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Global Integrability for Solutions to Obstacle Problems

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Abstract. Denote

$$\mathcal{K}_{\psi,\theta}(\Omega)=\left\{v\in W^{1,p}(\Omega): v\geq\psi,\text{ a.e. and } v-\theta\in W_0^{1,p}(\Omega)\right\},$$

where ψ is any function in $\Omega\subset\mathbb{R}^N$, $N\geq 2$, with values in $\mathbb{R}\cup\{\pm\infty\}$ and θ is a measurable function. This paper deals with global integrability for $u\in\mathcal{K}_{\psi,\theta}$ such that

$$\int_\Omega\langle\mathcal{A}(x,\nabla u),\nabla(w-u)\rangle\mathrm{d}x\geq\int_\Omega\langle F,\nabla(w-u)\rangle\mathrm{d}x,\ \forall\, w\in\mathcal{K}_{\psi,\theta}(\Omega),$$

with $|\mathcal{A}(x,\xi)|\approx|\xi|^{p-1}$, $1<p<N$. Some global integrability results are obtained.

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1 Introduction and statement of results

Throughout this paper we let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$. Let $1<p<N$ and we consider elliptic equations of the form

$$-\mathrm{div}\mathcal{A}(x,\nabla u(x))=-\mathrm{div}F(x), \tag{1.1}$$

where $\mathcal{A}(x,\xi):\ \Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following structural conditions: there exist $0<\alpha_1\leq\gamma<\infty$ and $0<\alpha_2\leq\gamma<\infty$, such that for almost all $x\in\Omega$

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and all $\xi,\zeta\in\mathbb{R}^N$,

$$\langle\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta),\xi-\zeta\rangle\geq\begin{cases}\alpha_1(1+|\xi|^2+|\zeta|^2)^{\frac{p-2}{2}}|\xi-\zeta|^2, & \text{if } 1<p<2,\\ \alpha_2|\xi-\zeta|^p, & \text{if } p\geq 2\end{cases}\tag{1.2}$$

and

$$|\mathcal{A}(x,\xi)|\leq\gamma|\xi|^{p-1}.\tag{1.3}$$

As far as the vector $F\in\mathbb{R}^N$ in (1.1) is concerned, we assume

$$F\in L^{p'}(\Omega;\mathbb{R}^N),\qquad p'=\frac{p}{p-1}.\tag{1.4}$$

Remark 1.1. The prototype of (1.1) is the nonhomogeneous p-harmonic equation

$$-\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))=-\mathrm{div}F(x).$$

By [1], the operator

$$\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi\tag{1.5}$$

satisfies

$$\langle|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta,\xi-\zeta\rangle\geq\begin{cases}(p-1)(1+|\xi|^2+|\zeta|^2)^{\frac{p-2}{2}}|\xi-\zeta|^2, & \text{if } 1<p<2,\\ 2^{2-p}|\xi-\zeta|^p, & \text{if } p\geq 2\end{cases}$$

and

$$\big||\xi|^{p-2}\xi\big|=|\xi|^{p-1},$$

that is, the operator (1.5) satisfies (1.2) with $\alpha_1=p-1$, $\alpha_2=2^{2-p}$, and (1.3) with $\gamma=1$.

We consider obstacle problems of (1.1): suppose that ψ is any function in Ω with values in $\mathbb{R}\cup\{\pm\infty\}$ and that θ is a measurable function such that

$$\theta_*=\max\{\psi,\theta\}\in\theta+W_0^{1,p}(\Omega).\tag{1.6}$$

Denote

$$\mathcal{K}_{\psi,\theta}(\Omega)=\Big\{v\in W^{1,p}(\Omega):v\geq\psi,\text{ a.e. and } v-\theta\in W_0^{1,p}(\Omega)\Big\}.$$

The function ψ is an obstacle and θ determines the boundary values. Note that $\theta_*\in\mathcal{K}_{\psi,\theta}(\Omega)$.

Definition 1.1. *A solution to the $\mathcal{K}_{\psi,\theta}$-obstacle problem is a function $u\in\mathcal{K}_{\psi,\theta}(\Omega)$ such that*

$$\int_\Omega\langle\mathcal{A}(x,\nabla u),\nabla(w-u)\rangle\mathrm{d}x\geq\int_\Omega\langle F,\nabla(w-u)\rangle\mathrm{d}x\tag{1.7}$$

whenever $w\in\mathcal{K}_{\psi,\theta}(\Omega)$.