

Explicit H^1 -Estimate for the Solution of the Lamé System with Mixed Boundary Conditions

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Received 20 May 2019; Accepted 27 February 2020

Abstract. In this paper we consider the Lamé system on a polygonal convex domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit L^2 norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the H^1 norm of this solution. Note that the obtained upper-bound is not optimal.

AMS Subject Classifications: 35J57, 74B05

Chinese Library Classifications: O175.27

Key Words: Lamé system; Korn's inequality; Poincaré's inequality; inequality of trace; explicit estimates.

1 Introduction

Let Ω be a bounded open connected subset of \mathbb{R}^2 . The static equilibrium of a deformable structure occupying Ω is governed by the Lamé linear elasto-static system, see [1]. In this paper, we restrict the study to a convex domain Ω whose boundary has a polygonal shape that possesses $m+1$ edges with $m \geq 2$. We denote $\Gamma = \cup_{i=0}^m \Gamma_i$ its boundary and $d(\Omega)$ its diameter. Moreover, we assume that all the edges Γ_i have strictly positive measure. The system under consideration is given by

$$\begin{cases} Lu = f & \text{a.e in } \Omega, \\ \sigma(u) \cdot \vec{n}_i = g_i & \text{on } (\Gamma - \Gamma_0) \cap \Gamma_i, \quad 1 \leq i \leq m, \\ u = 0 & \text{on } \Gamma_0. \end{cases} \quad (1.1)$$

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We need to assume that the edges Γ_i which form the boundary Γ fulfill a condition similar to assumption (H_2) in ([2], Theorem 2.3). Actually, for our purpose, a stronger condition is needed and it is formulated in (1.5) below. The vector function $u = (u^1, u^2)$ satisfying the system (1.8) describes a displacement in the plane. In this model we impose a homogeneous Dirichlet condition on Γ_0 and a Neumann condition on the remaining part of the boundary. The equality on the boundary is understood in the sense of the trace. We denote L the Lamé operator defined by

$$Lu := -\operatorname{div}\sigma(u) = -\operatorname{div}[2\mu\varepsilon(u) + \lambda\operatorname{Tr}\varepsilon(u)Id]. \quad (1.2)$$

We assume the data functions f and g at the right hand sides to satisfy $f \in [L^2(\Omega)]^2$ and $g \in [H^{\frac{1}{2}}(\Gamma - \Gamma_0)]^2$. The vector \vec{n}_i represents the outside normal to Γ_i . We write μ and λ the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor ε is defined by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u). \quad (1.3)$$

The weak form of problem (1.1) is (see [1,3]): Find $u \in V$ such that $\forall v \in V$

$$\int_{\Omega} 2\mu\varepsilon(u)\varepsilon(v) + \lambda\operatorname{div}u\operatorname{div}v\,dx = \int_{\Omega} f v\,dx + \int_{\Gamma - \Gamma_0} g v\,d\sigma(x), \quad (1.4)$$

where

$$V = \left\{ v \in [H^1(\Omega)]^2; \quad v = 0 \quad \text{on } \Gamma_0 \right\}.$$

The existence and uniqueness issue of the solution of (1.4) in V is classic, (see [3]).

If we denote θ the interior angle between the edges Γ_j and Γ_k , $0 \leq j, k \leq m$ such that $\bar{\Gamma}_j \cap \bar{\Gamma}_k \neq \emptyset$ and if we denote γ the interior angle between the Neumann part of the boundary $\Gamma_N := \Gamma - \Gamma_0$ and the Dirichlet part of the boundary $\Gamma_D := \Gamma_0$, then we impose

$$0 < \theta < \pi, \quad 0 < \gamma < \pi. \quad (1.5)$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (1.4). Precisely in that case we have, following ([2], Theorem 2.3) stated at the bottom of page 330, $u \in [H^{\frac{3}{2} + \iota}(\Omega)]^2$ for some positive $\iota > 0$, which implies in particular, using the appropriate Sobolev embedding and since Ω is a locally Lipschitz domain, see part II of ([4], Theorem 4.12, page 85), that $u \in [C^{0, \frac{1}{2} + \iota}(\bar{\Omega})]^2$ i.e. u is $(\frac{1}{2} + \iota)$ -holder continuous. One should notice that condition (1.5) are met since the domain considered in our case is convex. Let us denote

$$\|\varepsilon(u)\|_{0,\Omega} := \left(\int_{\Omega} \varepsilon(u)\varepsilon(u)\,dx \right)^{\frac{1}{2}}; \quad \|\nabla u\|_{0,\Omega} := \left(\int_{\Omega} |\nabla u^1|^2 + |\nabla u^2|^2\,dx \right)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [5], the trace and the Poincaré's inequalities, one easily gets from (1.4) the following estimate

$$\|\nabla u\|_{0,\Omega} \leq \frac{1}{c_k} \frac{1}{2\mu} \left(c_p \|f\|_{0,\Omega} + c_{p,t} \|g\|_{\frac{1}{2}, \Gamma - \Gamma_0} \right), \quad (1.6)$$