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Fixed Point Theorems in Relational Metric Spaces with an Application to Boundary Value Problems

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Abstract. In this paper, we establish fixed point theorems for generalized nonlinear contractive mappings using the concept of w-distance in relational metric spaces. Thus we generalize the recent results of Senapati and Dey [J. Fixed Point Theory Appl. 19, 2945-2961 (2017)] and many other important results relevant to this literature. In order to revel the usefulness of such investigations, an application to first order periodic boundary value problem are given. Moreover, we furnish a non-trivial example to demonstrate the validity of our generalization over previous existing results.

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1 Introduction

The classical Banach contraction principle (Bcp) has many inferences and huge applicability in mathematical theory and because of this, Bcp has been improved and generalized in various metric settings. One such interesting and important setting is to establish fixed point results in metric spaces equipped with an arbitrary binary relation. Utilizing the notions of various kind of binary relations such as partial order, strict order, near order, tolerance etc. on metric spaces, many researcher are doing their research during several years (see [1-16]) and attempting to obtain new extensions of the celebrated Bcp. Among these extensions, we must quote the one due to Alam and Imdad [8], where some relation theoretic analogues of standard metric notions (such as continuity and completeness)

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were used. Further, Ahmadullah et al. [14] extended the above setting for nonlinear contractions and obtained an extension of the Boyd-Wong [17] fixed point theorem to such spaces.

On the other hand, recently Senapati and Dey [11] improved and refined the main result of Alam and Imdad [8], Ahmadullah et al. [14] and many others, by utilizing the notion of w-distance in relational metric spaces, that is, metric spaces endowed with an arbitrary binary relation. Moreover, for further motivation of research in this direction, we refer some important recent generalizations of w-distance with applications to boundary value problem as well (see, e.g., [19-21]). It is our aim in this paper to give an extension of these results to nonlinear φ-contraction and explore the possibility of their application in finding a solution of first order periodic boundary value problem too.

2 Preliminaries

Throughout this chapter, $\mathcal{R}$ stands for a non-empty binary relation, $\mathbb{N}_0$ stands for the the set of whole numbers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and R for the set of all real numbers.

Definition 2.1. ([8]). *Let $\mathcal{R}$ be a binary relation on a non-empty set X and $x,y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$. We denote it by $[x,y] \in \mathcal{R}$.*

Definition 2.2. ([8]). *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called an $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$.*

Definition 2.3. ([8]). *Let X be a non-empty set and T a self-mapping on X. A binary relation $\mathcal{R}$ on X is called T-closed if for any $x,y \in X, (x,y) \in \mathcal{R}$ implies $(Tx,Ty) \in \mathcal{R}$.*

Definition 2.4. ([14]). *Let (X,d) be a metric space and $\mathcal{R}$ a binary relation on X. We say that (X,d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

The following notion is a generalization of d-self-closedness of a partial order relation ($\preceq$) (defined by Turinici [5-6]).

Definition 2.5. ([8]). *Let (X,d) be a metric space. A binary relation $\mathcal{R}$ on X is called d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.*

Definition 2.6. ([14]). *Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and $x \in X$. A self-mapping T on X is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

The notion of $\mathcal{R}$-lower semi-continuity (briefly, $\mathcal{R}$-LSC) of a function is defined by Senapati and Dey [11] as follows: