JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 3, pp. 235-248

doi: 10.4208/jpde.v33.n3.4
September 2020

On the Radius of Spatial Analyticity for the Inviscid Boussinesq Equations

CHENG Feng[1,*] and XU Chao-Jiang[2,3]

[1] *Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China.*
[2] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China.*
[3] *Université de Rouen, CNRS UMR 6085, Laboratoire de Mathématiques, 76801 Saint-Etienne du Rouvray, France.*

Received 10 March 2020; Accepted 10 May 2020

Abstract. In this paper, we study the problem of analyticity of smooth solutions of the inviscid Boussinesq equations. If the initial datum is real-analytic, the solution remains real-analytic on the existence interval. By an inductive method we can obtain a lower bound on the radius of spatial analyticity of the smooth solution.

AMS Subject Classifications: 35Q35, 35B65, 76B03

Chinese Library Classifications: O175.29

Key Words: Boussinesq equations; analyticity; radius of analyticity.

1 Introduction

In this paper, we consider the following multi-dimensional invisicd Boussinesq equation on the torus $\mathbb{T}^d$,

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla p=\theta e_d, \\ \partial_t\theta+(u\cdot\nabla)\theta=0, \\ \operatorname{div} u=0, \\ u(x,0)=u_0(x),\ \theta(x,0)=\theta_0(x), \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `fengcheng@hubu.edu.cn` (F. Cheng), `xuchaojiang@nuaa.edu.cn` (C. J. Xu)

with $\mathrm{div}\, u_0 = 0$. Here, $u = (u_1, \ldots, u_d)$ is the velocity field, p the scalar pressure, and θ the scalar density. e_d denotes the vertical unit vector $(0, \ldots, 0, 1)$. The Boussinesq systems play an important role in geophysical fluids such as atmospheric fronts and oceanic circulation (see, e.g., [1–3]). Moreover, the Boussinesq systems are important for the study of the Rayleigh-Benard convection, see [4, 5].

Besides the physical importance, the invisicd Boussinesq equations can also be viewed as simplified model compared with the Euler equation. In the case $d = 2$, the 2D inviscid Boussinesq equations share some key features with the 3D Euler equations such as the vortex stretching mechanism. It was also pointed out in [6] that the 2D invisicid Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows outside the symmetric axis.

The inviscid Boussinesq equations have been studied by many authors through the years, for instance [7–16]. Specially, Chae and Nam [8] studied local existence and uniqueness of the inviscid Boussinesq equation and some blow-up criterion in the Sobolev space, Yuan [11] and Liu et al. [14] in the Besov space, Chae and Kim [7] and Cui et al. [15] in the Hölder spaces, Xiang and Yan [16] in the Triebel-Lizorkin-Lorentz spaces. It was remarked that the global regularity for the inviscid Boussinesq equations even in two dimensions is a challenging open problem in mathematical fluid mechanics.

In this paper, we are concerned with the analyticity of smooth solutions of the inviscid Boussinesq equations (1.1). The analyticity of the solution for Euler equations in the space variables, for analytic initial data is an important issue, studied in [17–23]. In particular, Kukavica and Vicol [21] studied the analyticity of solutions for the Euler equations and obtained that the radius of analyticity $\tau(t)$ of any analytic solution $u(t,x)$ has a lower bound

$$\tau(t) \geq C(1+t)^{-2} \exp\left(-C_0 \int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} \mathrm{d}s \right) \tag{1.2}$$

for a constant $C_0 > 0$ depending on the dimension and $C > 0$ depending on the norm of the initial datum in some finite order Sobolev space. The same authors in [22] obtained a better lower bound for $\tau(t)$ for the Euler equations in a half space replacing $(1+t)^{-2}$ by $(1+t)^{-1}$ in (1.2). In [24], we have investigated the Gevrey analyticity of the smooth solution for the ideal MHD equations following the method of [21]. The approach used in [21–24] relies on the energy method in infinite order Gevrey-Sobolev spaces. Recently, Cappiello and Nicola [25] developed a new inductive method to simplify the proof of [21, 22]. In this paper, we shall apply this inductive method to study the analyticity of smooth solution for the inviscid Boussinesq equations. The main additional difficulty arises from the estimate of the weak coupling term $u \cdot \nabla \theta$.

The paper is organized as follows. In Section 2, we will give some notations and state our main results. In Section 3, we first recall some known results and then give some lemmas which are needed to prove the main Theorem. In Section 4, we finish the proof of Theorem 2.1.