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Blowup Behavior of Solutions to an ω-diffusion Equation on the Graph

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Abstract. In this article, we discuss the blowup phenomenon of solutions to the ω-diffusion equation with Dirichlet boundary conditions on the graph. Through Banach fixed point theorem, comparison principle, construction of auxiliary function and other methods, we prove the local existence of solutions, and under appropriate conditions the blowup time and blowup rate estimation are given. Finally, numerical experiments are given to illustrate the blowup behavior of the solution.

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1 Introduction

In this paper, we mainly study the blowup phenomenon of the following problem

$$\begin{cases} u_t(x,t)=\Delta_w u(x,t)+e^{\beta t}\left(u^p(x,t)-\lambda u^q(x,t)\right), & (x,t)\in S\times(0,\infty),\\ u(x,t)=0, & (x,t)\in\partial S\times[0,\infty),\\ u(x,0)=u_0(x), & x\in V,\end{cases} \tag{1.1}$$

where p, q, β, λ are all greater than 0, and the initial value $u_0(x)\not\equiv 0$ is non-negative. The graph $G(V,E,w)$ is a simple weighted graph with limited connectivity. The set V of vertices on the graph consists of two disjoint subsets S and ∂S. E is the set of edges of graph G, and weighted function $w: V\times V\to[0,\infty)$ satisfies:

$$\omega(x,x)=0,\ \forall x\in V,\qquad \omega(x,y)=\omega(y,x),\ \forall x,y\in V,\qquad \omega(x,y)=0\Leftrightarrow(x,y)\notin E.$$

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We define the Laplace operator on the graph as that in reference [1]

$$\Delta_\omega u(x,t)=\sum_{y\in V}\omega(x,y)[u(y,t)-u(x,t)]\,, \quad \forall x\in V.$$

In daily life, after the fireworks ignite the fuse, it rely on the instantaneous burst of gunpowder to generate energy and present colorful scenery. Lithium-ion batteries can cause fires, explosions and other safety accidents under poor temperature and humidity conditions. Ion conductors stimulate polarization phenomenon and other common phenomena can all be attributed to the singular solution models of differential equations. The study of these models will have a great guiding effect on our lives, therefore, the study of the singular phenomenon of the solution has always been a hot issue for mathematicians. As early as 1966, Fujita [2] conducted a pioneering study on the blowup phenomenon of the solution of the semilinear reaction-diffusion equation

$$\begin{cases} u_t=\Delta u+u^{1+\alpha}, & (x,t)\in R^m\times(0,T),\\ u(x,0)=u_0(x), & x\in R^m, \end{cases}$$

on the condition $\alpha>0$ and get some interesting conclusion. When $0<m\alpha<2$, the Cauchy problem has no non-trivial solution as a whole. When $2<m\alpha$ and the initial value is small enough, the solution exists as a whole, and the solution will blow up in a finite time if the initial value is large enough. This great research urges more and more scholars to study the blowup behavior actively, and explore many complex models ([3-8]) consequently.

In recent years, some scholars have begun to pay attention to the study of singular solutions of evolution equations defined on the network structure. Many objects and their interrelationships are generally represented by a network. In a power system, the network is composed of a number of components and a circuit that transmits electrical signals through certain requirements. In mathematics, the weighted graph is another name for the network. Vertices represent objects and edges represent connections between objects, which is widely used to analyze discrete objects. Scholars have also conducted some in-depth discussions on different boundary value problems. The *ω-Laplace* equation on the graph is obtained by modeling the power grid, which has a wide range of applications in various fields. It can be used to simulate the energy flow in the network or the vibration of molecules in [1,9-12], and it can also be used to study dynamic systems and image processing in [13,14]. Chung [1] first defined some concepts of calculus in discrete cases, such as directional derivative, gradient, etc., and he proved that the overall uniqueness of the solution to the inverse problem, and the solubility of boundary value problems of the first and second types under appropriate monotonic conditions.

In [15], Chung also studied a class of *ω-heat* equations with nonlinear source terms

$$\begin{cases} u_t=\Delta u-u^q, & (x,t)\in\Omega\times(0,\infty),\\ u\equiv 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ u(\cdot,0)=u_0\ge 0, & x\in\overline{\Omega}, \end{cases}$$