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Study of a Generalized Nonlinear Euler-Poisson-Darboux System: Numerical and Bessel Based Solutions

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Abstract. In this paper a nonlinear Euler-Poisson-Darboux system is considered. In a first part, we proved the genericity of the hypergeometric functions in the development of exact solutions for such a system in some special cases leading to Bessel type differential equations. Next, a finite difference scheme in two-dimensional case has been developed. The continuous system is transformed into an algebraic quasi linear discrete one leading to generalized Lyapunov-Sylvester operators. The discrete algebraic system is proved to be uniquely solvable, stable and convergent based on Lyapunov criterion of stability and Lax-Richtmyer equivalence theorem for the convergence. A numerical example has been provided at the end to illustrate the efficiency of the numerical scheme developed in section 3. The present method is thus proved to be more accurate than existing ones and lead to faster algorithms.

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1 Introduction

In the present paper a nonlinear Euler-Poisson-Darboux system is studied for two folds. In a first part, exact solutions based on general hypergeometric series and on Bessel functions in some special cases have been developed. The application of such forms showed

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that the system may be transformed in some cases to Bessel type differential equations. The general system of coupled equations is characterized by the the presence of some cross-correlated nonlinearities characterized by the presence of simultaneous superlinear two power laws which interchange the role according to the dynamical system studied. This type of problems is very important in plasma physics and engineering as it is used to model telegraphic phenomena, turbulense especially for plasma and for accelerating electrons.

In a second part, Lyapunov-Sylvester algebraic operators have been applied to approximate the numerical solutions of the generalized Euler-Poisson-Darboux (EPD) system in two-dimensional case. The present article is precisely devoted to the development of a numerical method based on two-dimensional finite difference scheme to approximate the solution of the generalized EPD system in $\mathbb{R}^2$ in the presence of mixed power laws nonlinearities. Denote for $a\in\mathbb{R}$, $\Gamma_a(x)=2a/x$ and for λ,γ in $\mathbb{R}$, $F_{\lambda,\gamma}(x)=(\Gamma_\lambda(x),\Gamma_\gamma(x))$. We consider the evolutive system

$$\begin{cases} u_{tt}+\Gamma_a(t)v_t=\Delta u+<F_{\lambda,\gamma},\nabla v>+|u|^{p-1}v, \\ v_{tt}+\Gamma_a(t)u_t=\Delta v+<F_{\lambda,\gamma},\nabla u>+|v|^{q-1}u, \end{cases} \tag{1.1}$$

with initial conditions

$$u(x,y,t_0)=u_0(x,y) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,t_0)=u_1(x,y), \quad (x,y)\in\Omega, \tag{1.2}$$

and boundary conditions

$$\frac{\partial u}{\partial \eta}(x,y,t)=0, \quad ((x,y),t)\in\partial\Omega\times(t_0,+\infty), \tag{1.3}$$

on a rectangular domain $\Omega=[L_0,L_1]\times[L_0,L_1]$ in $\mathbb{R}^2$. $t_0\geq 0$ is a real parameter fixed as the initial time, u_t is the first order partial derivative in time, u_{tt} is the second order partial derivative in time, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator on $\mathbb{R}^2$. $\frac{\partial}{\partial\eta}$ is the outward normal derivative operator along the boundary $\partial\Omega$. Finally, u, u_0 and u_1 are real valued functions with u_0 and u_1 are $\mathcal{C}^2$ on $\overline{\Omega}$. u and v are the unknown candidates assumed to be $\mathcal{C}^4$ on $\overline{\Omega}$. p and q are real parameters such that $p,q>1$.

In the present work, existence and multiplicity of the solutions of problem (1.1)-(1.3) are developed in some special cases. We showed that special functions such as hypergeometric series and Bessel function are generic for developing such solutions. Section 3 is devoted to the development of a two-dimensional discrete scheme to transform the continuous problem (1.1)-(1.3) to a discrete one. A system of generalized Lyapunov-Sylvester equations is obtained. The solvability of such a discrete system is proved next in Section 4. Section 5 is concerned with the consistency, stability and the convergence of the discrete Lyapunov-Sylvester problem obtained in Section 3. The crucial idea is the application of the truncation error for consistency, Lyapunov cretirion for stability and the