JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 3, pp. 222-234

doi: 10.4208/jpde.v33.n3.3
September 2020

Bounds for the Blow-Up Time on the Pseudo-Parabolic Equation with Nonlocal Term

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Received 25 February 2020; Accepted 29 April 2020

Abstract. We investigate the initial boundary value problem of the pseudo-parabolic equation $u_t-\triangle u_t-\triangle u=\phi_u u+|u|^{p-1}u$, where ϕ_u is the Newtonian potential, which was studied by Zhu et al. (Appl. Math. Comput., 329 (2018) 38-51), and the global existence and the finite time blow-up of the solutions were studied by the potential well method under the subcritical and critical initial energy levels. We in this note determine the upper and lower bounds for the blow-up time. While estimating the upper bound of blow-up time, we also find a sufficient condition of the solution blowing-up in finite time at arbitrary initial energy level. Moreover, we also refine the upper bounds for the blow-up time under the negative initial energy.

AMS Subject Classifications: 35K70, 35B44

Chinese Library Classifications: O175.26

Key Words: Pseudo-parabolic equation; Newtonian potential; bounds of lifespan; blow-up; concavity method.

1 Introduction

Our purpose is to consider the bounds of lifespan on the problem

$$\begin{cases} u_t-\triangle u_t-\triangle u=\phi_u u+|u|^{p-1}u, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$, T is the maximal existence time of solutions, $p \in (1,5)$, $\phi_u u$ is a nonlocal term and ϕ_u is the Newtonian potential, which is defined as

$$\phi_u = \int_\Omega \frac{u^2(y)}{4\pi|x-y|} \mathrm{d}y, \qquad x \in \mathbb{R}^3$$

[1, Chapter 4].

The elliptic systems with the nonlocal term $\phi_u u$ have been focused in physics and mathematics, such as the Schrödinger-Poisson systems [2-4], the Schrödinger-Poisson-Slater systems [5-7] and the Klein-Gordon-Maxwell equations [8, 9]. Lately, the local and global existence of the solutions on the parabolic equation with such potential are studied by Ianni in [10], and the finite time blow-up and global existence of the solutions on the pseudo-parabolic equations with such potential are obtained by Zhu et al. in [11] under subcritical and critical initial energy levels.

Problem (1.1) with a general nonlinear term $f(u)$ describes a variety of important physical processes, such as the unidirectional propagation of nonlinear, dispersive, long waves [12], the analysis of non-stationary processes in semiconductors [13] and the aggregation of populations [14]. Moreover, problem (1.1) can be regarded as a Sobolev type equation or a Sobolev-Galpern type equation [15].

Based on the works of Showalter et al. in [16] and Ting in [17], the pseudo-parabolic equations have been extensively studied (such as [18-22] and references therein), and during this period, many authors in their work mainly focus on the estimate of the blow-up time (such as [11, 23-27] and references therein). The existing research results show that the bounds for the blow-up time and the finite time blow-up of the solutions with arbitrary initial energy level on problem (1.1) are still unknown. Inspired by [11, 24, 28], we decide in this paper to deal with this two unknown problems.

The rest part of this article is organized as follows. In Section 2, we introduce some Notations, first-order differential inequalities and the existence theorem of weak solutions. In Section 3, we estimate the upper bound for the blow-up time on problem (1.1). In Section 4, we determine the lower bound for the blow-up time on problem (1.1).

2 Preliminaries

Throughout the paper, we denote by $\|u\|_{H_0^1} = \left(\int_\Omega |u|^2 + |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$ the $H_0^1(\Omega)$ norm and by $\|u\|_l = \left(\int_\Omega |u|^l \mathrm{d}x\right)^{\frac{1}{l}}$ the $L^l(\Omega)$ norm for $1 \le l \le \infty$.

To convenience, we now introduce

$$\int_\Omega \phi_u u^2 \mathrm{d}x \le C\|u\|_{H_0^1}^4 \tag{2.1}$$

[11, Remark 2.3], the embedding inequality

$$\|u\|_p \le \mathcal{C}\|\nabla u\|_q, \tag{2.2}$$