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A Nash Type Result for Divergence Parabolic Equation Related to Hörmander's Vector Fields

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Abstract. In this paper we consider the divergence parabolic equation with bounded and measurable coefficients related to Hörmander's vector fields and establish a Nash type result, i.e., the local Hölder regularity for weak solutions. After deriving the parabolic Sobolev inequality, (1,1) type Poincaré inequality of Hörmander's vector fields and a De Giorgi type Lemma, the Hölder regularity of weak solutions to the equation is proved based on the estimates of oscillations of solutions and the isomorphism between parabolic Campanato space and parabolic Hölder space. As a consequence, we give the Harnack inequality of weak solutions by showing an extension property of positivity for functions in the De Giorgi class.

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1 Introduction

Schauder theory for the solutions to linear elliptic and parabolic equations with C^α coefficients or VMO coefficients has been completed. De Girogi has followed the local Hölder continuity for the solutions to the divergence elliptic equation with bounded and measurable coefficients

$$-\sum_{i,j=1}^{n} D_i\left(a^{ij}(x)D_ju\right)=0, \qquad x\in\mathbb{R}^n,$$

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and given the a priori estimate of Hölder norm (see [1]). Nash in [2] derived independently the similar result for the solutions to the parabolic equation with a different approach from [1]. Hereafter Moser in [3] developed a new method (nowadays it is called the Moser iteration method) and proved again results above-mentioned to elliptic and parabolic equations. These important works break a new path for the study of regularity for weak solutions to partial differential equations.

In [4], Fabes and Stroock proved the Harnack inequality for linear parabolic equations by going back to Nash's original technique in [2]. A very interesting approach has been raised by De Benedetto (see [5]) for proving a Harnack inequality of functions belonging to parabolic De Giorgi classes. The approach was used to derive the Hölder continuity of solutions to linear second order parabolic equations with bounded and measurable coefficients (see [6]). Giusti [7] applied the approach to give a proof of the Harnack inequality in the elliptic setting.

Square sum operators constructed by vector fields satisfying the finite rank condition were introduced by Hörmander (see [8]), who deduced that such operators are hypoelliptic. Many authors carried on researches to such operators and acquired numerous important results ([9–14]). Nagel, Stein and Wainger ([15]) concluded the deep properties of balls and metrics defined by vector fields of this type. Many other authors obtained very appreciable results related to Hörmander's square sum operators, for instance fundamental solutions ([16]), the Poincaré inequality ([17]), potential estimates ([18]), also see [4,19–21], etc. All these motivate the study to degenerate elliptic and parabolic equations formed from Hörmander's vector fields. Schauder estimates to degenerate elliptic and parabolic equations related to noncommutative vector fields have been handled in [22–24], etc. Bramanti and Brandolini in [25] investigated Schauder estimates to the following Hörmander type nondivergence parabolic operator

$$H=\partial_t-\sum_{i,j=1}^{q}a_{ij}(t,x)X_iX_j-\sum_{i=1}^{q}b_i(t,x)X_i-c(t,x),$$

where coefficients $a_{ij}(t,x),b_i(t,x)$ and $c(t,x)$ are C^α.

In this paper, we are concerned with the divergence parabolic equation with bounded and measurable coefficients related to Hörmander's vector fields and try to establish a Nash type result for weak solutions which will play a crucial role for corresponding nonlinear problems.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $\mathcal{Q}_T=\Omega\times(0,T)\subset\mathbb{R}^n\times\mathbb{R}$, $T>0$. Consider the following divergence parabolic equation

$$u_t+X_j^*\left(a^{ij}(x,t)X_iu\right)+b_i(x,t)X_iu+c(x,t)u=f(x,t)-X_i^*f^i(x,t),\qquad (x,t)\in\mathcal{Q}_T \tag{1.1}$$

where

$$X_i=\sum_{k=1}^{n}b_{ik}(x)\frac{\partial}{\partial x_k}(b_{ik}(x)\in C^\infty(\Omega),\quad i=1,\cdots,q,\ q\leq n)$$