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Gelfand-Shilov Smoothing Effect for the Radially Symmetric Spatially Homogeneous Landau Equation under the Hard Potential $\gamma=2$

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Abstract. Based on the spectral decomposition for the linear and nonlinear radially symmetric homogeneous non-cutoff Landau operators under the hard potential $\gamma=2$ in perturbation framework, we prove the existence and Gelfand-Shilov smoothing effect for solution to the Cauchy problem of the symmetric homogenous Landau equation with small initial datum.

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1 Introduction

In this work, we consider the spatially homogeneous Landau equation

$$\begin{cases} \partial_t f = Q_L(f,f), \\ f|_{t=0} = f_0 \geq 0, \end{cases}$$

where $f=f(t,v)$ is the density distribution function depending on the variables $v\in\mathbb{R}^3$ and the time $t\geq 0$. The Landau bilinear collision operator is given by

$$Q_L(g,f)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} a(v-v_*)\big(g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\big)\mathrm{d}v_*,$$

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where $a=(a_{i,j})_{1\leq i,j\leq 3}$ stands for the nonnegative symmetric matrix

$$a(v)=(|v|^2\mathbf{I}-v\otimes v)|v|^\gamma \in M_3(\mathbb{R}),\quad -3<\gamma<+\infty.$$

This equation is obtained as a limit of the Boltzmann equation, when all the collisions become grazing. See [1,2]. We shall consider the Cauchy problem of radially symmetric homogeneous non-cutoff Landau equation under the hard potential case $\gamma=2$ with the natural initial datum $f_0\geq 0$

$$\int_{\mathbb{R}^3} f_0(v)\mathrm{d}v=1;\quad \int_{\mathbb{R}^3} v_j f_0(v)\mathrm{d}v=0,\ \ j=1,2,3;\quad \int_{\mathbb{R}^3}|v|^2 f_0(v)\mathrm{d}v=3. \tag{1.1}$$

Consider the fluctuation $f(t,v)=\mu(v)+\sqrt{\mu}(v)g(t,v)$ near the absolute Maxwellian

$$\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}},$$

the Cauchy problem is reduced to

$$\begin{cases}\partial_t g+\mathcal{L}(g)=\mathbf{\Gamma}(g,g), & t>0,\ v\in\mathbb{R}^3,\\ g|_{t=0}=g^0, \end{cases} \tag{1.2}$$

with $g^0(v)=\mu^{-\frac{1}{2}}f_0(v)-\sqrt{\mu}(v)$, where

$$\mathbf{\Gamma}(g,g)=\mu^{-\frac{1}{2}}Q_{\mathcal{L}}(\sqrt{\mu}g,\sqrt{\mu}g),\quad \mathcal{L}(g)=-\mu^{-\frac{1}{2}}\Big(Q_{\mathcal{L}}(\sqrt{\mu}g,\mu)+Q_{\mathcal{L}}(\mu,\sqrt{\mu}g)\Big).$$

The linear operator $\mathcal{L}$ is nonnegative with the null space

$$\mathcal{N}=\mathrm{span}\left\{\sqrt{\mu},\sqrt{\mu}v_1,\sqrt{\mu}v_2,\sqrt{\mu}v_3,\sqrt{\mu}|v|^2\right\}.$$

The projection function $\mathbf{P}:\mathcal{S}'(\mathbb{R}^3)\to\mathcal{N}$ is well defined. The assumption of the initial datum f_0 in (1.1), transforms to be $g^0\in\mathcal{N}^\perp$. We introduce the symmetric Gelfand-Shilov spaces, for $0<s\leq 1$,

$$S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3)=\Big\{u\in\mathcal{S}'(\mathbb{R}^3);\ \exists c>0,\ e^{c\mathcal{H}^s}u\in L^2(\mathbb{R}^3)\Big\};$$

where $\mathcal{H}=-\Delta+\frac{|v|^2}{4}$. This spaces can be also characterized through the decomposition into the Hermite basis $(\Psi_\alpha)_{\alpha\in\mathbb{N}^3}$,

$$f\in S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3)\ \Leftrightarrow\ f\in L^2(\mathbb{R}^3),\ \exists\epsilon_0>0,\ \left\|\Big(e^{\epsilon_0|\alpha|^s}(f,\Psi_\alpha)_{L^2}\Big)_{\alpha\in\mathbb{N}^3}\right\|_{l^2}<+\infty.$$

For more details, see in [3, Theorem 2.1] or in [4, Proposition 2.1].

The existence and regularity of the solution to Cauchy problem for the spatially homogeneous Landau equation with hard potentials has already been treated in [5,6]. We