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Positive Ground State Solutions for a Critical Nonlocal Problem in Dimension Three

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Abstract. In this paper, we are interested in the following nonlocal problem with critical exponent

$$\begin{cases} -\left(a-b\displaystyle\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=\lambda|u|^{p-2}u+|u|^4u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases}$$

where a,b are positive constants, $2<p<6$, Ω is a smooth bounded domain in $\mathbb{R}^3$ and $\lambda>0$ is a parameter. By variational methods, we prove that problem has a positive ground state solution u_b for $\lambda>0$ sufficiently large. Moreover, we take b as a parameter and study the asymptotic behavior of u_b when $b\searrow 0$.

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Key Words: Nonlocal problem; critical exponent; positive solutions; variational methods.

1 Introduction

We investigate the existence of positive ground state solutions of the following critical nonlocal problem

$$\begin{cases} -\left(a-b\displaystyle\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=\lambda|u|^{p-2}u+|u|^4u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where a,b are positive constants, $2<p<6$, Ω is a smooth bounded domain in $\mathbb{R}^3$ ($2^*=6$ is the critical exponent in dimension three) and $\lambda>0$ is a parameter.

The energy functional associated to (1.1) is defined by

$$I_{b,\lambda}(u)=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{\lambda}{p}\int_\Omega |u|^p\mathrm{d}x-\frac{1}{6}\int_\Omega |u|^6\mathrm{d}x,$$

where $\|u\|^2=\int_\Omega |\nabla u|^2\mathrm{d}x$. Then $I_{b,\lambda}$ is well defined on $H_0^1(\Omega)$ and belongs to $C^1(H_0^1(\Omega),\mathbb{R})$. Obviously, critical points of $I_{b,\lambda}$ are the weak solutions of (1.1). Here, we call $u\in H_0^1(\Omega)$ is a weak solution of (1.1), if for any $\phi\in H_0^1(\Omega)$, it holds

$$\left(a-b\|u\|^2\right)\int_\Omega \nabla u\nabla\phi\mathrm{d}x-\lambda\int_\Omega |u|^{p-2}u\phi\mathrm{d}x-\int_\Omega |u|^4u\phi\mathrm{d}x=0.$$

Moreover, a positive solution of (1.1) is called a positive ground state solution of (1.1), if it possesses the least energy among all positive solutions.

In (1.1), if we replace $a-b\int_\Omega|\nabla u|^2\mathrm{d}x$ by $a+b\int_\Omega|\nabla u|^2\mathrm{d}x$, it becomes to the following critical Kirchhoff type problem

$$\begin{cases}-\left(a+b\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u=\lambda|u|^{p-2}u+|u|^4u, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}\tag{1.2}$$

As well known that Kirchhoff type problem is presented by Kirchhoff [1] in 1883 as an extension of the classical d'Alembert wave equation for free vibration of elastic strings. Such kind of problem is often viewed as being nonlocal due to the appearance of the term $\int_\Omega|\nabla u|^2\mathrm{d}x\Delta u$, which implies that the equation is no longer a pointwise identity. This causes some mathematical difficulties which make the study of such problem particularly interesting. During the past decade, there are many interesting existence results of positive solutions to (1.2) via variational methods. For example, considering a general nonlinearity instead of $\lambda|u|^{p-2}u$ with $2<p<6$, Alves et al. [2] and Figueiredo [3] obtained the existence and multiplicity of positive solutions of (1.2) under $\lambda>0$ is sufficiently large. In the case $1<p<2$, Sun and Liu [4] got the existence of positive solution to (1.2) when $\lambda>0$ is sufficiently small. For the singular case (that is, $0<p<1$), Lei et al. [5] proved that problem (1.2) has at least two positive solution for $\lambda>0$ small enough. For more related results, we refer to [6-10] and the references therein.

On the other hand, the subcritical case of problem (1.1) has been considered in the past few years by some scholars, for example [11-21]. In particular, the first existence and multiplicity results of solutions for problem (1.1) without critical exponent was done by Yin and Liu [11]. As they indicated in their paper, nonlocal problem like (1.1) (involving a negative nonlocal term) provokes some difficulties essential different from those of the Kirchhoff type problem. In [12], Lei et al. showed the existence of at least two positive