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On Free Boundary Problem for the Non-Newtonian Shear Thickening Fluids

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Abstract. The aim of this paper is to explore the free boundary problem for the Non-Newtonian shear thickening fluids. These fluids not only have vacuum, but also have strong nonlinear properties. In this paper, a class of approximate solutions is first constructed, and some uniform estimates are obtained for these approximate solutions. Finally, the existence of free boundary problem solutions is proved by these uniform estimates.

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1 Introduction

It is well known that the non-newtonian shear thickening flows can be described by the following equations (for example, see [1-6])

$$\rho_t+(\rho u)_x=0, \tag{1.1}$$

$$(\rho u)_t+(\rho u^2)_x-((|u_x|^2+\mu)^{(p-2)/2}u_x)_x+(A\rho^\gamma)_x=0, \tag{1.2}$$

where $p>2$, $A>0$, $\mu>0$ and $\gamma>1$ are some given positive constants, and ρ,u,ρ^γ represent the density, velocity and pressure for the non-Newtonian fluids, respectively.

We assume that the initial density ρ_0 is some given nonnegative function satisfying $\text{supp}\rho_0=[a_0,b_0]$ for some constants a_0 and b_0, and $||\rho_0||_{L^1(a_0,b_0)}=1$. Let $x=a(t)$ and $x=b(t)$ represent the free boundary which is the interface between fluid and vacuum, and then

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have $\rho(a(t),t)=\rho(b(t),t)=0$, and $a'(t)=u(a(t),t)$ with $a(0)=a_0$, and $b'(t)=u(b(t),t)$ with $b(0)=b_0$.

We introduce the Lagrange coordinate transformation

$$s=t, \quad y=\int_{a(t)}^{x}\rho(z,t)\mathrm{d}z. \tag{1.3}$$

Clearly, the left dividing line $\Gamma_0: x=a(t)$ for the interface is a straight line $\Gamma_0: y=0$ in Lagrange coordinates. In addition, in the right dividing line $\Gamma_1: x=b(t)$ for the interface, we have

$$y=\int_{a(t)}^{b(t)}\rho(z,t)\mathrm{d}z=\int_{a_0}^{b_0}\rho_0(z)\mathrm{d}z=1. \tag{1.4}$$

Therefore, the right dividing line $\Gamma_1: x=b(t)$ for the interface is a straight line $\Gamma_1: y=1$ in Lagrange coordinates. In particular, in Lagrange coordinates, the original equations (1.1)-(1.2) are transformed into the following equations

$$\rho_s+\rho^2u_y=0, \tag{1.5}$$

$$u_s-(((\rho u_y)^2+\mu)^{(p-2)/2}\rho u_y)_y+(A\rho^\gamma)_y=0. \tag{1.6}$$

This paper is to solve the above equations (1.5)-(1.6) in $Q_S\equiv(0,1)\times(0,S)(S>0)$ with the following initial condition

$$(\rho(y,0),u(y,0))=(\rho_0(y),u_0(y)), \quad y\in[0,1], \tag{1.7}$$

and the following boundary condition

$$(\rho u_y)(0,s)=(\rho u_y)(1,s)=0, \quad s\geq 0, \tag{1.8}$$

where the initial density $\rho_0=\rho_0(y)$ and the initial velocity $u_0=u_0(y)$ have the following properties [A1]-[A3]:

[A1] The initial density $\rho_0\in C(-\infty,+\infty)\cap C^1(0,1)$ satisfies

$$\rho_0(y)>0 \quad \forall y\in(0,1), \quad \rho_0(y)=0 \quad \forall y\in(-\infty,0]\cup[1,+\infty). \tag{1.9}$$

[A2] The initial velocity $u_0\in C^3(-\infty,+\infty)$ satisfies $u_{0y}(0)=u_{0y}(1)=0$.

[A3] The initial value (ρ_0,u_0) also has the following property:

$$\begin{aligned}M_0\equiv&1+||\rho_0(y)||_{L^\infty(-\infty,+\infty)}+||\rho_0^{-1}(y)||_{L^1(-\infty,+\infty)}+||\rho_0'(y)||_{L^2(-\infty,+\infty)}\\&+||u_0(y)||_{W^{3,\infty}(-\infty,+\infty)}<+\infty.\end{aligned}$$

Our main results are the following theorems.