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On Regularization of a Source Identification Problem in a Parabolic PDE and its Finite Dimensional Analysis

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Abstract. We consider the inverse problem of identifying a general source term, which is a function of both time variable and the spatial variable, in a parabolic PDE from the knowledge of boundary measurements of the solution on some portion of the lateral boundary. We transform this inverse problem into a problem of solving a compact linear operator equation. For the regularization of the operator equation with noisy data, we employ the standard Tikhonov regularization, and its finite dimensional realization is done using a discretization procedure involving the space $L^2(0,\tau;L^2(\Omega))$. For illustrating the specification of an a priori source condition, we have explicitly obtained the range space of the adjoint of the operator involved in the operator equation.

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1 Introduction

Let $d\geq 1$ and Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. For a fixed $\tau>0$ we denote the cylindrical domain $\Omega\times[0,\tau]$ by Ω_τ and its lateral surface $\partial\Omega\times[0,\tau]$ by $\partial\Omega_\tau$. Let Σ be a relatively open subset of $\partial\Omega$. We denote the boundary surface $\Sigma\times[0,\tau]$ by Σ_τ. For

$$f\in L^2(0,\tau;L^2(\Omega)),\quad g\in L^2(0,\tau;L^2(\partial\Omega)),\quad h\in L^2(\Omega),$$

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we consider the parabolic PDE

$$\begin{cases} u_t - \nabla\cdot(Q(x)\nabla u) = f & \text{in } \Omega_\tau, \\ Q(x)\nabla u\cdot\vec{n} = g & \text{on } \partial\Omega_\tau, \\ u(\cdot,0) = h & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $Q\in(L^\infty(\Omega))^{d\times d}$ is a symmetric matrix with entries from $L^\infty(\Omega)$ satisfying the uniform ellipticity condition, that is, there exist a constant $\kappa_0>0$ such that

$$Q\xi\cdot\xi \geq \kappa_0|\xi|^2 \quad \text{a.e on } \Omega, \text{ and for all } \xi\in\mathbb{R}^d, \tag{1.2}$$

where $|\xi|^2 = {\xi_1}^2+\ldots+{\xi_d}^2$ for $\xi=(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d$ and $\vec{n}$ is the outward unit normal to $\partial\Omega$.

Throughout the paper, for a Banach space X, $\phi\in L^2(0,\tau;X)$ means ϕ is an X-valued function on $[0,\tau]$ such that $t\mapsto\|\phi(t)\|_X$ belongs to $L^2[0,\tau]$. Also, throughout we use the standard notations of the function spaces $L^2(\Omega)$ and the Sobolev spaces $H^1(\Omega)$ (see [1–3]).

For results related to existence and uniqueness of the classical solution corresponding to the forward problem associated with (1.1), namely, that of finding u satisfying (1.1) from the knowledge of f,g,h as considered above, one may refer to [4–6]. In certain cases, a classical solution may not exist for the forward problem, but we may have a *weak solution*. In [5, Theorem 2.4], the authors have given an existence result for a weak solution of (1.1). We first state the existence result precisely, whose proof follows along similar lines as in [5, Theorem 2.4].

Theorem 1.1. ([5, Theorem 2.4]) *Let $f\in L^2(0,\tau;L^2(\Omega)), g\in L^2(0,\tau;L^2(\partial\Omega))$ and $h\in L^2(\Omega)$. Also, let $Q\in(L^\infty(\Omega))^{d\times d}$ be symmetric satisfying the uniform ellipticity condition* (1.2)*. Then there exists a unique $u\in L^2(0,\tau;H^1(\Omega))$ with $u_t\in L^2(0,\tau;(H^1(\Omega))')$ satisfying*

$$\langle u_t(\cdot,t),\varphi\rangle + \int_\Omega Q\nabla u(\cdot,t)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega f(\cdot,t)\,\varphi\,\mathrm{d}x + \int_{\partial\Omega} g\varphi\,\mathrm{d}x \tag{1.3}$$

for all $\varphi\in H^1(\Omega)$ and for a.a.(almost all) $t\in[0,\tau]$ with $u(\cdot,0)=h$ a.e. in Ω. Further, there exists a constant $C_1>0$, independent of f, such that

$$\begin{aligned} &\|u\|_{L^2(0,\tau;H^1(\Omega))} + \|u_t\|_{L^2(0,\tau;(H^1(\Omega))')} \\ \leq\, & C_1\Big(\|f\|_{L^2(0,\tau;L^2(\Omega))} + \|g\|_{L^2(0,\tau;L^2(\partial\Omega))} + \|h\|_{L^2(\Omega)}\Big). \end{aligned} \tag{1.4}$$

In (1.3), the notation $\langle\cdot,\cdot\rangle$ stands for the duality action between $H^1(\Omega)$ and $(H^1(\Omega))'$, where $(H^1(\Omega))'$ stands for the dual of $H^1(\Omega)$. Also, u_t denotes the distributional derivative of u with respect to t, that is, u_t is the unique element in $L^2(0,\tau;(H^1(\Omega))')$ such that

$$\int_0^\tau \varphi'(t)u(t)\,\mathrm{d}t = -\int_0^\tau \varphi(t)u_t(t)\,\mathrm{d}t \quad \text{for all } \varphi\in C_c^\infty(0,\tau).$$