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Infinitely Many Solutions for the Fractional Nonlinear Schrödinger Equations of a New Type

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Abstract. This paper, we study the multiplicity of solutions for the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = u^p,\ \ u>0,\ \ x\in\mathbb{R}^N,\ \ u\in H^s(\mathbb{R}^N),$$

with $s\in(0,1)$, $N\geq 3$, $p\in(1,\frac{2N}{N-2s}-1)$ and $\lim_{|y|\to+\infty}V(y)>0$. By assuming suitable decay property of the radial potential $V(y)=V(|y|)$, we construct another type of solutions concentrating at infinite vertices of two similar equilateral polygonal with infinitely large length of sides. Hence, besides the length of each polygonal, we must consider one more parameter, that is the height of the podetium, simultaneously. Another difficulty lies in the non-local property of the operator $(-\Delta)^s$ and the algebraic decay involving the approximation solutions make the estimates become more subtle.

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1 Introduction

In this paper, we consider the following nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = u^p,\quad u>0,\ \ x\in\mathbb{R}^N,\ \ u\in H^s(\mathbb{R}^N), \tag{1.1}$$

where $s\in(0,1), p\in(1,2_s^*-1), 2_s^*=\frac{2N}{N-2s}$, $(-\Delta)^s$ is the nonlocal operator defined as

$$(-\Delta)^s u = c(N,s)P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

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where $P.V.$ is the principal value and $c(N,s)=\pi^{2s+\frac{N}{2}}\Gamma(s+\frac{N}{2})/\Gamma(-s)$. For more details on the fractional Laplace operator, we can refer to [1,2].

The fractional Laplace operator appears in many fields such as biological modeling, physics and mathematical finance, and can be regarded as an infinitesimal generator of a stable Lévy process [3]. The fractional Laplace problem has recently been extensively studied, such as [4–22] Brezis-Nirenberg problems on lower dimensions, local uniqueness and periodicity for the prescribed scalar curvature problem, solutions of critical exponent via Local Pohozaev identities and the references therein.

When $s=1$, the classical nonlinear Schrödinger equation

$$-\Delta u+V(x)u=u^p, \quad u>0 \text{ in } \mathbb{R}^N \tag{1.2}$$

was derived from the Bose-Einstein condensates (BEC) originated in 1924-1925, when Einstein predicted that, below a critical temperature, part of the bosons would occupy the same quantum state to form a condensate ([23–25]). New experimental advances make many mathematicians study again the following of Gross-Pitaevskii (GP) equations proposed by Gross [26] and Pitaevskii [27] in the 1960s:

$$i\partial_t\psi(x,t)=-\Delta\psi(x,t)+\bar{V}(x)\psi(x,t)-a|\psi(x,t)|^2\psi(x,t), \quad x\in\mathbb{R}^N, \tag{1.3}$$

where $\bar{V}(x)\geq 0$ is a real-valued potential. If we want to find a solution for (1.3) of the form $\psi(x,t)=u(x)e^{-i\mu t}$, where μ represents the chemical potential of the condensate and $u(x)$ is a function independent of time, then u satisfies (1.2) with $V(x)=\bar{V}(x)-\mu$.

For the problem

$$\begin{cases}-\Delta u+V(x)u=u^p, \ u>0 \text{ in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\end{cases}$$

with $p\in(1,\frac{N+2}{N-2})$, $N\geqslant 3$, one refer to [4,8,15,16,28–30] for many well-known results. In particular, Duan and Musso in [28] constructed a new type of solutions which are different from Wei and Yan obtained in [30], which first developed the technique of applying the reduction argument for non-singularly perturbed elliptic problems.

It is well-known ([29,31]) that the ground state solution U of the fractional problem

$$(-\Delta)^s u+u=u^p, \ u>0, \ x\in\mathbb{R}^N, \ u(0)=\max_{x\in\mathbb{R}^N}u(x)$$

satisfies that

$$\frac{C_1}{1+|x|^{N+2s}}\leqslant U(x)\leqslant\frac{C_2}{1+|x|^{N+2s}}.$$

Moreover, the linearized operator $L_0=(-\Delta)^s+1-pU^{p-1}$ is non-degenerate, that is, its kernel is given by $kerL_0=\text{span}\{\partial_{x_1}U,\partial_{x_2}U,...,\partial_{x_N}U\}$.