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Boundedness of Toeplitz Type Operators Associated to Fractional and Pseudo-Differential Operators on Orlicz Space

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Abstract. In this paper, the boundedness from Lebesgue space to Orlicz space of certain Toeplitz type operator related to the fractional and pseudo-differential operators is obtained.

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1 Introduction

As the development of singular integral operators (see [1,2]), their commutators have been well studied. In [3-5], the authors proved that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1<p<\infty$. Chanillo (see [6,7]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [8], Janson proved the boundedness for the commutators generated by the singular integral operators and BMO functions from Lebesgue spaces to Orlicz spaces. In [9], some pseudo-differential operators are introduced and the boundedness for the operators are obtained (see [10-12]). In this paper, we will introduce the Toeplitz type operator associated to the fractional and pseudo-differential operators and prove the boundedness properties of the Toeplitz type operator from Lebesgue space to Orlicz space.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function

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of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\mathrm{d}y, \tag{1.1}$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)\mathrm{d}x$. It is well-known that (see [1,2])

$$f^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|\mathrm{d}y. \tag{1.2}$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|\mathrm{d}y. \tag{1.3}$$

We write that $M_pf=(M(f^p))^{1/p}$ for $0<p<\infty$. For $1\le r<\infty$ and $0<\eta<n$, let

$$M_{\eta,r}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q|f(y)|^r\mathrm{d}y\right)^{1/r}. \tag{1.4}$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO}=||f^{\#}||_{L^{\infty}}$. More generally, let ρ be a non-decreasing positive function on $[0,+\infty)$ and define $BMO_\rho(R^n)$ as the space of all functions f such that

$$\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)-f_Q|\mathrm{d}y\le C\rho(r). \tag{1.5}$$

For $\beta>0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta}=\sup_{x\neq y}|f(x)-f(y)|/|x-y|^{\beta}<\infty. \tag{1.6}$$

For f, m_f denotes the distribution function of f, that is $m_f(t)=|\{x\in R^n:|f(x)|>t\}|$.

Let ρ be a non-decreasing convex function on $[0,+\infty)$ with $\rho(0)=0$. ρ^{-1} denotes the inverse function of ρ. The Orlicz space $L_\rho(R^n)$ is defined by the set of functions f such that $\int_{R^n}\rho(\lambda|f(x)|)\mathrm{d}x<\infty$ for some $\lambda>0$. The norm is given by

$$||f||_{L_\rho}=\inf_{\lambda>0}\lambda^{-1}\left(1+\int_{R^n}\rho(\lambda|f(x)|)\mathrm{d}x\right). \tag{1.7}$$

2 The main Theorem

In this paper, we will study the following pseudo-differential operators. We say a symbol $\sigma(x,\xi)$ is in the class $S^m_{\rho,\delta}$ and write $\sigma\in S^m_{\rho,\delta}$, if for $x,\xi\in R^n$,

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial\xi^\beta}\sigma(x,\xi)\right|\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}. \tag{2.1}$$