On Local Wellposedness of the Schrödinger-Boussinesq System

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Abstract. In this paper we prove that the Schrödinger-Boussinesq system with solution \((u,v,(-\partial_x)^{-2}v_t)\) is locally wellposed in \(H^s \times H^s \times H^{s-1}, s \geq -1/4\). The local wellposedness is obtained by the transformation from the problem into a nonlinear Schrödinger type equation system and the contraction mapping theorem in a suitably modified Bourgain type space inspired by the work of Kishimoto, Tsugawa. This result improves the known local wellposedness in \(H^s \times H^s \times H^{s-1}, s > -1/4\) given by Farah.

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1 Introduction

We study the Cauchy problem of the Schrödinger-Boussinesq system(SBq) in \((t,x) \in \mathbb{R} \times \mathbb{R}:

\begin{align*}
\partial_t u + \partial_x^2 u &= iv, \\
\partial_t^2 v - \partial_x^2 v + \partial_x^4 v + \partial_x^2 v^2 &= \partial_x^2 |u|^2, \\
u(0,x) &= u_0(x),
\end{align*}

(1.1) (1.2) (1.3)

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\[ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad \text{(1.4)} \]

where function \( u \) is complex-valued, while \( v \) can be complex-valued or real-valued.

The system above appears in the study of interaction of solitons in optics and can be considered as a model describing interactions between short and intermediate long waves, see Makhankov [1], Yajima et al. [2], Nishikawa et al. [3]. The short wave term \( u(t,x) \) is described by the Schrödinger equation (1.1) with the potential \( v(t,x) \) satisfying the Boussinesq equation (1.2) and representing the intermediate long wave. When \( \partial_x^2 v^2 \) in (1.2) is replaced with \( \partial_x^2 (\beta |v|^{p-1} v) \), where \( \beta \in \mathbb{R} \) is a constant, the (1.1)-(1.4) is called generalized Schrödinger-Boussinesq system and is studied in Linares and Navas [4] and so on. The results about wellposedness, numerical results, stability of solitary waves etc. for the generalized Schrödinger-Boussinesq system can be found in [5–8] and the references therein.

As for the local wellposedness, Linares and Navas [4] obtained that the solution \((u,v)\) of the generalized Schrödinger-Boussinesq system is locally wellposed in \( C([-T,T]; L^2(\mathbb{R})) \cap L^4([-T,T]; L^\infty(\mathbb{R})) \). The main method of [4] is \( L^p-L^q \) estimates, more precisely, Linares and Navas derived the results by using the global smoothing effects established in [9] for the generalized Boussinesq equation and the standard Strichartz estimates for the Schrödinger equation. We remark that the results of [4] is suit for the Schrödinger-Boussinesq system as well, meanwhile from the method used in [4], it is not possible to obtain corresponding local wellposedness results of the generalized Schrödinger-Boussinesq system in higher dimensions, as the smoothing effects in [9] are the special case of Strichartz type estimates for the generalized Boussinesq equation and do not fit with the Sobolev embedding inequalities in higher dimensions.

Farah [10] has researched following Schrödinger-Boussinesq system

\[
\begin{cases}
\partial_t u + \partial_{xx} u = vu,
\partial_t^2 v - \partial_x^2 v + \partial_t^4 v = -\partial_x^2 |u|^2, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = (v_1)_x(x)
\end{cases}
\quad \text{(1.5)}
\]

and used the classic Bourgain space method to obtain low regularity local wellposedness of \((u,v, (\partial_x^{-1})^s v_t)\) in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > -1/4 \), which can be applied to (1.1)-(1.4) as well. See also Farah, Pastor [11] for the periodic case. Farah [10] introduced norms

\[
\|u\|_{X^{s,b}} = \left\| \left\langle \tau + \xi^2 \right\rangle^b \left\langle \xi^s \right\rangle^s \hat{u} \right\|_{L^2_{\tau,x}} \quad \text{for the Schrödinger equation},
\quad \text{(1.6)}
\]

\[
\|v\|_{X^{s,b}} = \left\| \left\langle |\tau| - \gamma(\xi) \right\rangle^b \left\langle \xi^s \right\rangle^s \hat{v} \right\|_{L^2_{\tau,x}} \quad \text{for the Boussinesq equation},
\quad \text{(1.7)}
\]

and applied the property

\[
\frac{1}{c} \leq \frac{1 + |x-y|}{1 + |x - \sqrt{y^2+y}|} \leq c, \quad \text{for } y \geq 0 \text{ and some constant } c > 0
\quad \text{(1.8)}
\]