Nonlinear Parabolic Equations with Singular Coefficient with Respect to the Unknown and with Diffuse Measure Data

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Abstract. An existence and uniqueness result of a renormalized solution for a class of doubly nonlinear parabolic equations with singular coefficient with respect to the unknown and with diffuse measure data is established. A comparison result is also proved for such solutions.

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1 Introduction

In the present paper we establish an existence and uniqueness result of a renormalized solution for a class of nonlinear parabolic equations of the type

\[
\begin{aligned}
\frac{\partial b(u)}{\partial t} - \text{div} (\mathcal{A}(x,u) \nabla u) + \lambda u &= \mu & \text{in } Q, \\
B(u(t=0)) &= b(u_0) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega \times (0,T).
\end{aligned}
\]

(1.1)

In Problem (1.1) the framework is the following: $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$), $T$ is a positive real number and $Q = \Omega \times (0,T)$ with the lateral boundary $\partial \Omega \times (0,T)$ and $u_0 \in L^1(\Omega)$, $u_0 \leq m \text{ a.e. in } \Omega$, where $m$ is a positive real number. The function $b$ is assumed

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to be a $C^1$-function defined on $\mathbb{R}$, such that which is strictly increasing. The Carathéodory function $A$ defined on $\Omega \times ]-\infty, m[,$ satisfies

$$\lim_{s \to m^-} A(x,s) = +\infty \quad \text{for a.e. } x \in \Omega.$$

The first difficulty in solving this equation is defining the field $A(x,u)\nabla u$ on the subset $\{(x,t); u(x,t) = m\}$ of $Q$, since, on this set, $A(x,u) = +\infty$.

The second difficulty is represented here by the presence of the measure data $\mu$ and the nonlinear term $b(u)$. To overcome these difficulties, we use in this paper the framework of renormalized solutions. A large number of papers was then devoted to the study of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and in different contexts: in addition to the references already mentioned see, [1–12].

The existence and uniqueness of a renormalized solution of (1.1) have been proved in [13] (see also [14]) in the stationary case where $A(x,u) = d(u) + A(u)$ with $d(r) = (d_i(r))_{1 \leq i \leq N}$ is a diagonal matrix defined on an interval $]-\infty, m[$ such that there exists an index $p \in \{1, \cdots, N\}$ such that $\lim_{r \to m^-} d_p(r) = +\infty$, where the matrix $A(r) \in C^0(\mathbb{R}, \mathbb{R}^{N \times N})$ and where $\mu \in L^2(\Omega)$ (see also [15, 16]).

In the stationary and evolution cases of $u_t - \text{div}(A(x,t,u)\nabla u) = f$ in $Q$, where the matrix $A(x,t,s)$ blows up (uniformly with respect to $(x,t)$) as $s \to m^-$ and where $f \in L^1(Q)$, the existence of renormalized solution has been proved in [2].

We call a finite measure $\mu$ diffuse if it does not charge sets of zero 2-capacity (see Section 2 for the definition) and $\mathcal{M}_0(Q)$ will denote the subspace of all diffuse measures in $Q$. According to a representation theorem for diffuse measures proved in [17], for every $\mu \in \mathcal{M}_0(Q)$, there exist $f \in L^1(Q)$, $g \in L^2(0,T;H^1_{0}(\Omega))$ and $\chi \in L^2(0,T;H^{-1}(\Omega))$ such that $\mu = f + \chi + g$, in $\mathcal{D}'(Q)$.

In the case of

$$u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (d_i(u) \frac{\partial u}{\partial x_i}) = \mu,$$

where the coefficients $d_i(s)$ are continuous on an interval $]-\infty, m[$ such that there exists an index $p \in \{1, \cdots, N\}$ such that $\lim_{r \to m^-} d_p(r) = +\infty$, the data $u_0 \in L^1(\Omega)$ such that $u_0 \leq m$, and $\mu \in \mathcal{M}_0(Q)$, the existence of renormalized solution has been proved in [18].

When $b$ is assumed to satisfy $0 < b_0 \leq b'(r) \leq b_1$, $\forall r \in \mathbb{R}$ and $A(x,u)\nabla u$ is replaced by $a(x,t,\nabla u)$, and $\mu \in \mathcal{M}_0(Q)$, the existence and uniqueness of renormalized solution have been established in [19].

In the case of

$$u_t - \text{div}(a(t,x,u,\nabla u)) + \text{div}(\Phi(u)) = f + \text{div}g \quad \text{in } Q,$$

where $\Phi$ is a continuous function, $f \in L^1(Q)$ and $g \in (L^{p'}(Q))^N$ the existence of renormalized solution has been proved in [20].