases} \tag{2.29}$$

and

$$\sum_{i=1}^{3}(aa_i + bb_i + cc_i)^2 \neq 0. \tag{2.30}$$

3. Three-dimensional polynomial differential systems having an isolated invariant tours $\mathbb{T}^2$

In this section, we consider the smallest degree of three-dimensional polynomial differential systems (1.2) having an isolated two-dimensional invariant torus $\mathbb{T}^2$ in two possible relative positions between the torus $\mathbb{T}^2$ and the equilibrium $O(0,0,0)$. We first consider the case that $O(0,0,0) \in \mathbb{T}^2$, that is $r = 1$, and we denote the invariant torus with $r = 1$ by $\mathbb{T}^2_0$.

Theorem 3.1. *If the n-order system* (1.2) *has an isolated invariant torus* $\mathbb{T}^2_0$ *with* $r = 1$, *then the smallest degree of system* (1.2) *is two, that is,* $n = 2$. *Further, a quadratic polynomial differential system* (2.13) *has an isolated invariant torus* $\mathbb{T}^2_0$ *if and only if*

(i) the coefficients of the linear term of system (2.13) *satisfy*

$$a_1 = a_3 = b_2 = b_3 = c_1 = c_2 = c_3 = 0, \text{ and } a_2 + b_1 = 0; \tag{3.1}$$

(ii) the coefficients of the quadratic term of system (2.13) *satisfy*

$$\begin{cases} a_{11} = a_{33} = b_{22} = b_{33} = c_{12} = c_{13} = c_{23} = 0, \\ a_{12} + b_{11} = 0, \quad a_{22} + b_{12} = 0, \quad a_{23} + b_{13} = 0, \\ 2a_{13} + c_{11} = 0, \quad 2b_{23} + c_{22} = 0, \quad 2a_{13} = 2b_{23} = c_{33}; \end{cases} \tag{3.2}$$

(iii) $a_{13} \neq 0$.

Proof. Assume that system (1.2) has an isolated invariant torus $\mathbb{T}_0^2$ with $r = 1$. Then let

$$H(x_1, x_2, x_3) = ((x_1^2 + x_2^2) - 1)^2 + x_3^2 - 1 = (x_1^2 + x_2^2)^2 + x_3^2 - 2(x_1^2 + x_2^2).$$

There exists a nonzero polynomial $K(x_1, x_2, x_3)$ with degree less than $n - 1$ such that

$$\frac{dH(x_1, x_2, x_3)}{dt}|_{(1.2)} = H(x_1, x_2, x_3)K(x_1, x_2, x_3). \tag{3.3}$$

Hence, we have

$$\begin{aligned} &4x_1(x_1^2 + x_2^2 - 1)f_1 + 4x_2(x_1^2 + x_2^2 - 1)f_2 + 2x_3 f_3 \\ =&((x_1^2 + x_2^2)^2 - 2(x_1^2 + x_2^2) + x_3^2)(\sum_{k=0}^{n-1} K^{(k)}(x_1, x_2, x_3)). \end{aligned} \tag{3.4}$$

If (3.4) holds for $n = 1$, then

$$a_1 = a_3 = b_2 = b_3 = c_1 = c_2 = c_3 = 0, \text{ and } a_2 + b_1 = 0 \tag{3.5}$$

by comparing the coefficients of quadratic polynomial terms on both sides of (3.4). However, linear system (2.18) with (3.5) has only invariant cylindrical surfaces and no invariant tori. Therefore, $n \geq 2$ if system (1.2) has an isolated invariant torus $\mathbb{T}_0^2$.

We now give the sufficient and necessary conditions for quadratic system (2.13) having an isolated invariant torus $\mathbb{T}_0^2$.

Let us first assume that system (2.13) has an invariant torus $\mathbb{T}_0^2$ with $r = 1$. Then $H(x_1, x_2, x_3)$ is an invariant algebraic surface and its cofactor $K(x_1, x_2, x_3)$ can be written as $K(x_1, x_2, x_3) = K_0 + K_1(x_1, x_2, x_3)$, where K_0 is constant and $K_1(x_1, x_2, x_3)$ is a homogeneous polynomial with degree one. Applying Darboux theory one gets

$$\begin{aligned} &((x_1^2 + x_2^2 - 1)^2 + x_3^2 - 1)(K_0 + K_1(x_1, x_2, x_3)) \\ =&4x_1(x_1^2 + x_2^2 - 1)(\sum_{i=1}^{3} a_i x_i + \sum_{1 \leq i \leq j \leq 3} a_{ij} x_i x_j) \\ &+ 4x_2(x_1^2 + x_2^2 - 1)(\sum_{i=1}^{3} b_i x_i + \sum_{1 \leq i \leq j \leq 3} b_{ij} x_i x_j) \\ &+ 2x_3(\sum_{i=1}^{3} c_i x_i + \sum_{1 \leq i \leq j \leq 3} c_{ij} x_i x_j). \end{aligned} \tag{3.6}$$

Hence, the homogeneous polynomial with degree two and degreee four on both sides of equality (3.6) should be equal, respectively, then we obtain

$$(-2(x_1^2+x_2^2)+x_3^2)K_0 = -4x_1(\sum_{i=1}^{3} a_i x_i) - 4x_2(\sum_{i=1}^{3} b_i x_i) + 2x_3(\sum_{i=1}^{3} c_i x_i), \tag{3.7}$$

and

$$(x_1^2+x_2^2)^2 K_0 = 4x_1(x_1^2+x_2^2)(\sum_{i=1}^{3} a_i x_i) + 4x_2(x_1^2+x_2^2)(\sum_{i=1}^{3} b_i x_i). \tag{3.8}$$

And the homogeneous polynomial with degree three and degree five on both sides of equality (3.6) should be equal, respectively, then we obtain

$$\begin{aligned}(-2(x_1^2+x_2^2)+x_3^2)K_1(x_1,x_2,x_3) =& -4x_1(\sum_{1\le i\le j\le 2} a_{ij}x_ix_j) \\ &- 4x_2(\sum_{1\le i\le j\le 2} b_{ij}x_ix_j) + 2x_3(\sum_{1\le i\le j\le 2} c_{ij}x_ix_j),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(x_1^2+x_2^2)^2 K_1(x_1,x_2,x_3) =& 4x_1(x_1^2+x_2^2)(\sum_{1\le i\le j\le 2} a_{ij}x_ix_j) \\ &+ 4x_2(x_1^2+x_2^2)(\sum_{1\le i\le j\le 2} b_{ij}x_ix_j).\end{aligned} \tag{3.10}$$

By comparing the coefficients of the same polynomial terms in (3.7) and (3.8), respectively, we get

$$\begin{cases} K_0 = 2c_3 = 2a_1 = 2b_2, \\ a_2 + b_1 = 0, \\ 2a_3 - c_1 = 0, \\ 2b_3 - c_2 = 0, \end{cases} \tag{3.11}$$

and

$$\begin{cases} K_0 = 4a_1 = 4b_2 = 2a_1 + 2b_2, \\ a_2 + b_1 = 0, \\ a_3 = b_3 = 0. \end{cases} \tag{3.12}$$

Combining (3.11) and (3.12), and simplifying them, we obtain the condition (3.1).

Plugging (3.9) into (3.10), and making replacement and simplification, we have

$$\begin{aligned}&4x_1x_3^2(\sum_{1\le i\le j\le 2} a_{ij}x_ix_j) + 4x_2x_3^2(\sum_{1\le i\le j\le 2} b_{ij}x_ix_j) \\ =& 4x_1^3(\sum_{1\le i\le j\le 2} a_{ij}x_ix_j) + 4x_1^2x_2(\sum_{1\le i\le j\le 2} b_{ij}x_ix_j) + 2x_1^2x_3(\sum_{1\le i\le j\le 2} c_{ij}x_ix_j) \\ &+ 4x_1x_2^2(\sum_{1\le i\le j\le 2} a_{ij}x_ix_j) + 4x_2^3(\sum_{1\le i\le j\le 2} b_{ij}x_ix_j) + 2x_2^2x_3(\sum_{1\le i\le j\le 2} c_{ij}x_ix_j).\end{aligned} \tag{3.13}$$

By comparing the coefficients of the same polynomial term on both sides of (3.13), we have

$$\begin{cases} 0 = 4a_{11}, \quad 0 = 4a_{12} + 4b_{11}, \quad 0 = 4a_{13} + 2c_{11}, \\ 0 = 4a_{22} + 4b_{12} + 4a_{11}, \quad 0 = 4a_{23} + 4b_{13} + 2c_{12}, \\ 4a_{11} = 4a_{33} + 2c_{13}, \quad 0 = 4b_{22} + 4a_{12} + 4b_{11}, \\ 0 = 4b_{23} + 2c_{22} + 4a_{13} + 2c_{11}, \quad 4a_{12} + 4b_{11} = 4b_{33} + 2c_{23}, \\ 4a_{13} = 2c_{33}, \quad 0 = 4a_{22} + 4b_{12}, \quad 0 = 4a_{23} + 4b_{13} + 2c_{12}, \\ 4a_{22} + 4b_{12} = 4a_{33} + 2c_{13}, \quad 4a_{23} + 4b_{13} = 0, \quad 4a_{33} = 0, \\ 4b_{22} = 0, \quad 0 = 4b_{23} + 2c_{22}, \quad 4b_{22} = 4b_{33} + 2c_{23}, \\ 4b_{23} = 2c_{33}, \quad 4b_{33} = 0. \end{cases} \tag{3.14}$$

By rearranging and simplifying the relations in (3.14), we obtain the condition (3.2). Since $K(x_1, x_2, x_3) \not\equiv 0$, $a_{13} \neq 0$, which is the condition (*iii*) in this theorem.

On the other hand, if the parameters of system (2.13) satisfy conditions (3.1) and (3.2), then it can be checked that

$$\frac{dH(x_1, x_2, x_3)}{dt}|_{(2.13)} = 4a_{13}x_3H(x_1, x_2, x_3),$$

which implies that $\mathbb{T}_0^2$ with $r = 1$ is invariant for the flow of system (2.13). Note that $a_{13} \neq 0$ by condition (*iii*) in this theorem. Thus, system (2.13) has an isolated invariant torus $\mathbb{T}_0^2$. The proof is finished. □

Last we consider the case that $O(0, 0, 0) \notin \mathbb{T}^2$, that is $0 < r \neq 1$. We denote the invariant torus with $0 < r \neq 1$ by $\mathbb{T}_1^2$.

Theorem 3.2. *If the n-order system (1.2) has an isolated invariant torus* $\mathbb{T}_1^2$ *with* $0 < r \neq 1$*, then the smallest degree of system (1.2) is three, that is,* $n = 3$*. Further, a cubic polynomial differential system (2.1) has an isolated invariant torus* $\mathbb{T}_1^2$ *if and only if*

(*i*) *the coefficients* a_i, b_i *and* c_i *of linear terms of system (2.1) satisfy the following conditions*

$$a_1 = b_2 = a_2 + b_1 = 0. \tag{3.15}$$

(*ii*) *the coefficients* a_{ij}, b_{ij}, c_{ij} *of quadratic terms of system (2.1) satisfy the following equalities*

$$\begin{cases} b_{22} = b_{23} = b_{33} = a_{11} = a_{13} = a_{33} = 0, \\ b_{11} + a_{12} = 0, \\ b_{12} + a_{22} = 0, \\ b_{13} + a_{23} = 0, \\ c_{11} = c_{12} = c_{13} = c_{23} = c_{22} = c_{33} = 0. \end{cases} \tag{3.16}$$

(*iii*) *the coefficients* a_{ijk}, b_{ijk} *and* c_{ijk} *of the cubic terms of system (2.1) satisfy the* *following equalities.*

$$\begin{cases} a_{111} = b_{222} = a_{333} = b_{333} = 0, \\ a_{112} + b_{111} = 0, \\ a_{122} + b_{112} = 0, \\ a_{222} + b_{122} = 0, \\ a_{233} + b_{133} = 0, \\ a_{113} = a_{223} + b_{123} = \frac{1}{2}c_{133} = \frac{1}{2}dc_1 - r^2 da_3, \\ a_{123} + b_{113} = b_{223} = \frac{1}{2}c_{233} = \frac{1}{2}dc_2 - r^2 db_3, \\ a_{133} = b_{233} = \frac{1}{2}c_{333} = \frac{1}{2}dc_3, \end{cases} \tag{3.17}$$

and

$$\begin{cases} c_{111} = c_{122} = (2r^4 d - 2)a_3 - r^2 dc_1, \\ c_{112} = c_{222} = (2r^4 d - 2)b_3 - r^2 dc_2, \\ c_{113} = c_{223} = -r^2 dc_3, \\ c_{123} = 0, \end{cases} \tag{3.18}$$

where $d := (r^4 - 1)^{-1}$.

(iv)

$$c_3^2 + (c_1 - 2r^2 a_3)^2 + (c_2 - 2r^2 b_3)^2 \neq 0. \tag{3.19}$$

Proof. Assume that $\mathbb{T}_1^2$ is an invariant algebraic surface of system (1.2). Let

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2 - r^2)^2 + x_3^2 - 1,\ 0 < r \neq 1.$$

Then $F(x_1, x_2, x_3)$ is a Darboux polynomial of system (1.2) and there exists a cofactor $K(x_1, x_2, x_3) = \sum_{j=0}^{n-1} K^{(j)}(x_1, x_2, x_3)$, such that

$$\frac{dF(x_1, x_2, x_3)}{dt}|_{(1.2)} = \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} f_i(x_1, x_2, x_3) = F(x_1, x_2, x_3)K(x_1, x_2, x_3), \tag{3.20}$$

where $K^{(j)}(x_1, x_2, x_3)$ is a homogeneous polynomial with degree j in the variables x_1, x_2, and x_3. By a direct computation, we have

$$\sum_{i=1}^{3} \frac{\partial F}{\partial x_i} f_i(x_1, x_2, x_3) = 4x_1(x_1^2 + x_2^2 - r^2)f_1 + 4x_2(x_1^2 + x_2^2 - r^2)f_2 + 2x_3 f_3, \tag{3.21}$$

which is a polynomial without constant term and linear terms. And

$$\begin{aligned} F(x_1, x_2, x_3)K(x_1, x_2, x_3) =& ((x_1^2 + x_2^2 - r^2)^2 + x_3^2 - 1) \sum_{j=0}^{n-1} K^{(j)}(x_1, x_2, x_3) \\ =& (r^4 - 1) \sum_{j=0}^{n-1} K^{(j)}(x_1, x_2, x_3) \\ &+ ((x_1^2 + x_2^2)^2 - 2(x_1^2 + x_2^2)r^2 + x_3^2) \sum_{j=0}^{n-1} K^{(j)}(x_1, x_2, x_3). \end{aligned} \tag{3.22}$$

Since $0 < r \neq 1$, $r^4 - 1 \neq 0$. From (3.20), we derive that

$$K^{(0)}(x_1, x_2, x_3) = K^{(1)}(x_1, x_2, x_3) \equiv 0.$$

Therefore, the degree of the cofactor $K(x_1, x_2, x_3)$ is at least two, which yields that $n \geq 3$.

In the following we consider a cubic polynomial differential system (2.1) and characterize system (2.1) having an isolated invariant torus $\mathbb{T}_1^2$.

We first assume that $\mathbb{T}_1^2$ is an isolated invariant algebraic surface of system (2.1); then, in view of (3.21) and (3.22), we derive that

$$\begin{aligned}
&((x_1^2 + x_2^2 - r^2)^2 + x_3^2 - 1)K^{(2)}(x_1, x_2, x_3) \\
=&((x_1^2 + x_2^2)^2 - 2r^2(x_1^2 + x_2^2) + x_3^2 + r^4 - 1)K^{(2)}(x_1, x_2, x_3) \qquad (3.23)\\
=&4x_1(x_1^2 + x_2^2 - r^2)(\sum_{i=1}^{3} a_i x_i + \sum_{1\leq i\leq j\leq 2} a_{ij} x_i x_j + \sum_{1\leq i\leq j\leq k\leq 2} a_{ijk} x_i x_j x_k) \\
&+ 4x_2(x_1^2 + x_2^2 - r^2)(\sum_{i=1}^{3} b_i x_i + \sum_{1\leq i\leq j\leq 2} b_{ij} x_i x_j + \sum_{1\leq i\leq j\leq k\leq 2} b_{ijk} x_i x_j x_k) \\
&+ 2x_3(\sum_{i=1}^{3} c_i x_i + \sum_{1\leq i\leq j\leq 2} c_{ij} x_i x_j + \sum_{1\leq i\leq j\leq k\leq 2} c_{ijk} x_i x_j x_k),
\end{aligned}$$

where $K^{(2)}(x_1, x_2, x_3)$ is a homogeneous polynomial with degree two.

Note that the polynomial on the left hand of (3.23) contains only terms with even degrees. So the coefficients of polynomial terms with odd degrees on the right hand of (3.23) should be zero. This leads that

$$\begin{cases} b_{22} = b_{23} = b_{33} = a_{11} = a_{13} = a_{33} = 0, \\ b_{11} + a_{12} = 0, \\ b_{12} + a_{22} = 0, \\ b_{13} + a_{23} = 0, \end{cases} \qquad (3.24)$$

and

$$\begin{cases} c_{33} = b_{22} = a_{11} = 0, \\ c_{11} - 2r^2 a_{13} = 0, \\ c_{13} - 2r^2 a_{33} = 0, \\ c_{23} - 2r^2 b_{33} = 0, \\ c_{22} - 2r^2 b_{23} = 0, \\ c_{12} - 2r^2 b_{13} - 2r^2 a_{23} = 0, \\ b_{11} + a_{12} = 0,\ b_{12} + a_{22} = 0. \end{cases} \qquad (3.25)$$

Further simplified by combining conditions in (3.24) and (3.25), it gives that

$$c_{11} = c_{12} = c_{13} = c_{23} = c_{22} = c_{33} = 0. \qquad (3.26)$$

Taking into account (3.26) and (3.24), we give the condition (3.16).

Now we compare the polynomial terms of degree two in (3.23), and obtain that

$$(r^4 - 1)K^{(2)}(x_1, x_2, x_3) = 2x_3 \sum_{i=1}^{3} c_i x_i - 4r^2(x_2 \sum_{i=1}^{3} b_i x_i + x_1 \sum_{i=1}^{3} a_i x_i). \qquad (3.27)$$

Again comparing the polynomial terms of degree four in (3.23), one gets

$$
\begin{aligned}
&(x_3^2-2r^2(x_1^2+x_2^2))K^{(2)}(x_1,x_2,x_3) \qquad (3.28)\\
=&2x_3\sum_{1\le i\le j\le k\le 2}c_{ijk}x_ix_jx_k+4x_1(x_1^2+x_2^2)(\sum_{i=1}^{3}a_ix_i)+4x_2(x_1^2+x_2^2)(\sum_{i=1}^{3}b_ix_i)\\
&-4r^2x_2(\sum_{1\le i\le j\le k\le 2}b_{ijk}x_ix_jx_k)-4r^2x_1(\sum_{1\le i\le j\le k\le 2}a_{ijk}x_ix_jx_k).
\end{aligned}
$$

Lastly, we compare the polynomial terms of degree six in (3.23) and obtain

$$
(x_1^2+x_2^2)K^{(2)}(x_1,x_2,x_3)=4x_1\sum_{1\le i\le j\le k\le 2}a_{ijk}x_ix_jx_k+4x_2\sum_{1\le i\le j\le k\le 2}b_{ijk}x_ix_jx_k. \qquad (3.29)
$$

Let $d=(r^4-1)^{-1}$. Then equality (3.27) shows that $K^{(2)}(x_1,x_2,x_3)$ is determined by the linear terms of system (2.1). Equality (3.29) tells us that polynomial terms with coefficients a_{ijk} and b_{ijk} are determined by $K^{(2)}(x_1,x_2,x_3)$. And equality (3.28) yields that polynomial terms with coefficient c_{ijk} are controlled by linear terms of system (2.1) and polynomial terms with coefficients a_{ijk} and b_{ijk}.

By comparing the coefficients in the same polynomial term in (3.29), we have

$$
\begin{cases}
a_{111}=-r^2da_1,\\
a_{112}+b_{111}=-r^2d(a_2+b_1),\\
a_{113}=\frac{1}{2}dc_1-r^2da_3,\\
a_{122}+b_{112}=-r^2d(b_2+a_1),\\
a_{123}+b_{113}=\frac{1}{2}dc_2-r^2db_3,\\
a_{133}=\frac{1}{2}dc_3,\\
a_{222}+b_{122}=-r^2d(b_1+a_2),\\
a_{223}+b_{123}=\frac{1}{2}dc_1-r^2da_3,\\
b_{222}=-r^2db_2,\\
b_{223}=\frac{1}{2}dc_2-r^2db_3,\\
b_{233}=\frac{1}{2}dc_3,\\
a_{233}+b_{133}=0,\ \ a_{333}=b_{333}=0.
\end{cases} \qquad (3.30)
$$

Now, we compare the coefficients of the same polynomial terms in (3.28) and obtain the following 15 equalities.

$$\begin{cases} 4r^2a_{111} = (4-8r^4d)a_1, \\ 4r^2(a_{112}+b_{111}) = (4-8r^4d)(a_2+b_1), \\ 2c_{111} - 4r^2a_{113} = (8r^4d-4)a_3 - 4r^2dc_1, \\ 4r^2(a_{122}+b_{112}) = (4-8r^4d)(a_1+b_2), \\ 4r^2(a_{123}+b_{113}) - 2c_{112} = (4-8r^4d)b_3 + 4r^2dc_2, \\ 4r^2a_{133} - 2c_{113} = 4r^2d(a_1+c_3), \\ 4r^2(a_{222}+b_{122}) = (4-8r^4d)(a_2+b_1), \\ 4r^2(a_{223}+b_{123}) - 2c_{122} = (4-8r^4d)a_3 + 4r^2dc_1, \\ 4r^2(a_{233}+b_{133}) - 2c_{123} = 4r^2d(a_2+b_1), \\ -4r^2a_{333} + 2c_{133} = -4r^2da_3 + 2dc_1, \\ 4r^2b_{222} = (4-8r^4d)b_2, \\ -4r^2b_{223} + 2c_{222} = (8r^4d-4)b_3 - 4r^2dc_2, \\ 4r^2b_{233} - 2c_{223} = 4r^2d(b_2+c_3), \\ 4r^2b_{333} - 2c_{233} = 4r^2db_3 - 2dc_2, \\ 2c_{333} = 2dc_3. \end{cases} \tag{3.31}$$

Plugging (3.30) into (3.31) we have

$$\begin{cases} a_1 = a_{111} = 0, \\ a_2+b_1 = a_{112}+b_{111} = 0, \\ c_{111} = (2r^4d-2)a_3 - r^2dc_1, \\ a_1+b_2 = a_{122}+b_{112} = 0, \\ c_{112} = (2r^4d-2)b_3 - r^2dc_2, \\ c_{113} = -r^2dc_3, \\ a_2+b_1 = a_{222}+b_{122} = 0, \\ c_{122} = (2r^4d-2)a_3 - r^2dc_1, \\ c_{123} = 0, \\ c_{133} = dc_1 - 2r^2da_3, \\ b_2 = b_{222} = 0, \\ c_{222} = (2r^4d-2)b_3 - r^2dc_2, \\ c_{223} = -r^2dc_3, \\ c_{233} = -2r^2db_3 + dc_2, \\ c_{333} = dc_3. \end{cases} \tag{3.32}$$

Combining and simplifying (3.32) and (3.30), we obtain the conditions (3.15), (3.17) and (3.18) in the theorem. Note that $\mathbb{T}_1^2$ is isolated. $K^{(2)}(x_1,x_2,x_3) \not\equiv 0$, which leads that the condition (3.19) holds.

On the other hand, if all parameters of system (2.1) satisfy conditions (3.15)-(3.18), then it can be checked that

$$\frac{dF(x_1,x_2,x_3)}{dt}|_{(2.1)} = 2F(x_1,x_2,x_3)K(x_1,x_2,x_3),$$

where

$$K(x_1,x_2,x_3)=(r^4-1)^{-1}(x_3(\sum_{i=1}^{3}c_ix_i)-2r^2(x_2(\sum_{i=1}^{3}b_ix_i)+x_1(\sum_{i=1}^{3}a_ix_i))). \quad (3.33)$$

And if condition (3.19) holds, then $K(x_1,x_2,x_3)\not\equiv 0$. Thus, $\mathbb{T}_1^2$ is isolated invariant torus of system (2.13). The proof is finished.

□

4. Discussion

In the paper, we characterize the simplest three-dimensional polynomial differential system having an equilibrium and a 2-dimensional orientable smooth compact invariant manifold: sphere $\mathbb{E}^2$ or torus $\mathbb{T}^2$ in $\mathbb{R}^3$. A natural question is how the relative position between the equilibrium and sphere $\mathbb{E}^2$ or torus $\mathbb{T}^2$ in $\mathbb{R}^3$ affect the global dynamics of this system.

When the compact invariant algebraic surface is sphere $\mathbb{E}^2$, if the equilibrium $O(0,0,0)$ is the center of $\mathbb{E}^2$ and it is a hyperbolic unstable equilibrium, we can obtain that sphere $\mathbb{E}^2$ is a global attractor of the system in $\mathbb{R}^3\setminus\{(0,0,0)\}$ under some conditions, for example, see [16, 17]. When the compact invariant algebraic surface is torus $\mathbb{T}^2$, the conditions that guarantee this torus is a global attractor of the system in $\mathbb{R}^3$, and richer dynamics of the system with sphere $\mathbb{E}^2$ or torus $\mathbb{T}^2$ awaits further research in the future.

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